(L, M)-NEIGHBORHOOD SPACES

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Abstract. We introduce the notions of (L, M)-neighborhood spaces and (2, M)-fuzzifying neighborhood spaces. We investigate the relations among (L, M)-neighborhood spaces, (L, M)-topological spaces and (2, M)-fuzzifying neighborhood spaces.

1. Introduction and preliminaries

Höhle [8-11] introduced the notions of L-fuzzy topology and L-filters on a completely quasi-monoidal lattice (including GL-monoid) L instead of a completely distributive lattice or a unit interval as the extensions of fuzzy topologies [3,16,18] and fuzzy filters [1,2,4-7]. Kotzé [14] introduced an (L, M)-topological space as a general approach where L and M are frames with 0 and 1. Kim et al. [12] introduced notions of (L, M)-topological spaces as an extension of that of Kotzé [11]. Here, L is a completely distributive lattice with 0 and 1 and M is a strictly two-sided, commutative quantale as an extension of a frame.

In this paper, we introduce notions of (L, M)-neighborhood spaces and (2, M)-fuzzifying neighborhood spaces with respect to Kim [12] as an extension of Demirci [4]. We investigate the relations among (L, M)-neighborhood spaces, (L, M)-topological spaces and (2, M)-fuzzifying neighborhood spaces.

In this paper, let X be a nonempty set and L = (L, ≤, ∨, ∧, ') a completely distributive lattice with the least element 0 and the greatest element 1 in L with an order reversing involution '. The family L^X denotes the set of all fuzzy subsets of a given set X. For each α ∈ L,

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let $\alpha$ denote the constant fuzzy sets of $X$. We denote the characteristic function of a subset $A$ of $X$ by $1_A$. A fuzzy point $x_t$ for $t \in L(t \neq 0)$ is an element of $L^X$ such that

$$x_t(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x. \end{cases}$$

The set of all fuzzy points in $X$ is denoted by $Pt(X)$. We say that $x_t q \lambda$ if $x_t \leq \lambda$. If $x_t \leq \lambda$, we denote $x_t q^\lambda$.

Let $M = (M, \leq, \land, \lor, \bot, \top)$ be a completely distributive lattice with the least element $\bot$ and the greatest element $\top$ in $M$.

**Definition 1.1** ([8-11,17]). A triple $(M, \leq, \odot)$ is called a \textit{strictly two-sided, commutative quantale} (stsc-quantale, for short) iff it satisfies the following properties:

- (M1) $(M, \odot)$ is a commutative semigroup,
- (M2) $a = a \odot \top$, for each $a \in M$,
- (M3) $\odot$ is distributive over arbitrary joins, i.e.,

$$\left( \bigvee_{i \in \Gamma} a_i \right) \odot b = \bigvee_{i \in \Gamma} (a_i \odot b).$$

**Remark 1.2.** ([8-11,13,17])

1. Each frame is a stsc-quantale. In particular, the unit interval $([0, 1], \leq, \land, 0, 1)$ is a stsc-quantale.
2. Every left continuous t-norm on $([0, 1], \leq, t)$ with $\odot = t$ is a stsc-quantale.
3. Every GL-monoid is a stsc-quantale.

**Definition 1.3** ([12,14]). A map $T : L^X \to M$ is called an $(L, M)$-\textit{topology} on $X$ if it satisfies the following conditions:

- (LO1) $T(\emptyset) = T(\top) = \bot$,
- (LO2) $T(\mu_1 \land \mu_2) \geq T(\mu_1) \odot T(\mu_2)$, for all $\mu_1, \mu_2 \in L^X$,
- (LO3) $T(\bigvee_{i \in \Lambda} \mu_i) \geq \bigwedge_{i \in \Lambda} T(\mu_i)$, for any $\{\mu_i\}_{i \in \Lambda} \subset L^X$.

The pair $(X, T)$ is called an $(L, M)$-\textit{topological space}.

Let $(X, T_1)$ and $(Y, T_2)$ be $(L, M)$-topological spaces. A map $\phi : (X, T_1) \to (Y, T_2)$ is called \textit{LF-continuous} iff $T_2(\lambda) \leq T_1(\phi^{-}(\lambda))$, for all $\lambda \in L^Y$. 
Remark 1.4 ([12]). Let $L = \{0, 1\}$ be given and $2^X \cong P(X)$ in a sense $1_A \in 2^X$ iff $A \in P(X)$. A map $\tau : P(X) \to M$ is called a $(2, M)$-fuzzifying topology on $X$ if it satisfies the following conditions:

(O1) $\tau(X) = \tau(\emptyset) = \top$,
(O2) $\tau(A \cap B) \geq \tau(A) \odot \tau(B)$, for all $A, B \in P(X)$,
(O3) $\tau(\bigcup_{i \in \Lambda} A_i) \geq \bigwedge_{i \in \Lambda} \tau(A_i)$, for any $\{A_i\}_{i \in \Lambda} \subseteq P(X)$.

The pair $(X, \tau)$ is called a $(2, M)$-fuzzifying topological space.

Let $(X, \tau_1)$ and $(Y, \tau_2)$ be $(2, M)$-fuzzifying topological spaces. A map $\phi : (X, \tau_1) \to (Y, \tau_2)$ is called fuzzifying continuous iff $\tau_2(A) \leq \tau_1(\phi^{-1}(A))$, $\forall A \in P(Y)$.

\section{$(L, M)$-filter spaces}

Definition 2.1. A map $F : L^X \to M$ is called an $(L, M)$-filter on $X$ if it satisfies the following conditions:

(LF1) $F(\emptyset) = \bot$ and $F(\top) = \top$.
(LF2) $F(\lambda \land \mu) \geq F(\lambda) \odot F(\mu)$ for all $\lambda, \mu \in L^X$.
(LF3) If $\lambda \leq \mu$, then $F(\lambda) \leq F(\mu)$ for all $\lambda, \mu \in L^X$.

The pair $(X, F)$ is called an $(L, M)$-filter space. Let $(X, F_1)$ and $(Y, F_2)$ be $(L, M)$-filter spaces. A map $\phi : (X, F_1) \to (Y, F_2)$ is called a filter map iff $F_2(\mu) \leq F_1(\phi^{-1}(\mu))$, for all $\mu \in L^Y$.

Remark 2.2. In the sense in Remark 1.4, a map $F : P(X) \to M$ is called a $(2, M)$-fuzzifying filter on $X$ if it satisfies the following conditions:

(F1) $F(X) = \top$ and $F(\emptyset) = \bot$,
(F2) $F(A \cap B) \geq F(A) \odot F(B)$, for all $A, B \in P(X)$,
(F3) If $A \subset B$, then $F(A) \subset F(B)$ for any $A, B \in P(X)$.

The pair $(X, F)$ is called an $(2, M)$-fuzzifying filter spaces. Let $(X, F_1)$ and $(Y, F_2)$ be $(2, M)$-fuzzifying filter spaces. A map $\phi : (X, F_1) \to (Y, F_2)$ is called a fuzzifying filter map iff $F_2(A) \leq F_1(\phi^{-1}(A))$, for all $A \in P(Y)$.

Remark 2.3. (1) If $L = ([0, 1], \land)$ and $M = \{0, 1\}$, $(L, M)$-filter space is the concept of Chang [3].

(2) If $L = \{0, 1\}$ and $M = ([0, 1], \odot = \land)$, $(L, M)$-filter space is the concept of generalised filter [1,2].
(3) If $L$ and $M$ are frames with 0 and 1, $(L, M)$-filter space is the concept of Gähler [5,6].

**Theorem 2.4.** Let $(X, \mathcal{F})$ be an $(L, M)$-filter space. We define a function $\mathcal{T}_F : L^X \to M$ as follows:

$$
\mathcal{T}_F(\lambda) = \begin{cases} 
\mathcal{F}(\lambda), & \text{if } \lambda \neq 0, \\
\top, & \text{if } \lambda = 0.
\end{cases}
$$

Then $(X, \mathcal{T}_F)$ is an $(L, M)$-topological space.

**Proof.** We only show the condition (LO3). For $\lambda_j \in L^X$, since $\lambda_j \leq \bigvee_{j \in J} \lambda_j$ for all $j \in J$, we have $\mathcal{F}(\lambda_j) \leq \mathcal{F}(\bigvee_{j \in J} \lambda_j)$, so

$$
\bigwedge_{j \in J} \mathcal{T}_F(\lambda_j) \leq \mathcal{T}_F(\bigvee_{j \in J} \lambda_j).
$$

\hfill \Box

**Theorem 2.5.** Let $(X, F)$ be a $(2, M)$-fuzzifying filter space. We define a function $\mathcal{F}_F : L^X \to M$ as follows:

$$
\mathcal{F}_F(\lambda) = \bigwedge_{r \in L} F(\lambda_r),
$$

where $\lambda_r = \{x \in X : \lambda(x) \geq r\}$ for $r \in L - \{0\}$. Then $\mathcal{F}_F$ is an $(L, M)$-filter.

**Proof.** (LF1) Clear.

(LF2) For each $\lambda, \mu \in L^X$, we have

$$
\mathcal{F}_F(\lambda \wedge \mu) = \bigwedge_{r \in L} F((\lambda \wedge \mu)_r) = \bigwedge_{r \in L} F(\lambda_r \wedge \mu_r)
\geq \bigwedge_{r \in L} \left( F(\lambda_r) \odot F(\mu_r) \right) \geq \bigwedge_{r \in L} F(\lambda_r) \odot \bigwedge_{r \in L} F(\mu_r)
= \mathcal{F}_F(\lambda) \odot \mathcal{F}_F(\mu).
$$

(LF3) If $\lambda \leq \mu$, then $\lambda_r \subseteq \mu_r$. Thus

$$
\mathcal{F}_F(\lambda) = \bigwedge_{r \in L} F(\lambda_r) \leq \bigwedge_{r \in L} F(\mu_r) = \mathcal{F}_F(\mu)
$$

\hfill \Box
LEMMA 2.5. Let $A \in P(X)$ and $\alpha \in L - \{0\}$. Then $F_F(\alpha \cdot 1_A) = F(A)$.

THEOREM 2.6. Let $(X, F_1)$, $(Y, F_2)$ be $(2, M)$-fuzzifying filter spaces. A map $\phi : (X, F_1) \to (Y, F_2)$ is a fuzzifying filter map iff $\phi : (X, F_1) \to (Y, F_{F_2})$ is a filter map.

Proof. For each $\mu \in L^Y$, we have

$$F_{F_1}(\phi^-(\mu)) = \bigwedge_{r \in L} F_1((\phi^-(\mu))_r) \geq \bigwedge_{r \in L} F_2(\mu_r) = F_{F_2}(\mu).$$

Conversely, suppose there exists $A \in P(X)$ such that $F_1(\phi^{-1}(A)) \not\geq F_2(A)$. It implies $F_{F_1}(1_{\phi^{-1}(A)}) = F_1(\phi^{-1}(A)) \not\geq F_2(A) = F_{F_2}(1_A)$. \(\square\)

EXAMPLE 2.7. Let $X = \{x, y, z\}$ be a set. Define a binary operation $\otimes$ on $M = [0, 1]$ by $x \otimes y = \max\{0, x + y - 1\}$. Then $M = ([0, 1], \leq, \otimes)$ is a stsc-quantale. Define a $(2, M)$-fuzzifying topology $F : P(X) \to [0, 1]$ as follows:

$$F(A) = \begin{cases} 1, & \text{if } A = X \\ 0.8, & \text{if } A = \{x, y\}, \\ 0.6, & \text{if } A = \{y\}, \\ 0.7, & \text{if } B = \{y, z\}, \\ 0, & \text{otherwise.} \end{cases}$$

For $\lambda, \mu \in [0, 1]^X$ with

$\lambda(x) = 0.3, \lambda(y) = 0.7, \lambda(z) = 0.5, \mu(x) = 0.7, \mu(y) = 0.2, \mu(z) = 0.5,$

we have

$$(\lambda)_r \in \{\{y\}, \{y, z\}, X\}, \ (\mu)_r \in \{\{x\}, \{x, z\}, X\}.$$

Hence $F_F(\lambda) = 0.6$ and $F_F(\mu) = 0.$
3. \((L, M)\)-neighborhood spaces.

**Definition 3.1.** An \((L, M)\)-neighborhood system on \(X\) is a set \(Q = \{Q_{x_t} \mid x_t \in Pt(X)\}\) of maps \(Q_{x_t} : L^X \to M\) such that for each \(\lambda, \mu \in L^X\), we have

- (LN1) \(Q_{x_t}\) is an \((L, M)\)-filter on \(X\).
- (LN2) \(Q_{x_t}(\lambda) > \perp\) implies \(x_t q \lambda\).
- (LN3) \(Q_{x_t}(\lambda) = \bigvee_{x_t q \mu \leq \lambda} \left(\bigwedge y q \mu Q_{y_t}(\mu)\right)\).

The pair \((X, Q)\) is called an \((L, M)\)-neighborhood space.

Let \((X, Q_1)\) and \((Y, Q_2)\) be \((L, M)\)-neighborhood spaces. A function \(\phi : (X, Q_1) \to (Y, Q_2)\) is called an \(\text{LN}\)-map if for all \(\lambda \in L^Y\) and for all \(x_t \in Pt(X)\),

\[
\phi^{\leftarrow}(x_t q \lambda) \geq Q_{\phi^{-1}(x_t)}(\lambda). 
\]

**Remark 3.2.** By the sense of Remark 2.2, since \(x_1 q 1_A\) iff \(x \in A\), a map \(N_x : P(X) \to M\) is called a \((2, M)\)-fuzzifying neighborhood of \(x \in X\) if it satisfies the following conditions:

- (N1) \(N_x\) is a \((2, M)\)-filter on \(X\).
- (N2) \(N_x(A) > \perp\) implies \(x \in A\).
- (N3) \(N_x(A) = \bigvee_{x \in B \subseteq A} \left(\bigwedge y \in B N_y(B)\right)\).

A set \(N = \{N_x \mid x \in X\}\) is called a \((2, M)\)-fuzzifying neighborhood system on \(X\). A map \(f : (X, N_1) \to (Y, N_2)\) is called a \(N\)-fuzzifying map if for each \(A \in P(Y)\) and for each \(x \in X\), \((N_1)_x(f^{-1}(A))\) ≥ \((N_2)_f(x)(A)\).

**Theorem 3.3.** Let \((X, T)\) be an \((L, M)\)-topological space and \(x_t \in Pt(X)\). Define a map \(Q^T_{x_t} : L^X \to M\) as:

\[
Q^T_{x_t}(\lambda) = \begin{cases} 
\bigvee \{T(\mu) \mid x_t q \mu \leq \lambda\} & \text{if } x_t q \lambda, \\
0 & \text{if } x_t q \lambda.
\end{cases}
\]

Then
1. \(Q^T = \{Q^T_{x_t} \mid x_t \in Pt(X)\}\) is an \((L, M)\)-neighborhood system on \(X\),
2. if \(t < s\) for \(t, s \in L\), then \(Q^T_{x_t}(\lambda) \leq Q^T_{x_s}(\lambda)\).

**Proof.** (1) (LF1) and (LF3) are easily proved.
(2) Suppose there exist \(\lambda, \mu \in L^X\) such that

\[
Q^T_{x_t}(\lambda \land \mu) \not\geq Q^T_{x_t}(\lambda) \circ Q^T_{x_t}(\mu).
\]
By the definition of \( Q^T_{x_t}(\lambda) \) and (M3) of Definition 1.1, there exists \( \lambda_1 \in L^X \) with \( x_t \preceq \lambda_1 \leq \lambda \) such that

\[
Q^T_{x_t}(\lambda \land \mu) \not\geq T(\lambda_1) \odot Q^T_{x_t}(\mu).
\]

Again, by the definition of \( Q^T_{x_t}(\mu) \) and (M3) of Definition 1.1, there exists \( \mu_1 \in L^X \) with \( x_t \preceq \mu_1 \leq \mu \) such that

\[
Q^T_{x_t}(\lambda \land \mu) \not\geq T(\lambda_1) \odot T(\mu_1).
\]

Since \( x_t \preceq (\lambda_1 \land \mu_1) \leq \lambda \land \mu \), we have

\[
Q^T_{x_t}(\lambda \land \mu) \geq T(\lambda_1 \land \mu_1) \geq T(\lambda_1) \odot T(\mu_1).
\]

It is a contradiction. Hence

\[
Q^T_{x_t}(\lambda \land \mu) \geq Q^T_{x_t}(\lambda) \odot Q^T_{x_t}(\mu), \quad \forall \lambda, \mu \in L^X
\]

So, \( Q_{x_t} \) is an \((L, M)\)-filter on \( X \).

(LN2) It is easy from the definition of \( Q^T_{x_t} \).

(LN3) For all \( \lambda \in L^X \) with \( x_t \preceq \lambda \), we have

\[
T(\mu) \leq \bigwedge \{ Q^T_{y_s}(\mu) \mid y_s \preceq \lambda \} \leq Q^T_{x_t}(\mu) \leq Q^T_{x_t}(\lambda).
\]

Therefore,

\[
Q^T_{x_t}(\lambda) = \bigvee_{x_t \preceq \lambda} T(\mu) \leq \bigvee_{x_t \preceq \lambda} \left( \bigwedge_{y_s \preceq \lambda} Q^T_{y_s}(\mu) \right) \leq Q^T_{x_t}(\lambda).
\]

This means that \( Q^T_{x_t}(\lambda) = \bigvee_{x_t \preceq \lambda} \left( \bigwedge_{y_s \preceq \lambda} Q^T_{y_s}(\mu) \right) \).

(2) For \( t < s \) with \( t, s \in L \) and \( \forall \lambda \in L^X \), since

\[
\{ \mu \in L^X \mid x_t \preceq \mu \leq \lambda \} \subset \{ \rho \in L^X \mid x_s \preceq \rho \leq \lambda \},
\]

we have \( Q^T_{x_t}(\lambda) \leq Q^T_{x_s}(\lambda) \). \( \square \)
Example 3.4. Let \( X = \{ x, y \} \) be a set and \( L = M = [0, 1] \) a completely distributive lattice. Define a binary operation \( \otimes \) on \( M = [0, 1] \) by \( x \otimes y = \max\{0, x + y - 1\} \). Then \((0, 1], \leq, \otimes)\) is a stsc-quantale. Let \( \mu, \rho \in [0, 1]^X \) be defined as follows:

\[
\mu(x) = 0.6, \quad \mu(y) = 0.3 \quad \rho(x) = 0.5, \quad \rho(y) = 0.7.
\]

We define an \((L, M)\)-topology \( T : [0, 1]^X \rightarrow [0, 1] \) as follows:

\[
T(\lambda) = \begin{cases}
1, & \text{if } \lambda = \overline{\mathbb{1}}, \overline{\mathbb{0}} \\
0.8, & \text{if } \lambda = \mu, \\
0.3, & \text{if } \lambda = \rho, \\
0.7, & \text{if } \lambda = \mu \lor \rho, \\
0.2, & \text{if } \lambda = \mu \land \rho, \\
0, & \text{otherwise}.
\end{cases}
\]

We obtain \( Q_{T_{x,0.5}}^T, Q_{T_{y,0.8}}^T : [0, 1]^X \rightarrow [0, 1] \) as:

\[
Q_{T_{x,0.5}}^T(\lambda) = \begin{cases}
1 & \text{if } \lambda = \overline{\mathbb{1}}, \\
0.8 & \text{if } \mu \leq \lambda \neq \overline{\mathbb{1}}, \\
0 & \text{otherwise}.
\end{cases}
\]

\[
Q_{T_{y,0.8}}^T(\lambda) = \begin{cases}
1 & \text{if } \lambda = \overline{\mathbb{1}}, \\
0.8 & \text{if } \mu \leq \lambda \neq \overline{\mathbb{1}}, \\
0.3 & \text{if } \rho \leq \lambda \neq \mu, \\
0.2 & \text{if } \rho \land \mu \leq \lambda \neq \rho, \\
0 & \text{otherwise}.
\end{cases}
\]

From Theorem 3.3, we can obtain the following corollary.

**Corollary 3.5.** Let \((X, \tau)\) be a \((2, M)\)-fuzzifying topological space. We define a function \((\mathcal{N}_\tau)_x : P(X) \rightarrow M\) by

\[
(\mathcal{N}_\tau)_x(A) = \bigvee_{x \in B \subseteq A} \tau(B).
\]

Then \((\mathcal{N}_\tau)_x\) is a \((2, M)\)-fuzzifying neighborhood of \( x \in X \).
Theorem 3.6. Let $Q = \{Q_{x_t} : L^X \to M \mid x_t \in Pt(X)\}$ be a family of $Q_{x_t}$ satisfying (LN1) and (LN2) of Definition 3.1. We define a map $T^Q : L^X \to M$ as follows:

$$T^Q(\lambda) = \begin{cases} \bigwedge \{Q_{x_t}(\lambda) \mid x_t q \lambda\} & \text{if } \lambda \neq \overline{0} \\ \top & \text{if } \lambda = \overline{0} \end{cases}$$

Then we have the following properties.

1. $T^Q$ is an $(L, M)$-topology on $X$.
2. If $Q = \{Q_{x_t} \mid x_t \in Pt(X)\}$ is an $(L, M)$-neighborhood system on $X$, then $Q^T = Q_{x_t}$, for all $x_t \in Pt(X)$.
3. If $Q_1$ and $Q_2$ are $(L, M)$-neighborhood systems on $X$ such that $T^{Q_1} = T^{Q_2}$, then $Q_1 = Q_2$.

Proof. (1) (LO1) is trivial.
(LO2) For $\lambda, \mu \in L^X$, we have

$$T^Q(\lambda \land \mu)$$

$$= \bigwedge \{Q_{x_t}(\lambda \land \mu) \mid x_t q(\lambda \land \mu)\}$$

$$\geq \bigwedge \{Q_{x_t}(\lambda) \lor Q_{x_t}(\mu) \mid x_t q (\lambda \land \mu)\}$$

$$\geq \left( \bigwedge \{Q_{x_t}(\lambda) \mid x_t q(\lambda \land \mu)\} \right) \lor \left( \bigwedge \{Q_{x_t}(\mu) \mid x_t q (\lambda \land \mu)\} \right)$$

$$= T^Q_{x_t}(\lambda) \lor T^Q_{x_t}(\mu).$$

(LO3) Suppose $T^Q(\bigvee_{j \in J} \mu_j) \neq \bigwedge_{j \in J} T^Q(\mu_j)$. Then there exists a family $\{\mu_j \mid x_t q(\bigvee_{j \in J} \mu_j)\}$ such that

$$Q_{x_t}(\bigvee_{j \in J} \mu_j) \neq \bigwedge_{j \in J} Q_{x_t}(\mu_j).$$

Since $x_t q(\bigvee_{j \in J} \mu_j)$, there exists $j \in J$ such that $x_t q \mu_j$ such that

$$Q_{x_t}(\bigvee_{j \in J} \mu_j) \neq Q_{x_t}(\mu_j).$$
It is a contradiction for a filter $Q_{x_t}$. Hence the result holds.

(2) 
$$Q_{x_t}^T(\lambda) = \bigvee \{ T^Q(\mu) \mid x_t q \mu \leq \lambda \}$$
$$= \bigvee \left\{ \bigwedge \{ Q_{y_s}(\mu) \mid y_s q \mu \} \mid x_t q \mu \leq \lambda \right\}$$
$$= Q_{x_t}(\lambda) \quad \text{(by (LN 3))}.$$ 

(3) Since $T^Q_1 = T^Q_2$, for $\lambda \in L^X$ and $x_t \in Pt(X)$, we have

$$\left( Q_1 \right)_{x_t}(\lambda) = \bigvee \left\{ \bigwedge \{ (Q_1)_{y_s}(\mu) \mid y_s q \mu \} \mid x_t q \mu \leq \lambda \right\}$$
$$= \bigvee \{ T^Q_1(\mu) \mid x_t q \mu \leq \lambda \}$$
$$= \bigvee \{ T^Q_2(\mu) \mid x_t q \mu \leq \lambda \}$$
$$= \bigvee \left\{ \bigwedge \{ (Q_2)_{y_s}(\mu) \mid y_s q \mu \} \mid x_t q \mu \leq \lambda \right\}$$
$$= (Q_2)_{x_t}(\lambda).$$

Hence $Q_1 = Q_2$. \qed

**Corollary 3.7.** Let $N_x : P(X) \to M$ be a map satisfying (N1) and (N2) for all $x \in X$. We define a map $\tau_N : P(X) \to M$ by

$$\tau_N(A) = \bigwedge_{x \in A} N_x(A).$$

Then:

1. $(X, \tau_N)$ is a $(2, M)$-fuzzifying topological space,
2. if $N$ is a $(2, M)$-fuzzifying neighborhood system, then $(N_{\tau_N})_x = N_x$,
3. if $N_1$ and $N_2$ are $(2, M)$-fuzzifying neighborhood systems on $X$ such that $\tau_{N_1} = \tau_{N_2}$, then $N_1 = N_2$.

The following lemma is easily proved.

**Lemma 3.8.** If $x_t q \lambda$, then there exists $\mu_{x_t} \in L^X$ such that $x_t q \mu_{x_t} \leq \lambda$. Thus $\lambda = \bigvee_{x_t q \lambda} \mu_{x_t}$. 
THEOREM 3.9. Let $(X, T)$ be an $(L, M)$-topological space and $Q^T$ an $(L, M)$-neighborhood system in $(X, T)$. Then $T = T^{Q^T}$.

Proof. Since $Q_{x_t}^T(\lambda) = \vee\{T(\mu) \mid x_t q \mu \leq \lambda\} \geq T(\lambda)$ for all $x_t q \lambda$, we have $\wedge\{Q_{x_t}^T(\lambda) \mid x_t q \lambda\} \geq T(\lambda)$. So, $T^{Q^T} \geq T$.

Conversely, there exists $\lambda \in L^X$ such that $T^{Q^T}(\lambda) \not\leq T(\lambda)$. For each $x_t \in P(X)$ with $x_t q \lambda$, if $x_t q \mu_{x_t} \leq \lambda$, then by Lemma 3.8, we get

$$\lambda = \bigvee_{x_t q \lambda} \mu_{x_t}.$$  

Thus, $\wedge\{T(\mu_{x_t}) \geq T(\lambda)\} \geq T^{Q^T}(\lambda)$.

It is a contradiction. Thus, $T \geq T^{Q^T}$. □

COROLLARY 3.10. Let $(X, \tau)$ be a $(2, M)$-fuzzifying topological space and $\mathcal{N}_\tau$ a $(2, M)$-fuzzifying neighborhood system in $(X, \tau)$. Then $\tau = \tau_{\mathcal{N}_\tau}$.

THEOREM 3.11. Let $(X, Q_1), (Y, Q_2)$ be $(L, M)$-neighborhood spaces. A mapping $\phi : (X, Q_1) \to (Y, Q_2)$ is an LN-map iff $\phi : (X, T^{Q_1}) \to (Y, T^{Q_2})$ is LF-continuous.

Proof. Since $\forall \lambda \in L^Y, \forall x_t \in P(X), x_t q \phi^- (\lambda)$ if and only if $(\phi^-(x_t) = \phi(x_t) q \lambda$ and

$$\{y_t \in P_t(Y) \mid y_t q \lambda\} \supset \{\phi(x_t) \in P_t(Y) \mid x_t \in P_t(X), \phi(x_t) q \lambda\},$$

we have

$$T^{Q_2}(\lambda) = \bigwedge\{(Q_2)_{y_t} (\lambda) \mid y_t q \lambda\}$$

$$\leq \bigwedge\{(Q_2)_{\phi^-(x_t)} (\lambda) \mid \phi^-(x_t) q \lambda\}$$

$$\leq \bigwedge\{(Q_1)_{x_t}(\phi^- (\lambda)) \mid x_t q \phi^- (\lambda)\}$$

$$= T^{Q_1}(\phi^-(\lambda)).$$
Thus, \( \phi : (X, T^{Q_1}) \to (Y, T^{Q_2}) \) is LF-continuous.

Conversely, since \( \forall \lambda \in L^Y, T^{Q_2}(\lambda) \leq T^{Q_1}(\phi^-(\lambda)) \), \( Q_1 = Q^{T^{Q_1}} \) and \( Q_2 = Q^{T^{Q_2}} \), we have

\[
(Q_2)_{\phi^-(x_t)}(\lambda) = \bigvee \{ T^{Q_2}(\mu) \mid \phi^-(x_t) \land \mu \leq \lambda \}
\leq \bigvee \{ T^{Q_2}(\mu) \mid x_t \land \phi^-(\mu) \leq \phi^-(\lambda) \}
\leq \bigvee \{ T^{Q_1}(\phi^-(\mu)) \mid x_t \land \phi^-(\mu) \leq \phi^-(\lambda) \}
\leq (Q_1)_{x_t}(\phi^-(\lambda)).
\]

Hence the proof is complete . \( \square \)

**Corollary 3.12.** Let \( (X, N_1) \) and \( (Y, N_2) \) be \((2, M)\)-fuzzifying neighborhood spaces. A map \( f : (X, N_1) \to (Y, N_2) \) is a \( N \)-fuzzifying map iff \( f : (X, \tau_{N_1}) \to (Y, \tau_{N_2}) \) is fuzzifying continuous.

From Theorems 3.9 and 3.11 we obtain the following corollaries.

**Corollary 3.13.** Let \( (X, T_1) \) and \( (Y, T_2) \) be \((L, M)\)-topological spaces. A mapping \( \phi : (X, T_1) \to (Y, T_2) \) is LF-continuous if and only if \( \phi : (X, Q^{T_1}) \to (Y, Q^{T_2}) \) is an LN-map.

**Corollary 3.14.** Let \( (X, \tau_1) \) and \( (Y, \tau_2) \) be \((2, M)\)-fuzzifying topological spaces. A map \( f : (X, \tau_1) \to (Y, \tau_2) \) is fuzzifying continuous iff \( f : (X, N_{\tau_1}) \to (Y, N_{\tau_2}) \) is an \( N \)-fuzzifying map.

**References**

(L, M)-neighborhood spaces


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