RIORDAN MATRICES WITH THE SPECIAL A-SEQUENCES IN THE BELL SUBGROUP

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Abstract. In this paper, we study the Riordan matrices with the A-sequence \((1,1,\ldots,1,0,\ldots)\) where 1’s appear in \(m\) times. As a result, we obtain new Riordan matrices and give their lattice path interpretations.

1. Introduction

The concept of the Riordan group \((R,\ast)\) has been introduced by Shapiro et. al. [5]. A Riordan matrix \(D = \{d_{n,k}\}_{n,k \in \mathbb{N}_0}\) where \(\mathbb{N}_0 := \{0,1,2,3,\ldots\}\) is defined by a pair of formal power series \(g(z) = g_0 + g_1 z + g_2 z^2 + \ldots\) and \(f(z) = f_1 z + f_2 z^2 + \ldots\) with \(g_0 \neq 0\) and \(f_1 \neq 0\) such that the generic element is

\[ d_{n,k} = [z^n]g(z)(f(z))^k, \]

where \([z^n]\) is the coefficient operator. Then \(D\) is an infinite lower triangular matrix with nonzero diagonal entries. We denote a Riordan matrix by \(D = (g(z), f(z))\). The concept of the Riordan matrix is useful to get some combinatorial sums and identities. If we give the operation \(\ast\) being the usual matrix multiplication to the set of all Riordan matrices \(R\) as follows:

\[ (g(z), f(z)) (h(z), \ell(z)) = (g(z)h(f(z)), \ell(f(z))) \]

then \((R,\ast)\) forms a group, which is called the Riordan group. The identity element is \(I = (1,z)\) and the inverse element is given by \((g(z), f(z))^{-1} = \left(\frac{1}{g(f(z))}, \overline{f}(z)\right)\) where \(\overline{f}(f(z)) = f(\overline{f}(z)) = z\). There are some important subgroups of the Riordan group [6]. In particular, we are interested

Received November 1, 2007.
2000 Mathematics Subject Classification: Primary: 05A15; secondary: 05A40.
Key words and phrases: Riordan array, A-sequence, lattice path.
in the Bell subgroup $B$ defined by

$$B = \left\{ \left( \frac{f(z)}{z}, f(z) \right) | f(z) = f_1 z + f_2 z^2 + \ldots, f_1 \neq 0 \right\}.$$ 

From now on, we will call the element in the Bell subgroup the Bell matrix.

Rogers [4] and Merlini et. al. [3] obtained a characterization of a Riordan matrix by some sequences as the following theorem (also see [1], [2]).

**Theorem 1.1.** Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then $D$ is a Riordan matrix if and only if there exist two sequences $A = \{a_0, a_1, a_2, \ldots\}$ and $Z = \{z_0, z_1, z_2, \ldots\}$ with $a_0 \neq 0$, $z_0 \neq 0$ such that

(i) $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}$ ($k, n = 0, 1, \ldots$),
(ii) $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}$, ($n = 0, 1, \ldots$).

It is known that if $A_D(z)$ and $Z_D(z)$ are the generating functions of the $A$- and $Z$-sequences of a Riordan matrix $D = (g(z), f(z))$ respectively, then it can be proven that $f(z)$ and $g(z)$ are the solutions of the following functional equations:

\[ (1) \quad f(z) = z A_D(f(z)), \quad \text{and} \quad g(z) = g(0)/(1 - z Z_D(f(z))). \]

A simple computation shows that if a Bell matrix has the $A$-sequence $\{a_0, a_1, a_2, \ldots\}$ then the $Z$-sequence is $\{a_1, a_2, \ldots\}$. Hence it suffices to consider $A$-sequence when we study a Bell matrix. Pascal matrix is an example of a Bell matrix with $A$-sequence $\{1, 1, 0, 0, \ldots\}$. In this paper, more generally, we observe a Bell matrix $D_m$ with $A$-sequence $A_m := \{1, \ldots, 1, 0, 0, \ldots\}$ where 1’s appear in $m$ times, $m \geq 3$. As a result, we give a combinatorial interpretation to $D_m$ for $m \geq 1$.

**2. Bell matrices with $A$-sequence $A_m$**

Let us consider a sequence $A_m = \{1, \ldots, 1, 0, 0, \ldots\}$ where 1’s appear in $m$ times. We denote a Bell matrix with the $A$-sequence $A_m$ by $D_m = [d_{n,k}^{(m)}]_{n,k \in \mathbb{N}_0} = \left( \frac{f_m(z)}{z}, f_m(z) \right)$. By the first equation in (1), $f_m(z)$ satisfies

\[ (2) \quad f_m(z) = z \left( 1 + f_m(z) + \ldots + (f_m(z))^{m-1} \right). \]
It is easy to see that $D_1 = (1, z)$ is the identity matrix, $D_2 =\left(\frac{1}{1-z}, \frac{z}{1-z}\right)$ is the Pascal matrix and $D_3 = \left(\frac{f_2(z)}{z}, f_2(z)\right)$ is the directed animal matrix (A064189)[7] where $f_2(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z}$. For the case $m \geq 4$, $D_m$ is unknown. But it is known that the 0-th column entry $d_{n,0}^{(m)}$ of $D_m (m \geq 3)$ counts the number of Dyck $n$-paths with no \(UU \ldots UD\), i.e. no arrangement of consecutive $m$ $U$'s (See A001006, A036765, A036766, A036767 in [7]). Equivalently, $d_{n,0}^{(m)} (n \geq 0, m \geq 3)$ can be interpreted by the number of all paths from $(0,0)$ to $(n,n-k)$ using the steps $H = (1,0) = \rightarrow$ and $V = (0,1) = \uparrow$ with no consecutive $m$ $V$'s. Further, this interpretation can be expanded to $d_{n,k}^{(m)}$ for $n \geq k \geq 0$, $m \geq 1$.

**Theorem 2.1.** Let $D_m = \{d_{n,k}^{(m)} \}_{n,k \geq 0}$ be a Bell matrix with the $A$-sequence $A_m = \{1, \ldots, 1, 0, 0, \ldots\}$ and $d_{0,0}^{(m)} = 1$. Then, $d_{n,k}^{(m)}$ counts the number of all paths from $(0,0)$ to $(n,n-k)$ using the horizontal step $H = (1,0)$ and the vertical step $V = (0,1)$ that has no consecutive $m$ $V$'s, which do not pass through the line $y = x$ for $n \geq k \geq 0$, $m \geq 1$.

**Proof.** We fix $m \geq 1$. Since $D_m$ is a Bell matrix with the $A$-sequence $A_m = \{1,1, \ldots, 1, 0, 0, \ldots\}$, the $Z$-sequence of $D_m$ is $Z_m = \{1,1, \ldots, 1, 0, 0, \ldots\}$ and hence

$$d_{n,0}^{(m)} = \sum_{\ell=0}^{m-2} d_{n-1, \ell}^{(m)} \text{ and } d_{n,k}^{(m)} = d_{n-1,k-1}^{(m)} + \sum_{\ell=0}^{m-2} d_{n-1,k+\ell}^{(m)} \text{ for } n \geq k \geq 1.$$  

Let $\tilde{d}_{n,k}^{(m)}$ be the number of all paths from $(0,0)$ to $(n,n-k)$ using the horizontal step $H = (1,0)$ and the vertical step $V = (0,1)$ that has no consecutive $m$ $V$’s (denoted by $V_1 \ldots V_m$), which do not pass through the line $y = x$ for $n \geq k \geq 0$, $m \geq 1$. In particular, $\tilde{d}_{n,0}^{(m)}$ counts the number of all paths from $(0,0)$ to $(n,n)$ using the steps $H$ and $V$ that has no $V_1 \ldots V_m$. We define $\tilde{d}_{0,0}^{(m)} := 1$. First, we notice that a path from $(0,0)$ to $(n,n-k)$ must pass through at least one of the points $(n-1,\ell)$ for $\ell = 0, 1, \ldots, n-k$. In counting process, to avoid the duplication, we may assume that the path starting at $(0,0)$ and arriving at $(n-1,\ell)$
only has a horizontal step as the next step. It is obvious that

\[ d_{n,0}^{(m)} = \sum_{\ell=0}^{n-1} d_{n-1,\ell}^{(m)} \text{ for } 1 \leq n < m \text{ and } d_{n,k}^{(m)} = \sum_{\ell=0}^{k} d_{n-1,k-\ell}^{(m)} \text{ for } 1 \leq k \leq n \leq m \]

since no path has the arrangement \( V_1 \ldots V_m \).

Now, let us consider \( d_{m,0}^{(m)} \). There are \( m-1 \) points \((m-1, 0), \ldots, (m-2, m-2)\) that the path from \((0, 0)\) to \((m, m)\) must pass through. If the path pass through the point \((m-1, 0)\), then it has the arrangement \( V_1 \ldots V_m \). Other points have no problem to pass through. So \( d_{m,0}^{(m)} = \sum_{\ell=0}^{m-2} d_{m-1,\ell}^{(m)} \).

On the other hand, let \( 1 \leq m < n \). Let us consider the path from \((0, 0)\) to \((n, n)\). To avoid the arrangement \( V_1 \ldots V_m \), no path must pass the points \((n-1, \ell)\) for \( \ell = 0, 1, \ldots, n-m \). Since the number of all paths from \((0, 0)\) to \((n-1, \ell)\) that has no \( V_1 \ldots V_m \) is \( d_{n-1,n-\ell-1}^{(m)} \), we have

\[ d_{n,0}^{(m)} = \sum_{\ell=n-m-1}^{n-1} d_{n-1,n-\ell-1}^{(m)} = \sum_{\ell=0}^{m-2} d_{n-1,\ell}^{(m)}. \]

Next, let us consider the path going from \((0, 0)\) to \((n, n-k)\). Similarly, to avoid \( V_1 \ldots V_m \), the path doesn’t have to pass the points \((n-1, \ell)\) for \( \ell = 0, 1, \ldots, n-k-m \). Thus,

\[ d_{n,k}^{(m)} = \sum_{\ell=n-k-m+1}^{n-k} d_{n-1,n-\ell-1}^{(m)} = \sum_{\ell=k-1}^{k+m-2} d_{n-1,\ell}^{(m)} = d_{n-1,1-k}^{(m)} + \sum_{\ell=0}^{m-2} d_{n-1,k+\ell}^{(m)} \]

Let \( d_{0,0}^{(m)} = d_{0,m}^{(m)} \). Then we obtain \( d_{n,k}^{(m)} = d_{n,k}^{(m)} \) for all \( n \geq k \geq 0 \) and hence \( d_{n,k}^{(m)} \) counts the number of all paths from \((0, 0)\) to \((n, n-k)\) using the horizontal step \( H = (1, 0) \) and the vertical step \( V = (0, 1) \) that has no consecutive \( m \) \( V \)'s, which do not pass through the line \( y = x \) for \( n \geq k \geq 0 \), \( m \geq 1 \).

\[ \square \]

3. Combinatorial identity for \( m = 4 \)

In this section, we consider the explicit form of the generating function (g.f.) \( f_m(z) \) in \( D_m = \left( \frac{f_m(z)}{z}, f_m(z) \right) \) and then we discuss some combinatorial identities.

Since \( f_m(z) \) is given by a solution of the equation (2), it is difficult to get the explicit form of \( f_m(z) \) for \( m \geq 4 \) directly. In some cases we can get the g.f. for the inverse matrix of \( D_m \) by the known Riordan matrix.
For example, let $m = 4$. Then

$$D_4 = \left( \frac{f_4(z)}{z}, f_4(z) \right) = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 5 \\ 13 \\ 36 \\ \vdots \end{bmatrix}$$

where undefined entries of $D_4$ are zeros.

Consider the Riordan matrix $X := \left( \frac{g(z)}{z}, g(z) \right)$ where

$$g(z) = \frac{1 - z + z^2 + z^3 - \sqrt{(1 - z^4)(1 - 2z - z^2)}}{2z}.$$  

By the matrix multiplication of a Riordan group we obtain $D_4X = \left( \frac{g(f)}{z}, g(f) \right)$ where $f := f_4(z)$ and

$$g(f) = \frac{1 - f + f^2 + f^3 - \sqrt{(1 - f^4)(1 - 2f - f^2)}}{2f}. \tag{3}$$

By applying (2) for $m = 4$ to (3), we can get

$$g(f) = z \left( \frac{1 - \sqrt{1 - 4z}}{2z} \right)^2,$$

which is known as the the generalized Catalan numbers ($A004149$)[7]. Hence we have

$$D_4X = \left( \left( \frac{1 - \sqrt{1 - 4z}}{2z} \right)^2, z \left( \frac{1 - \sqrt{1 - 4z}}{2z} \right)^2 \right) := B.$$

Equivalently,

$$D_4^{-1} = XB^{-1} = \left( \frac{h(z)}{z}, h(z) \right),$$

where

$$h(z) = 2z \cdot \frac{1 - z + z^2 + z^3 - \sqrt{(1 - z^4)(1 - 2z - z^2)}}{\left( 1 + z + z^2 + z^3 - \sqrt{(1 - z^4)(1 - 2z - z^2)} \right)^2}. \tag{4}$$
Theorem 3.1. Let $D_4 = [d_{n,k}^{(4)}]_{n,k \geq 0}$. Then

$$
\sum_{k=0}^{n} d_{n,k}^{(4)} \left( \sum_{j=0}^{\frac{n}{2}} \left( k + 2 \choose 2j + 2 \right) 2^j \right) = 4^n, \quad n \geq 0.
$$

Proof. Let us multiply the column vector $(1, 4, 4^2, \ldots)^T$ to $D_4^{-1} = H$. Then by (4), a simple computation shows that

$$(h(z), h(z)) \begin{bmatrix} 1 \\ 4 \\ 4^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ \vdots \end{bmatrix},$$

where the sequence $\{1, 3, 8, 20, \ldots\} =: \{p_n\}$ is the binomial transform of the alternating sequence of $2^n$ and $3 \cdot 2^n$ (A029744)[7] and has the explicit formula $p_n = \sum_{j=0}^{\frac{n+2}{2}} (n+2) 2^j$ where $n \geq 0$ (A048739)[7]. Therefore, from

$$D_4 \begin{bmatrix} 1 \\ 3 \\ 8 \\ 20 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \\ 20 & 5 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

we obtain (5).

Since $D_m$ is a invertible, there is always a sequence $\{s_n^{(m)}\}_{n \in \mathbb{N}_0}$ of some transforms satisfying

$$\sum_{k=0}^{n} d_{n,k}^{(m)} s_k^{(m)} = m^n, \quad m \geq 5.$$

Problem: Find the g.f. for a sequence $\{s_n^{(m)}\}_{n \in \mathbb{N}_0}$ for each $m \in \mathbb{N}_0$.

References


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