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RIORDAN MATRICES WITH THE SPECIAL A-SEQUENCES IN THE BELL SUBGROUP

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Abstract. In this paper, we study the Riordan matrices with the A-sequence $(1, 1, \ldots, 1, 0, \ldots)$ where 1's appear in m times. As a result, we obtain new Riordan matrices and give their lattice path interpretations.

1. Introduction

The concept of the *Riordan group* $(\mathcal{R}, *)$ has been introduced by Shapiro et. al. [5]. A *Riordan matrix* $D = \{d_{n,k}\}_{n,k \in \mathrm{N}_0}$ where $\mathrm{N}_0 := \{0, 1, 2, 3, \ldots\}$ is defined by a pair of formal power series $g(z) = g_0 + g_1 z + g_2 z^2 + \ldots$ and $f(z) = f_1 z + f_2 z^2 + \ldots$ with $g_0 \neq 0$ and $f_1 \neq 0$ such that the generic element is

$$d_{n,k} = [z^n] g(z)(f(z))^k,$$

where $[z^n]$ is the coefficient operator. Then D is an infinite lower triangular matrix with nonzero diagonal entries. We denote a Riordan matrix by $D = (g(z), f(z))$. The concept of the Riordan matrix is useful to get some combinatorial sums and identities. If we give the operation $*$ being the usual matrix multiplication to the set of all Riordan matrices $\mathcal{R}$ as follows :

$$(g(z), f(z))\,(h(z), \ell(z)) = (g(z)h(f(z)), \ell(f(z)))$$

then $(\mathcal{R}, *)$ forms a group, which is called the *Riordan group*. The identity element is $I = (1, z)$ and the inverse element is given by $(g(z), f(z))^{-1} = \left(\frac{1}{g(\overline{f}(z))}, \overline{f}(z)\right)$ where $\overline{f}(f(z)) = f(\overline{f}(z)) = z$. There are some important subgroups of the Riordan group [6]. In particular, we are interested

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in the *Bell subgroup* B defined by

$$B = \left\{ \left(\frac{f(z)}{z}, f(z) \right) | f(z) = f_1 z + f_2 z^2 + \dots,\ f_1 \neq 0 \right\}.$$

From now on, we will call the element in the Bell subgroup the *Bell matrix*.

Rogers [4] and Merlini et. al. [3] obtained a characterization of a Riordan matrix by some sequences as the following theorem(also see [1], [2]).

THEOREM 1.1. *Let $D = [d_{n,k}]$ be an infinite triangular matrix. Then D is a Riordan matrix if and only if there exist two sequences $A = \{a_0, a_1, a_2, \dots\}$ and $Z = \{z_0, z_1, z_2, \dots\}$ with $a_0 \neq 0$, $z_0 \neq 0$ such that*

(i) $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}$ $(k, n = 0, 1 \dots)$,
(ii) $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}$, $(n = 0, 1, \dots)$.

It is known that if $A_D(z)$ and $Z_D(z)$ are the generating functions of the A- and Z-sequences of a Riordan matrix $D = (g(z), f(z))$ respectively, then it can be proven that $f(z)$ and $g(z)$ are the solutions of the following functional equations :

$$f(z) = zA_D(f(z)), \quad \text{and} \quad g(z) = g(0)/(1 - zZ_D(f(z))). \tag{1}$$

A simple computation shows that if a Bell matrix has the A-sequence $\{a_0, a_1, a_2, \dots\}$ then the Z-sequence is $\{a_1, a_2, \dots\}$. Hence it suffices to consider A-sequence when we study a Bell matrix. Pascal matrix is an example of a Bell matrix with A-sequence $\{1, 1, 0, \dots\}$. In this paper, more generally, we observe a Bell matrix D_m with A-sequence $A_m := \{1, \dots, 1, 0, \dots\}$ where 1's appear in m times, $m \geq 3$. As a result, we give a combinatorial interpretation to D_m for $m \geq 1$.

2. Bell matrices with A-sequence A_m

Let us consider a sequence $A_m = \{1, \dots, 1, 0, 0, \dots\}$ where 1's appear in m times. We denote a Bell matrix with the A-sequence A_m by $D_m = [d_{n,k}^{(m)}]_{n,k \in \mathrm{N}_0} = \left(\frac{f_m(z)}{z}, f_m(z) \right)$. By the first equation in (1), $f_m(z)$ satiesfies

$$f_m(z) = z \left(1 + f_m(z) + \dots + (f_m(z))^{m-1} \right). \tag{2}$$

It is easy to see that $D_1 = (1, z)$ is the identity matrix, $D_2 = \left(\frac{1}{1-z}, \frac{z}{1-z}\right)$ is the Pascal matrix and $D_3 = \left(\frac{f_2(z)}{z}, f_2(z)\right)$ is the directed animal matrix (A064189)[7] where $f_2(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z}$. For the case $m \geq 4$, D_m is unknown. But it is known that the 0-th column entry $d^{(m)}_{n,0}$ of $D_m(m \geq 3)$ counts the number of Dyck n-paths with no $\underbrace{UU\ldots U}_{m}D$, i.e. no arrangement of consecutive m U's (See A001006, A036765, A036766, A036767 in [7]). Equivalently, $d^{(m)}_{n,0}(n \geq 0,\ m \geq 3)$ can be interpreted by the number of all paths from $(0,0)$ to $(n, n-k)$ using the steps $H = (1,0) =\rightarrow$ and $V = (0,1) =\uparrow$ with no consecutive m V's. Further, this interpretation can be expanded to $d^{(m)}_{n,k}$ for $n \geq k \geq 0$, $m \geq 1$.

Theorem 2.1. *Let $D_m = [d^{(m)}_{n,k}]_{n,k\geq 0}$ be a Bell matrix with the A-sequence $A_m = \{\underbrace{1,\ldots,1}_{m}, 0, 0, \ldots\}$ and $d^{(m)}_{0,0} = 1$. Then, $d^{(m)}_{n,k}$ counts the number of all paths from $(0,0)$ to $(n, n-k)$ using the horizontal step $H = (1,0)$ and the vertical step $V = (0,1)$ that has no consecutive m V's, which do not pass through the line $y = x$ for $n \geq k \geq 0$, $m \geq 1$.*

Proof. We fix $m \geq 1$. Since D_m is a Bell matrix with the A-sequence $A_m = \{\underbrace{1,1,\ldots,1}_{m}, 0, 0, \ldots\}$, the Z-sequence of D_m is $Z_m = \{\underbrace{1,1,\ldots,1}_{m-1}, 0, 0, \ldots\}$ and hence

$$d^{(m)}_{n,0} = \sum_{\ell=0}^{m-2} d^{(m)}_{n-1,\ell} \text{ and } d^{(m)}_{n,k} = d^{(m)}_{n-1,k-1} + \sum_{\ell=0}^{m-2} d^{(m)}_{n-1,k+\ell} \text{ for } n \geq k \geq 1.$$

Let $\hat{d}^{(m)}_{n,k}$ be the number of all paths from $(0,0)$ to $(n, n-k)$ using the horizontal step $H = (1,0)$ and the vertical step $V = (0,1)$ that has no consecutive m V's (denoted by $V_1 \ldots V_m$), which do not pass through the line $y = x$ for $n \geq k \geq 0$, $m \geq 1$. In particular, $\hat{d}^{(m)}_{n,0}$ counts the number of all paths from $(0,0)$ to (n,n) using the steps H and V that has no $V_1 \ldots V_m$. We define $\hat{d}^{(m)}_{0,0} := 1$. First, we notice that a path from $(0,0)$ to $(n, n-k)$ must pass through at least one of the points $(n-1, \ell)$ for $\ell = 0, 1, \ldots, n-k$. In counting process, to avoid the duplication, we may assume that the path starting at $(0,0)$ and arriving at $(n-1, \ell)$

only has a horizontal step as the next step. It is obvious that

$$\hat{d}^{(m)}_{n,0} = \sum_{\ell=0}^{n-1} \hat{d}^{(m)}_{n-1,\ell} \text{ for } 1 \leq n < m \text{ and } \hat{d}^{(m)}_{n,k} = \sum_{\ell=0}^{k} \hat{d}^{(m)}_{n-1,k-\ell} \text{ for } 1 \leq k \leq n \leq m$$

since no path has the arrangement $V_1 \dots V_m$.
Now, let us consider $\hat{d}^{(m)}_{m,0}$. There are $m-1$ points $(m-1, 0), \dots (m-1, m-2)$ that the path from $(0, 0)$ to (m, m) must pass through. If the path pass through the point $(m-1, 0)$, then it has the arrangement $V_1 \dots V_m$. Other points have no problem to pass through. So $\hat{d}^{(m)}_{m,0} = \sum_{\ell=0}^{m-2} \hat{d}^{(m)}_{m-1,\ell}$.
On the other hand, let $1 \leq m < n$. Let us consider the path from $(0, 0)$ to (n, n). To avoid the arrangement $V_1 \dots V_m$, no path must pass the points $(n-1, \ell)$ for $\ell = 0, 1, \dots, n-m$. Since the number of all paths from $(0, 0)$ to $(n-1, \ell)$ that has no $V_1 \dots V_m$ is $\hat{d}^{(m)}_{n-1,n-\ell-1}$, we have $\hat{d}^{(m)}_{n,0} = \sum_{\ell=n-m-1}^{n-1} \hat{d}^{(m)}_{n-1,n-\ell-1} = \sum_{\ell=0}^{m-2} \hat{d}^{(m)}_{n-1,\ell}$. Next, let us consider the path going from $(0, 0)$ to $(n, n-k)$. Similarly, to avoid $V_1 \dots V_m$, the path doesn't have to pass the points $(n-1, \ell)$ for $\ell = 0, 1, \dots, n-k-m$. Thus, $\hat{d}^{(m)}_{n,k} = \sum_{\ell=n-k-m+1}^{n-k} \hat{d}^{(m)}_{n-1,n-\ell-1} = \sum_{\ell=k-1}^{k+m-2} \hat{d}^{(m)}_{n-1,\ell} = \hat{d}^{(m)}_{n-1,k-1} + \sum_{\ell=0}^{m-2} \hat{d}^{(m)}_{n-1,k+\ell}$.
Let $d^{(m)}_{0,0} = \hat{d}^{(m)}_{0,0}$. Then we obtain $d^{(m)}_{n,k} = \hat{d}^{(m)}_{n,k}$ for all $n \geq k \geq 0$ and hence $d^{(m)}_{n,k}$ counts the number of all paths from $(0, 0)$ to $(n, n-k)$ using the horizontal step $H = (1, 0)$ and the vertical step $V = (0, 1)$ that has no consecutive m V's, which do not pass through the line $y = x$ for $n \geq k \geq 0$, $m \geq 1$. □

3. Combinatorial identity for $m = 4$

In this section, we consider the explicit form of the generating function(g.f.) $f_m(z)$ in $D_m = \left(\frac{f_m(z)}{z}, f_m(z)\right)$ and then we discuss some combinatorial identities.

Since $f_m(z)$ is given by a solution of the equation (2), it is difficult to get the explicit form of $f_m(z)$ for $m \geq 4$ directly. In some cases we can get the g.f. for the inverse matrix of D_m by the known Riordan matrix.

For example, let $m = 4$. Then

$$D_4 = \left(\frac{f_4(z)}{z}, f_4(z)\right) = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 13 & 14 & 9 & 4 & 1 & \\ 36 & 40 & 28 & 14 & 5 & 1 \\ & & \cdots & & & \end{bmatrix}$$

where undefined entries of D_4 are zeros.

Consider the Riordan matrix $X := \left(\frac{g(z)}{z}, g(z)\right)$ where

$$g(z) = \frac{1 - z + z^2 + z^3 - \sqrt{(1-z^4)(1-2z-z^2)}}{2z}.$$

By the matrix multiplication of a Riordan group we obtain $D_4X = \left(\frac{g(f)}{z}, g(f)\right)$ where $f := f_4(z)$ and

$$g(f) = \frac{1 - f + f^2 + f^3 - \sqrt{(1-f^4)(1-2f-f^2)}}{2f}. \tag{3}$$

By applying (2) for $m = 4$ to (3), we can get

$$g(f) = z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^2,$$

which is known as the the generalized Catalan numbers ($A004149$)[7]. Hence we have

$$D_4X = \left(\left(\frac{1-\sqrt{1-4z}}{2z}\right)^2, z\left(\frac{1-\sqrt{1-4z}}{2z}\right)^2\right) := B.$$

Equivalently,

$$D_4^{-1} = XB^{-1} = \left(\frac{h(z)}{z}, h(z)\right),$$

where

$$h(z) = 2z\frac{1 - z + z^2 + z^3 - \sqrt{(1-z^4)(1-2z-z^2)}}{\left(1 + z + z^2 + z^3 - \sqrt{(1-z^4)(1-2z-z^2)}\right)^2}. \tag{4}$$

THEOREM 3.1. *Let* $D_4 = [d^{(4)}_{n,k}]_{n,k\geq 0}$. *Then*

$$\sum_{k=0}^{n} d^{(4)}_{n,k} \left(\sum_{j=0}^{\frac{k+2}{2}} \binom{k+2}{2j+2} 2^j \right) = 4^n, \quad n \geq 0. \tag{5}$$

Proof. Let us multiply the column vector $(1, 4, 4^2, \ldots)^T$ to $D_4^{-1} = H$. Then by (4), a simple computation shows that

$$\left(\frac{h(z)}{z}, h(z) \right) \begin{bmatrix} 1 \\ 4 \\ 4^2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \\ \vdots \end{bmatrix},$$

where the sequence $\{1, 3, 8, 20, \ldots\} =: \{p_n\}$ is the binomial transform of the alternating sequence of 2^n and $3 \cdot 2^n$ (A029744)[7] and has the explicit formula $p_n = \sum_{j=0}^{\frac{n+2}{2}} \binom{n+2}{2j+2} 2^j$ where $n \geq 0$ (A048739)[7]. Therefore, from

$$D_4 \begin{bmatrix} 1 \\ 3 \\ 8 \\ 20 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 2 & 2 & 1 & \\ 5 & 5 & 3 & 1 \\ & \cdots & & \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 8 \\ 20 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 4^2 \\ 4^3 \\ \vdots \end{bmatrix},$$

we obtain (5). $\square$

Since D_m is a invertible, there is always a sequence $\{s_n^{(m)}\}_{n\in\mathbf{N}_0}$ of some transforms satisfying

$$\sum_{k=0}^{n} d^{(m)}_{n,k} s^{(m)}_k = m^n, \quad m \geq 5.$$

Problem: Find the g.f. for a sequence $\{s_n^{(m)}\}_{n\in\mathbf{N}_0}$ for each $m \in N_0$.

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