

HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE.

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ABSTRACT. In this paper we consider the hyponormality of Toeplitz operators T_φ on the Bergman space $L_a^2(\mathbb{D})$ with symbol in the case of function $f + \bar{g}$ with polynomials f and g . We present some necessary conditions for the hyponormality of T_φ under certain assumptions about the coefficients of φ .

1. Introduction

A bounded linear operator A on a Hilbert space is said to be hyponormal if its selfcommutator $[A^*, A] := A^*A - AA^*$ is positive semi-definite. Let \mathbb{D} denote the open unit disk in the complex plane \mathbb{C} , dA the area measure on the complex plane \mathbb{C} . The space $L^2(\mathbb{D})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{D}} f(z) \overline{g(z)} dA(z).$$

The Bergman space $L_a^2(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of all analytic functions on \mathbb{D} . Let $L^\infty(\mathbb{D})$ be the space of bounded measurable function on \mathbb{D} . For $\varphi \in L^\infty(\mathbb{D})$, the multiplication operator M_φ on the Bergman space are defined by $M_\varphi(f) = \varphi \cdot f$, where f is in L_a^2 . If P denotes the orthogonal projection of $L^2(\mathbb{D})$ onto the Bergman space L_a^2 , the Toeplitz operator T_φ on the Bergman space is defined by

$$T_\varphi(f) = P(\varphi \cdot f),$$

where φ is measurable and f is in L_a^2 . It is clear that those operators are bounded if φ is in $L^\infty(\mathbb{D})$. The Hankel operator $H_\varphi : L_a^2 \rightarrow L_a^{2\perp}$ is defined by

$$H_\varphi(f) = (I - P)(\varphi \cdot f).$$

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Let $H^2(\mathbb{T})$ denote the Hardy space of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. For a given $\psi \in L^\infty(\mathbb{T})$, the Toeplitz operator on the Hardy space is the operator T_ψ on $H^2(\mathbb{T})$ defined by $T_\psi f = P_+(\psi \cdot f)$, where f is in $H^2(\mathbb{T})$ and P_+ denotes the orthogonal projection that maps $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$.

Basic properties of the Bergman space and the Hardy space can be found in [1] and [3]. The hyponormality of Toeplitz operators on the Hardy space has been studied by C. Cowen [2], T. Nakazi and K. Takahashi [7], K. Zhu [9], W.Y. Lee [4], [5], [6] and others. In [2], Cowen characterized the hyponormality of Toeplitz operator T_φ on $H^2(\mathbb{T})$ by properties of the symbol $\varphi \in L^\infty(\mathbb{T})$. It also exploited the fact that functions in $H^{2\perp}$ are conjugates of functions in zH^2 . For the Bergman space, $L_a^{2\perp}$ is much larger than the conjugates of functions in zL_a^2 , and no dilation theorem (similar to Sarason's theorem) is available. Indeed it is quite difficult to determine the hyponormality of T_φ . In fact the study of hyponormal Toeplitz operators on the Bergman space seems to be scarce from the literature. In [10], it was shown that if $\varphi(z) = a_{-m}\bar{z}^m + a_{-N}\bar{z}^N + a_m z^m + a_N z^N$ ($0 < m < N$) and if $a_m \bar{a}_N = a_{-m} \bar{a}_{-N}$, then

$$T_\varphi \text{ is hyponormal} \iff \begin{cases} \frac{1}{N+1}(|a_N|^2 - |a_{-N}|^2) \geq \frac{1}{m+1}(|a_{-m}|^2 - |a_m|^2) & \text{if } |a_{-N}| \leq |a_N| \\ N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2) & \text{if } |a_N| \leq |a_{-N}|. \end{cases}$$

In this paper we study the hyponormality of Toeplitz operators T_φ on the Bergman space $L_a^2(\mathbb{D})$ with symbols in the case of function $\varphi = \bar{g} + f$ with polynomial f and g . We will now consider the hyponormality of Toeplitz operators on the Bergman space with a symbol in the class of functions $\bar{g} + f$, where f and g are polynomials. Since the hyponormality of operators is translation invariant we may assume that $f(0) = g(0) = 0$. We shall list the well-known properties of Toeplitz operators T_φ on the Bergman space.

If f, g are in $L^\infty(\mathbb{D})$ then we can easily check that

- (i) $T_{f+g} = T_f + T_g$
- (ii) $T_f^* = T_{\bar{f}}$
- (iii) $T_{\bar{f}} T_g = T_{\bar{f}g}$ if f or g is analytic.

These properties enable to us establish several consequences of hyponormality.

PROPOSITION 1.1. ([8]) Let f, g be bounded and analytic in the $\bar{g} + f \in L^\infty(\mathbb{D})$. Then the followings are equivalent.

- (i) $T_{\bar{g}+f}$ is hyponormal.
- (ii) $H_{\bar{g}}^* H_{\bar{g}} \leq H_{\bar{f}}^* H_{\bar{f}}$.
- (iii) $\|(I - P)(\bar{g}k)\| \leq \|(I - P)(\bar{f}k)\|$ for any k in L_a^2 .
- (iv) $\|\bar{g}k\|^2 - \|P(\bar{g}k)\|^2 \leq \|\bar{f}k\|^2 - \|P(\bar{f}k)\|^2$ for any k in L_a^2 .
- (v) $H_{\bar{g}} = CH_{\bar{f}}$ where C is of norm less than or equal to one.

Let s, t be nonnegative integers and P be the orthogonal projection. Then we have

$$(1.1) \quad P(\bar{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t}, & \text{if } s \geq t \\ 0, & \text{if } s < t. \end{cases}$$

2. Main results

In this section we establish some necessary conditions for the hyponormality of Toeplitz operator T_φ on the Bergman space L_a^2 under certain additional assumption concerning the polynomial symbol φ .

In [11], it was shown that if $f(z) = a_m z^m + a_N z^N, g(z) = a_{-m} z^m + a_{-N} z^N$ ($0 < m < N$) and $|a_N| < |a_{-N}|$, then the hyponormality of $T_{\bar{g}+f}$ implies that

$$N^2(|a_{-N}|^2 - |a_N|^2) \leq m^2(|a_m|^2 - |a_{-m}|^2).$$

The following Theorem gives a necessary condition for hyponormality of $T_{\bar{g}+f}$ when $m = 1$ and $N = 2$.

THEOREM 2.1. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where*

$$f(z) = a_1z + a_2z^2 \quad \text{and} \quad g(z) = a_{-1}z + a_{-2}z^2.$$

If T_φ is hyponormal, then

- (i) $2(|a_2|^2 - |a_{-2}|^2) \geq 3(|a_{-1}|^2 - |a_1|^2)$.
- (ii) $\left(\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2)\right) \left(\frac{1}{12}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{4}(|a_2|^2 - |a_{-2}|^2)\right) \geq \frac{1}{9}|\overline{a_1}a_2 - \overline{a_{-1}}a_{-2}|^2$

Proof. Let T_φ is hyponormal operator. Observe that

$$\begin{aligned} \langle M_{\bar{f}}(c_0 + c_1z), M_{\bar{f}}(c_0 + c_1z) \rangle &= |a_1|^2 \left(\frac{1}{2}|c_0|^2 + \frac{1}{3}|c_1|^2 \right) + \\ &|a_2|^2 \left(\frac{1}{3}|c_0|^2 + \frac{1}{4}|c_1|^2 \right) + \frac{2}{3} \operatorname{Re}(\overline{a_1}a_2c_0\overline{c_1}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \langle M_{\bar{g}}(c_0 + c_1z), M_{\bar{g}}(c_0 + c_1z) \rangle &= |a_{-1}|^2 \left(\frac{1}{2}|c_0|^2 + \frac{1}{3}|c_1|^2 \right) + \\ &|a_{-2}|^2 \left(\frac{1}{3}|c_0|^2 + \frac{1}{4}|c_1|^2 \right) + \frac{2}{3} \operatorname{Re}(\overline{a_{-1}}a_{-2}c_0\overline{c_1}). \end{aligned}$$

It follows that

$$\langle T_{\bar{f}}(c_0 + c_1z), T_{\bar{f}}(c_0 + c_1z) \rangle = \frac{1}{4}|a_1|^2|c_1|^2$$

Similarly, we have

$$\langle T_{\bar{g}}(c_0 + c_1z), T_{\bar{g}}(c_0 + c_1z) \rangle = \frac{1}{4}|a_{-1}|^2|c_1|^2$$

Hence

$$\begin{aligned} \langle H_{\bar{f}}(c_0 + c_1z), H_{\bar{f}}(c_0 + c_1z) \rangle &= |a_1|^2 \left(\frac{1}{2}|c_0|^2 + \frac{1}{12}|c_1|^2 \right) + \\ &|a_2|^2 \left(\frac{1}{3}|c_0|^2 + \frac{1}{4}|c_1|^2 \right) + \frac{2}{3} \operatorname{Re}(\bar{a}_1 a_2 c_0 \bar{c}_1). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \langle H_{\bar{g}}(c_0 + c_1z), H_{\bar{g}}(c_0 + c_1z) \rangle &= |a_{-1}|^2 \left(\frac{1}{2}|c_0|^2 + \frac{1}{12}|c_1|^2 \right) + \\ &|a_{-2}|^2 \left(\frac{1}{3}|c_0|^2 + \frac{1}{4}|c_1|^2 \right) + \frac{2}{3} \operatorname{Re}(\bar{a}_{-1} a_{-2} c_0 \bar{c}_1). \end{aligned}$$

Hence if T_φ is hyponormal, then

$$\begin{aligned} (2.1) \quad &\langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}})(c_0 + c_1z), (c_0 + c_1z) \rangle \\ &= (|a_1|^2 - |a_{-1}|^2) \left(\frac{1}{2}|c_0|^2 + \frac{1}{12}|c_1|^2 \right) + (|a_2|^2 - |a_{-2}|^2) \left(\frac{1}{3}|c_0|^2 + \frac{1}{4}|c_1|^2 \right) \\ &+ \frac{2}{3} \operatorname{Re}(\bar{a}_1 a_2 c_0 \bar{c}_1 - \bar{a}_{-1} a_{-2} c_0 \bar{c}_1) \geq 0. \end{aligned}$$

Since $\operatorname{Re}(\bar{a}_1 a_2 c_0 \bar{c}_1 - \bar{a}_{-1} a_{-2} c_0 \bar{c}_1) \geq -|\bar{a}_1 a_2 - \bar{a}_{-1} a_{-2}| |c_0 \bar{c}_1|$, it follows from (2.1) that

$$\begin{aligned} &(|a_1|^2 - |a_{-1}|^2) \left(\frac{1}{2}|c_0|^2 + \frac{1}{12}|c_1|^2 \right) + (|a_2|^2 - |a_{-2}|^2) \left(\frac{1}{3}|c_0|^2 + \frac{1}{4}|c_1|^2 \right) \\ &- \frac{2}{3} |\bar{a}_1 a_2 - \bar{a}_{-1} a_{-2}| |c_0 \bar{c}_1| \geq 0. \end{aligned}$$

There are two cases to consider.

Case 1) If $c_1 = 0$, then

$$3(|a_1|^2 - |a_{-1}|^2) + 2(|a_2|^2 - |a_{-2}|^2) \geq 0.$$

Case 2) If $c_1 \neq 0$, then we have that

$$\begin{aligned} &\left| \frac{c_0}{c_1} \right|^2 \left(\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2) \right) - \frac{2}{3} \left| \frac{c_0}{c_1} \right| |\bar{a}_1 a_2 - \bar{a}_{-1} a_{-2}| \\ &+ \frac{1}{12}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{4}(|a_2|^2 - |a_{-2}|^2) \geq 0. \end{aligned}$$

Since $\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2) \geq 0$, it follows that

$$\left(\frac{1}{2}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{3}(|a_2|^2 - |a_{-2}|^2)\right) \left(\frac{1}{12}(|a_1|^2 - |a_{-1}|^2) + \frac{1}{4}(|a_2|^2 - |a_{-2}|^2)\right) \geq \frac{1}{9}|\overline{a_1}a_2 - \overline{a_{-1}}a_{-2}|^2$$

This completes the proof. \square

The following Theorem gives a necessary condition for hyponormality of $T_{\overline{g}+f}$ when $m > 1$ and $N = m + 1$.

THEOREM 2.2. *Let $\varphi(z) = \overline{g(z)} + f(z)$, where*

$$f(z) = a_m z^m + a_{m+1} z^{m+1} \quad \text{and} \quad g(z) = a_{-m} z^m + a_{-(m+1)} z^{m+1}.$$

If T_φ is hyponormal, then (i) $(m+1)(|a_{m+1}|^2 - |a_{-(m+1)}|^2) \geq (m+2)(|a_{-m}|^2 - |a_m|^2)$.

$$(ii) \quad \left(\frac{1}{m+1}(|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+2}(|a_{m+1}|^2 - |a_{-(m+1)}|^2)\right) \left(\frac{1}{m+2}(|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+3}(|a_{m+1}|^2 - |a_{-(m+1)}|^2)\right) \geq \frac{1}{(m+2)^2} |\overline{a_m}a_{m+1} - \overline{a_{-m}}a_{-(m+1)}|^2$$

Proof. Let T_φ be a hyponormal operator. Observe that

$$\begin{aligned} & \langle M_{\overline{f}}(c_0 + c_1 z), M_{\overline{f}}(c_0 + c_1 z) \rangle \\ &= |a_m|^2 \left(\frac{1}{m+1} |c_0|^2 + \frac{1}{m+2} |c_1|^2 \right) + \\ & |a_{m+1}|^2 \left(\frac{1}{m+2} |c_0|^2 + \frac{1}{m+3} |c_1|^2 \right) + \frac{2}{m+2} \operatorname{Re}(\overline{a_m} a_{m+1} c_0 \overline{c_1}). \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \langle M_{\overline{g}}(c_0 + c_1 z), M_{\overline{g}}(c_0 + c_1 z) \rangle \\ &= |a_{-m}|^2 \left(\frac{1}{m+1} |c_0|^2 + \frac{1}{m+2} |c_1|^2 \right) + \\ & |a_{-(m+1)}|^2 \left(\frac{1}{m+2} |c_0|^2 + \frac{1}{m+3} |c_1|^2 \right) + \frac{2}{m+2} \operatorname{Re}(\overline{a_{-m}} a_{-(m+1)} c_0 \overline{c_1}). \end{aligned}$$

It follows that

$$\langle T_{\bar{f}}(c_0 + c_1z), T_{\bar{f}}(c_0 + c_1z) \rangle = 0.$$

Similarly, we have

$$\langle T_{\bar{g}}(c_0 + c_1z), T_{\bar{g}}(c_0 + c_1z) \rangle = 0.$$

Hence

$$\begin{aligned} & \langle H_{\bar{f}}(c_0 + c_1z), H_{\bar{f}}(c_0 + c_1z) \rangle \\ &= |a_m|^2 \left(\frac{1}{m+1} |c_0|^2 + \frac{1}{m+2} |c_1|^2 \right) + \\ & |a_{m+1}|^2 \left(\frac{1}{m+2} |c_0|^2 + \frac{1}{m+3} |c_1|^2 \right) + \frac{2}{m+2} \operatorname{Re}(\bar{a}_m a_{m+1} c_0 \bar{c}_1). \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \langle H_{\bar{g}}(c_0 + c_1z), H_{\bar{g}}(c_0 + c_1z) \rangle \\ &= |a_{-m}|^2 \left(\frac{1}{m+1} |c_0|^2 + \frac{1}{m+2} |c_1|^2 \right) + \\ & |a_{-(m+1)}|^2 \left(\frac{1}{m+2} |c_0|^2 + \frac{1}{m+3} |c_1|^2 \right) + \frac{2}{m+2} \operatorname{Re}(\bar{a}_{-m} a_{-(m+1)} c_0 \bar{c}_1). \end{aligned}$$

Hence if T_φ is hyponormal, then

$$\begin{aligned} & \langle (H_{\bar{f}}^* H_{\bar{f}} - H_{\bar{g}}^* H_{\bar{g}})(c_0 + c_1z), (c_0 + c_1z) \rangle \\ (2.2) \quad &= (|a_m|^2 - |a_{-m}|^2) \left(\frac{1}{m+1} |c_0|^2 + \frac{1}{m+2} |c_1|^2 \right) \\ &+ (|a_{m+1}|^2 - |a_{-(m+1)}|^2) \left(\frac{1}{m+2} |c_0|^2 + \frac{1}{m+3} |c_1|^2 \right) \\ &+ \frac{2}{m+2} \operatorname{Re}(\bar{a}_m a_{m+1} c_0 \bar{c}_1 - \bar{a}_{-m} a_{-(m+1)} c_0 \bar{c}_1) \geq 0. \end{aligned}$$

Since

$$\operatorname{Re}(\bar{a}_m a_{m+1} c_0 \bar{c}_1 - \bar{a}_{-m} a_{-(m+1)} c_0 \bar{c}_1) \geq -|\bar{a}_m a_{m+1} - \bar{a}_{-m} a_{-(m+1)}| |c_0 \bar{c}_1|,$$

it follow from (2.2) that

$$\begin{aligned} & (|a_m|^2 - |a_{-m}|^2) \left(\frac{1}{m+1} |c_0|^2 + \frac{1}{m+2} |c_1|^2 \right) + (|a_{m+1}|^2 - |a_{-(m+1)}|^2) \\ & \left(\frac{1}{m+2} |c_0|^2 + \frac{1}{m+3} |c_1|^2 \right) - \frac{2}{m+2} |\overline{a_m} a_{m+1} - \overline{a_{-m}} a_{-(m+1)}| |c_0 \overline{c_1}| \\ & \geq 0. \end{aligned}$$

There are two cases to consider.

Case 1) If $c_1 = 0$, then

$$(m+2)(|a_m|^2 - |a_{-m}|^2) + (m+1)(|a_{m+1}|^2 - |a_{-(m+1)}|^2) \geq 0.$$

Cases 2) If $c_1 \neq 0$, then we have that

$$\begin{aligned} & \left| \frac{c_0}{c_1} \right| \left(\frac{1}{m+1} (|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+2} (|a_{m+1}|^2 - |a_{-(m+1)}|^2) \right) \\ & - \frac{2}{m+2} \left| \frac{c_0}{c_1} \right| |\overline{a_m} a_{m+1} - \overline{a_{-m}} a_{-(m+1)}| + \frac{1}{m+2} (|a_m|^2 - |a_{-m}|^2) \\ & + \frac{1}{m+3} (|a_{m+1}|^2 - |a_{-(m+1)}|^2) \geq 0. \end{aligned}$$

Since $\frac{1}{m+1} (|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+2} (|a_{m+1}|^2 - |a_{-(m+1)}|^2) > 0$, it follows that

$$\begin{aligned} & \left(\frac{1}{m+1} (|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+2} (|a_{m+1}|^2 - |a_{-(m+1)}|^2) \right) \\ & \left(\frac{1}{m+2} (|a_m|^2 - |a_{-m}|^2) + \frac{1}{m+3} (|a_{m+1}|^2 - |a_{-(m+1)}|^2) \right) \\ & \geq \frac{1}{(m+2)^2} |\overline{a_m} a_{m+1} - \overline{a_{-m}} a_{-(m+1)}|^2. \end{aligned}$$

This completes the proof. \square

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