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COEFFICIENT INEQUALITIES FOR HARMONIC EXTERIOR MAPPINGS

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Abstract. The purpose of this paper is to study harmonic univalent mappings defined in $\Delta = \{z : |z| > 1\}$ that map ∞ to ∞. Some coefficient estimates are obtained in a normalized class of mappings.

1. Introduction

Let Σ be the class of all complex-valued, harmonic, orientation-preserving, univalent mappings, for which $f(\infty) = \lim_{z\to\infty} f(z)$ exists as ∞,

$$f(z) = h(z) + \overline{g(z)} + A \log |z| \tag{1.1}$$

of $\Delta = \{z : |z| > 1\}$, where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in Δ and $A \in \mathbb{C}$. Hengartner and Schober[3] show that the Jacobian $|f_z|^2 - |f_{\bar z}|^2$ is positive, and

$$a(z) = \frac{\overline{f_{\bar z}}}{f_z} = \frac{2zg'(z) + \bar A}{2zh'(z) + A}$$

is analytic in Δ and satisfies $|a(z)| < 1$.

The coefficient problem for this class appears to be difficult. In the full class Σ, a few estimates are known only for lower order coefficients: $|A| \le 2$ and $|b_1| \le 1$ hold for the full class Σ, and $|b_2| \le \frac{1}{2}(1 - |b_1|^2) \le \frac{1}{2}$ holds if $A = 0$. These coefficient bounds[3] are all sharp and a consequence of Schwarz's lemma. If we restrict our attention to some subclass of Σ, we

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can obtain good results; for $f \in \Sigma$ with $f(\Delta) = \Delta$, $|1+b_1| \leq 1$, $|b_n| \leq \frac{1}{n}$ for $n \geq 2$, and $|a_n| \leq \frac{1}{n}$ for all n.. These sharp coefficient bounds are obtained by Jun[4]. In this paper, we shall consider the subclass

$$\Sigma_c = \{f \in \Sigma : f \text{ is convex in the direction of the imaginary axis}\}$$

of Σ. In order to get some coefficient estimates of harmonic univalent mappings in Σ_c, we will consider the analytic univalent function $h + g$, and the meromorphic function $F(\zeta) = -\frac{1}{\zeta} + \zeta$ which is the minus reciprocal of the square-root transform of the Koebe function.

2. Mappings which are convex in the direction of the imaginary axis

Definition 1. A set D is called *convex in the direction of the imaginary axis* if every line parallel to the imaginary axis has a connected intersection with D.

Definition 2. A mapping f is *convex in the direction of the imaginary axis* if $f(\Delta)$ is convex in the direction of the imaginary axis.

Let Σ_c be the class of all mappings $f \in \Sigma$ which is convex in the direction of the imaginary axis.

Theorem 2.1. *If $f = h + \bar{g} + A\log|z| \in \Sigma_c$ with $\Re\{A\} = 0$, then the analytic function $h + g$ is conformal univalent in Δ.*

Proof. Since f is univalent, there exists a mapping $z = z(w)$ such that $f(z(w)) = w$ and $z(f(z)) = z$. Thus we have $h+g = f - A\log|z| + i2\Im\{g\}$ and

$$h(z(w)) + g(z(w)) = w + i\phi(w) \tag{2.1}$$

where $\phi(w) = iA\log|z(w)| + 2\Im\{g(z(w))\}$ is a continuous real valued function. Since $a(z) = \frac{2zg'(z)-A}{2zh'(z)+A}$ satisfies $|a(z)| < 1$, we have $h'(z) + g'(z) \neq 0$ in Δ. Thus $h + g$ is conformal, and the mapping $h(z(w)) + g(z(w)) = w + i\phi(w)$ is locally univalent since $z(w)$ is 1-1. If $w_1 + i\phi(w_1) = w_2 + i\phi(w_2)$ with $w_1 \neq w_2 (w_1 = u_1 + iv_1, w_2 = u_2 + iv_2)$, then $u_1 = u_2 = u$ and $v_1 + \phi(u + iv_1) = v_2 + \phi(u + iv_2)$. The real valued function $\psi(v) = v + \phi(u + iv)$, which is defined on some interval I since f is convex in the direction of the imaginary axis, is not strictly monotonic and therefore not locally 1-1. Thus $w + i\phi(w) = h + g$ is 1-1 and so conformal univalent. $\square$

LEMMA 2.2. *If $f = h + \bar{g} + A\log|z| \in \Sigma$ with $\Re\{A\} = 0$ is convex in the direction of the imaginary axis, then the analytic function $h + g$ is also convex in the direction of the imaginary axis.*

Proof. Let $D = f(\Delta)$. The image of D under the mapping $w + i\phi(w)$ defined as in (2.1) is convex in the direction of the imaginary axis since the mapping $w + i\phi(w)$ maps vertical lines into themselves.. Therefore $w + i\phi(w) = h(z(w)) + g(z(w))$ is also convex in the direction of the imaginary axis. $\square$

Sharp coefficient bounds of the analytic univalent function $H(z) = z + \sum_{n=0}^{\infty} c_n z^{-n}$ in Δ are known only for $1 \le n \le 3$: $|c_1| \le 1$ [2], $|c_2| \le \frac{2}{3}$ [5], $|c_3| \le \frac{1}{2} + e^{-6}$ [1]. From these, we can easily get the lower order coefficient bounds for the harmonic univalent mapping $f \in \Sigma_c$ with $\Re\{A\} = 0$ as follows;

$$|a_1 + b_1| \le 1,\ \ |a_2 + b_2| \le \frac{2}{3},\ \ |a_3 + b_3| \le \frac{1}{2} + e^{-6}.$$

In the following Theorem 2.3, we obtain the coefficient bounds for all orders.

THEOREM 2.3. *Let $f = h + \bar{g} + A\log|z| \in \Sigma_c$ with $\Re\{A\} = 0$. If $h + g$ is real on the real axis, then*

$$\begin{aligned} &|a_1 + b_1| \le 1, \\ &|a_n + b_n| \le \frac{2\sqrt{2}}{n} \quad \textit{for } n > 1. \end{aligned}$$

Proof. Let $G(\zeta) = h(1/\zeta) + g(1/\zeta)$ on $0 < |\zeta| < 1$. Then the function $G(\zeta) = \frac{1}{\zeta} + \sum_{k=1}^{\infty}(a_k + b_k)\zeta^k$ is regular univalent and convex in the direction of the imaginary axis by Theorem 2.1 and Lemma 2.2. $G(\zeta)$ is also real on the real axis. Thus, on $|\zeta| = r$ $(0 < r < 1)$,

$$\Im\{\zeta G'(\zeta)\} = -\frac{\partial}{\partial\theta}\Re\{G(re^{i\theta})\} \begin{cases} > 0 & \text{for } 0 < \theta < \pi \\ < 0 & \text{for } \pi < \theta < 2\pi. \end{cases}$$

Therefore

$$\Re\{\frac{-\zeta^2 G'(\zeta)}{1 - \zeta^2}\} > 0 \ \ \text{for } |\zeta| < 1.$$

Let $F(\zeta) = -\frac{1}{\zeta} + \zeta = -\frac{1}{\zeta} + \sum_{k=0}^{\infty}\alpha_k\zeta^k$. Then $\Re\{\frac{\zeta G'(\zeta)}{F(\zeta)}\} > 0$ and thus there exists a bounded regular function $\omega(\zeta)$, with $\omega(0) = 0$ and $|\omega(\zeta)| <$

1 in $|\zeta| < 1$, such that

$$\frac{\zeta G'(\zeta)}{F(\zeta)} = \frac{1+\omega(\zeta)}{1-\omega(\zeta)},\ \omega'(0) = 0.$$

This implies that

$$[\zeta F(\zeta) + \zeta^2 G'(\zeta)]\omega(\zeta) = \zeta^2 G'(\zeta) - \zeta F(\zeta).$$

Let $G(\zeta) = \frac{1}{\zeta} + \sum_{k=1}^{\infty}(a_k + b_k)\zeta^k = \frac{1}{\zeta} + \sum_{k=0}^{\infty} c_k\zeta^k$, then we have

$$[-2 + \sum_{k=0}^{\infty}(kc_k + \alpha_k)\zeta^{k+1}]\omega(\zeta) = \sum_{k=0}^{\infty}(kc_k - \alpha_k)\zeta^{k+1},$$

$$\begin{aligned}
[-2 + \sum_{k=0}^{n-1}&(kc_k + \alpha_k)\zeta^{k+1}]\omega(\zeta) \\
&= \sum_{k=0}^{n}(kc_k - \alpha_k)\zeta^{k+1} - \sum_{k=n}^{\infty}(kc_k + \alpha_k)\omega(\zeta)\zeta^{k+1} \\
&\quad + \sum_{k=n+1}^{\infty}(kc_k - \alpha_k)\zeta^{k+1} \\
&= \sum_{k=0}^{n}(kc_k - \alpha_k)\zeta^{k+1} + \sum_{k=n+2}^{\infty}\beta_k\zeta^k,
\end{aligned}$$

where $\sum_{k=n+2}^{\infty}\beta_k\zeta^k$ converges in $|\zeta| < 1$. Let $\zeta = re^{i\theta}$ $(r < 1)$. Then integrations give

$$\begin{aligned}
4 + \sum_{k=0}^{n-1}|kc_k + \alpha_k|^2 &\geq 4 + \sum_{k=0}^{n-1}|kc_k + \alpha_k|^2 r^{2k+2} \\
&= \frac{1}{2\pi}\int_0^{2\pi}|-2 + \sum_{k=0}^{n-1}(kc_k + \alpha_k)\zeta^{k+1}|^2 d\theta \\
&\geq \frac{1}{2\pi}\int_0^{2\pi}|-2 + \sum_{k=0}^{n-1}(kc_k + \alpha_k)\zeta^{k+1}|^2|\omega(\zeta)|^2 d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}|\sum_{k=0}^{n}(kc_k - \alpha_k)\zeta^{k+1} + \sum_{k=n+2}^{\infty}\beta_k\zeta^k|^2 d\theta \\
&\geq \sum_{k=0}^{n}|kc_k - \alpha_k|^2 r^{2k+2},
\end{aligned}$$

$$4 + \sum_{k=0}^{n-1} |kc_k + \alpha_k|^2 \geq \sum_{k=0}^{n} |kc_k - \alpha_k|^2,$$

$$\begin{aligned} |nc_n - \alpha_n|^2 &\leq 4 + \sum_{k=0}^{n-1} (|kc_k + \alpha_k|^2 - |kc_k - \alpha_k|^2) \\ &= 4 + 4 \sum_{k=0}^{n-1} k\Re\{c_k \bar{\alpha}_k\}. \end{aligned} \tag{2.2}$$

From (2.2) with $n = 1$, we obtain

$$|c_1 - 1|^2 \leq 4, \ \ |c_1 - 1| \leq 2$$

and, for $n > 1$,

$$n^2 |c_n|^2 \leq 4 + 4|c_1| \leq 16. \tag{2.3}$$

Since the analytic function $h + g$ is univalent in Δ by Theorem 2.1, we have

$$\sum_{k=1}^{\infty} k|a_k + b_k|^2 \leq 1$$

by the area theorem. From this we get

$$|a_1 + b_1| \leq 1. \tag{2.4}$$

We now write

$$|a_n + b_n| \leq \frac{2\sqrt{2}}{n} \ \text{ for } n > 1,$$

by (2.3) and (2.4). □

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