COEFFICIENT INEQUALITIES FOR HARMONIC EXTERIOR MAPPINGS

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Abstract. The purpose of this paper is to study harmonic univalent mappings defined in $\Delta = \{ z : |z| > 1 \}$ that map $\infty$ to $\infty$. Some coefficient estimates are obtained in a normalized class of mappings.

1. Introduction

Let $\Sigma$ be the class of all complex-valued, harmonic, orientation-preserving, univalent mappings, for which $f(\infty) = \lim_{z \to \infty} f(z)$ exists as $\infty$,

$$f(z) = h(z) + \frac{g(z)}{z} + A \log |z|$$

of $\Delta = \{ z : |z| > 1 \}$, where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in $\Delta$ and $A \in \mathbb{C}$. Hengartner and Schober[3] show that the Jacobian $|f_z|^2 - |f_{\overline{z}}|^2$ is positive, and

$$a(z) = \frac{f_{\overline{z}}}{f_z} = \frac{2zg'(z) + \bar{A}}{2zh'(z) + A}$$

is analytic in $\Delta$ and satisfies $|a(z)| < 1$.

The coefficient problem for this class appears to be difficult. In the full class $\Sigma$, a few estimates are known only for lower order coefficients: $|A| \leq 2$ and $|b_1| \leq 1$ hold for the full class $\Sigma$, and $|b_2| \leq \frac{1}{2}(1 - |b_1|^2) \leq \frac{1}{2}$ holds if $A = 0$. These coefficient bounds[3] are all sharp and a consequence of Schwarz’s lemma. If we restrict our attention to some subclass of $\Sigma$, we

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can obtain good results; for \( f \in \Sigma \) with \( f(\Delta) = \Delta \), \( |1 + b_1| \leq 1 \), \( |b_n| \leq \frac{1}{n} \) for \( n \geq 2 \), and \( |a_n| \leq \frac{1}{n} \) for all \( n \). These sharp coefficient bounds are obtained by Jun[4]. In this paper, we shall consider the subclass \( \Sigma_c = \{ f \in \Sigma : f \text{ is convex in the direction of the imaginary axis} \} \) of \( \Sigma \). In order to get some coefficient estimates of harmonic univalent mappings in \( \Sigma_c \), we will consider the analytic univalent function \( h + g \), and the meromorphic function \( F(\zeta) = -\frac{1}{\zeta} + \zeta \) which is the minus reciprocal of the square-root transform of the Koebe function.

2. Mappings which are convex in the direction of the imaginary axis

**Definition 1.** A set \( D \) is called convex in the direction of the imaginary axis if every line parallel to the imaginary axis has a connected intersection with \( D \).

**Definition 2.** A mapping \( f \) is convex in the direction of the imaginary axis if \( f(\Delta) \) is convex in the direction of the imaginary axis.

Let \( \Sigma_c \) be the class of all mappings \( f \in \Sigma \) which is convex in the direction of the imaginary axis.

**Theorem 2.1.** If \( f = h + \bar{g} + A \log |z| \in \Sigma_c \) with \( \Re\{A\} = 0 \), then the analytic function \( h + g \) is conformal univalent in \( \Delta \).

**Proof.** Since \( f \) is univalent, there exists a mapping \( z = z(w) \) such that \( f(z(w)) = w \) and \( z(f(z)) = z \). Thus we have \( h + g = f - A \log |z| + 2\Im\{g\} \) and

\[
(2.1) \quad h(z(w)) + g(z(w)) = w + i\phi(w)
\]

where \( \phi(w) = iA \log |z(w)| + 2\Im\{g(z(w))\} \) is a continuous real valued function. Since \( a(z) = \frac{2g(z) - A}{2f(z) + A} \) satisfies \( |a(z)| < 1 \), we have \( h'(z) + g'(z) \neq 0 \) in \( \Delta \). Thus \( h + g \) is conformal, and the mapping \( h(z(w)) + g(z(w)) = w + i\phi(w) \) is locally univalent since \( z(w) \) is 1-1. If \( w_1 + i\phi(w_1) = w_2 + i\phi(w_2) \) with \( w_1 \neq w_2(w_1 = u_1 + iv_1, w_2 = u_2 + iv_2) \), then \( u_1 = u_2 = u \) and \( v_1 + \phi(u + iv_1) = v_2 + \phi(u + iv_2) \). The real valued function \( \psi(v) = v + \phi(u + iv) \), which is defined on some interval \( I \) since \( f \) is convex in the direction of the imaginary axis, is not strictly monotonic and therefore not locally 1-1. Thus \( w + i\phi(w) = h + g \) is 1-1 and so conformal univalent. \( \square \)
Lemma 2.2. If \( f = h + \bar{g} + A \log |z| \in \Sigma \) with \( \Re\{A\} = 0 \) is convex in the direction of the imaginary axis, then the analytic function \( h + g \) is also convex in the direction of the imaginary axis.

Proof. Let \( D = f(\Delta) \). The image of \( D \) under the mapping \( w + i\phi(w) \) defined as in (2.1) is convex in the direction of the imaginary axis since the mapping \( w + i\phi(w) \) maps vertical lines into themselves. Therefore \( w + i\phi(w) = h(z(w)) + g(z(w)) \) is also convex in the direction of the imaginary axis.

Sharp coefficient bounds of the analytic univalent function \( H(z) = z + \sum_{n=0}^{\infty} c_n z^{-n} \) in \( \Delta \) are known only for \( 1 \leq n \leq 3 \):
\[
|c_1| \leq 1 \quad [2], \\
|c_2| \leq \frac{2}{3} \quad [5], \\
|c_3| \leq \frac{1}{2} + e^{-6} \quad [1].
\]
From these, we can easily get the lower order coefficient bounds for the harmonic univalent mapping \( f \in \Sigma \) with \( \Re\{A\} = 0 \) as follows;
\[
|a_1 + b_1| \leq 1, \\
|a_2 + b_2| \leq \frac{2}{3}, \\
|a_3 + b_3| \leq \frac{1}{2} + e^{-6}.
\]

In the following Theorem 2.3, we obtain the coefficient bounds for all orders.

Theorem 2.3. Let \( f = h + \bar{g} + A \log |z| \in \Sigma \) with \( \Re\{A\} = 0 \). If \( h + g \) is real on the real axis, then
\[
|a_1 + b_1| \leq 1, \\
|a_n + b_n| \leq \frac{2\sqrt{2}}{n} \quad \text{for} \quad n > 1.
\]

Proof. Let \( G(\zeta) = h(1/\zeta) + g(1/\zeta) \) on \( 0 < |\zeta| < 1 \). Then the function
\[
G(\zeta) = \frac{1}{\zeta} + \sum_{k=1}^{\infty} (a_k + b_k)\zeta^k
\]
is regular univalent and convex in the direction of the imaginary axis by Theorem 2.1 and Lemma 2.2. \( G(\zeta) \) is also real on the real axis. Thus, on \( |\zeta| = r \quad (0 < r < 1) \),
\[
\Im\{\zeta G'(\zeta)\} = -\frac{\partial}{\partial \theta} \Re\{G(re^{i\theta})\} \begin{cases} > 0 & \text{for } 0 < \theta < \pi \\ < 0 & \text{for } \pi < \theta < 2\pi. \end{cases}
\]
Therefore
\[
\Re\{-\zeta^2 G'(\zeta)\} > 0 \quad \text{for} \quad |\zeta| < 1.
\]
Let \( F(\zeta) = -\frac{1}{\zeta} + \zeta = -\frac{1}{\zeta} + \sum_{k=0}^{\infty} \alpha_k \zeta^k \). Then \( \Re\{\frac{\zeta G'(\zeta)}{F(\zeta)}\} > 0 \) and thus there exists a bounded regular function \( \omega(\zeta) \), with \( \omega(0) = 0 \) and \( |\omega(\zeta)| < \).
1 in $|\zeta| < 1$, such that

$$\frac{\zeta G'(\zeta)}{F(\zeta)} = \frac{1 + \omega(\zeta)}{1 - \omega(\zeta)}$$

such that $\omega'(0) = 0$.

This implies that

$$[\zeta F(\zeta) + \zeta^2 G'(\zeta)]\omega(\zeta) = \zeta^2 G'(\zeta) - \zeta F(\zeta).$$

Let $G(\zeta) = \frac{1}{\zeta} + \sum_{k=1}^{\infty} (a_k + b_k)\zeta^k = \frac{1}{\zeta} + \sum_{k=0}^{\infty} c_k\zeta^k$, then we have

$$[-2 + \sum_{k=0}^{n-1}(kc_k + \alpha_k)\zeta^{k+1}]\omega(\zeta) = \sum_{k=0}^{n}(kc_k - \alpha_k)\zeta^{k+1},$$

where $\sum_{k=n+2}^{\infty} \beta_k \zeta^k$ converges in $|\zeta| < 1$. Let $\zeta = re^{i\theta}$ ($r < 1$). Then integrations give

$$4 + \sum_{k=0}^{n-1}|kc_k + \alpha_k|^2 \geq 4 + \sum_{k=0}^{n-1}|kc_k + \alpha_k|^2r^{2k+2}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left| -2 + \sum_{k=0}^{n-1}(kc_k + \alpha_k)\zeta^{k+1}\right|^2 d\theta$$

$$\geq \frac{1}{2\pi} \int_0^{2\pi} \left| -2 + \sum_{k=0}^{n-1}(kc_k + \alpha_k)\zeta^{k+1}\right|^2 |\omega(\zeta)|^2 d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=0}^{n}(kc_k - \alpha_k)\zeta^{k+1} + \sum_{k=n+2}^{\infty} \beta_k \zeta^k\right|^2 d\theta$$

$$\geq \sum_{k=0}^{n}|kc_k - \alpha_k|^2r^{2k+2},$$
4 + \sum_{k=0}^{n-1} |kc_k + \alpha_k|^2 \geq \sum_{k=0}^{n} |kc_k - \alpha_k|^2,

|nc_n - \alpha_n|^2 \leq 4 + \sum_{k=0}^{n-1} (|kc_k + \alpha_k|^2 - |kc_k - \alpha_k|^2)

(2.2)

= 4 + 4 \sum_{k=0}^{n-1} k \Re\{c_k \bar{\alpha}_k\}.

From (2.2) with \( n = 1 \), we obtain

\[ |c_1 - 1| \leq 4, \quad |c_1 - 1| \leq 2 \]

and, for \( n > 1 \),

(2.3) \quad n^2 |c_n|^2 \leq 4 + 4 |c_1| \leq 16.

Since the analytic function \( h + g \) is univalent in \( \Delta \) by Theorem 2.1, we have

\[ \sum_{k=1}^{\infty} k |a_k + b_k|^2 \leq 1 \]

by the area theorem. From this we get

(2.4) \quad |a_1 + b_1| \leq 1.

We now write

\[ |a_n + b_n| \leq \frac{2\sqrt{2}}{n} \quad \text{for} \ n > 1, \]

by (2.3) and (2.4).

\[ \square \]

References

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