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EXISTENCE OF SIX SOLUTIONS OF THE NONLINEAR SUSPENSION BRIDGE EQUATION WITH NONLINEARITY CROSSING THREE EIGENVALUES

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Abstract. Let $Lu = u_{tt} + u_{xxxx}$ and E be the complete normed space spanned by the eigenfunctions of L. We reveal the existence of six nontrivial solutions of a nonlinear suspension bridge equation $Lu + bu^+ = 1 + \epsilon h(x,t)$ in E when the nonlinearity crosses three eigenvalues. It is shown by the critical point theory induced from the limit relative category of the torus with three holes and finite dimensional reduction method.

1. Introduction and main result

In this paper we investigate the multiplicity of the nonlinear suspension bridge equation with Dirichlet boundary condition

$$u_{tt} + u_{xxxx} + bu^+ = 1 + \epsilon h(x,t) \qquad \text{in } [-\frac{\pi}{2}, \frac{\pi}{2}] \times R, \tag{1.1}$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \tag{1.2}$$

$$u \text{ is } \pi - \text{periodic in } t \text{ and even in } x \text{ and } t, \tag{1.3}$$

where $u^+ = \max\{0, u\}$. The suspension bridge equation is considered as a model of the nonlinear oscillations in differential equation. We consider a one-dimensional beam of length π suspended by cables. When the cables are stretched, there is a restoring force which is assumed to be proportional to the amount of the stretching. But when the beam moves in the opposite direction, then there is no restoring force exerted

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on it. If $u(x,t)$ denotes the displacement in the downward direction at position x and time t, then a simplified model is given by the equations (1.1) with (1.2) and (1.3). McKenna and Walter [11] proved that if $3 < b < 15$, then (1.1) with (1.2) and (1.3) has at least two solutions by degree theory. Choi and Jung [4] also proved that if $3 < b < 15$, then (1.1) with (1.2) and (1.3) has at least three solutions by the variational reduction method, with replacing the condition for $u(t,x)$ in (1.3) by

$$u \text{ is } \pi - \text{periodic in } t \text{ and even in } x. \tag{1.4}$$

Micheletti and Saccon [13] proved that there exists a number $\delta_k > 0$ such that for any b with $\Lambda_k^- - \delta_k < -b < \Lambda_k^-$ and $\Lambda_k^- < \Lambda_1^-$ (1.1) with free-ends boundary conditions, and replacing the right hand side of (1.1) by $c > 0$ has at least four nontrivial solutions via the critical point theory on the manifold with boundary induced from the limit relative category of the Torus with one hole. In this paper we improve these results: We prove that when the nonlinear part b crosses three eigenvalues, (1.1) with (1.2) and (1.3) has at least six nontrivial solutions.
To state main result explicitly we need the following notations:
The eigenvalue problem

$$u_{tt} + u_{xxxx} = \lambda u \tag{1.5}$$

with (1.2) and (1.3) has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \ldots) \tag{1.6}$$

and corresponding normalized eigenfunctions ϕ_{mn} $(m, n \geq 0)$ given by

$$\phi_{0n} = \frac{\sqrt{2}}{\pi} \cos(2n+1)x \qquad \text{for } \ n \geq 0, \tag{1.7}$$

$$\phi_{mn} = \frac{2}{\pi} \cos 2mt \cos(2n+1)x \qquad \text{for } \ m > 0, n \geq 0. \tag{1.8}$$

It is convenient for the following to rearrange the eigenvalues λ_{mn} by increasing magnitude: from now on we denote by $(\lambda_i^-)_{i \geq 1}$ the sequence of the negative eigenvalues of (1.5) with (1.2) and (1.3), by $(\lambda_i^+)_{i \geq 1}$ the sequence of the positive ones, so that

$$\ldots \leq \lambda_i^- \leq \ldots \leq \lambda_2^- \leq \lambda_1^- < \lambda_1^+ \leq \lambda_2^+ \leq \ldots \leq \lambda_i^+ \leq \ldots. \tag{1.9}$$

We note that each eigenvalue has a finite multiplicity and that $\lambda_i^- \to -\infty$, $\lambda_i^+ \to +\infty$ as $i \to \infty$.

THEOREM 1.1. *For any b with $\lambda_4^- < -b < \lambda_3^-$, there exists $\epsilon_0 > 0$ depending on h and b such that if $|\epsilon| < \epsilon_0$, problem (1.1) with (1.2) and (1.3) has at least six nontrivial solutions.*

We are looking for weak solutions of (1.1) with (1.2) and (1.3), that is, we are looking for critical points of a suitable functional $J \in C^1$ on the Hilbert space E. We prove our main result as follows: We first show that the functional J satisfies Three holes Torus-Sphere variational linking inequality and the limit relative category of Torus with three holes is 4, so by the critical point theory induced from the limit relative category of the torus with three holes, we show that the functional J has at least four nontrivial mountain pass type critical points. We also find two nontrivial critical points by the finite dimensional reduction method, so we obtain at least six nontrivial critical points of I. In section 5, we recall the critical point theory induced from the limit relative category.

2. Variational approach

Let Q be the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and E_0 the Hilbert space defined by

$$E_0 = \{u \in L^2(Q) |\ u \text{ is even in } \ x\} \qquad (2.1).$$

The set of functions $\{\phi_{mn}\}$ is an orthonormal base in E_0. We define a subspace E of E_0 as follows

$$E = \{u \in E_0 |\ u = \sum h_{mn}\phi_{mn},\ \sum |\lambda_{mn}| h_{mn}^2 < \infty\} \qquad (2.2)$$

with a norm

$$\|u\| = [\sum |\lambda_{mn}| h_{mn}^2]^{\frac{1}{2}}. \qquad (2.3)$$

Then this normed space E is complete. We consider an orthonormal system of eigenfunctions $\{e_i^-, e_i^+,\ i \geq 1\}$ associated with the eigenvalues $\{\lambda_i^-, \lambda_i^+,\ i \geq 1\}$ instead of the system $\{\phi_{mn}, m, n \geq 0\}$ Let us set

$$E^+ = \text{closure of span}\{\text{eigenfunctions with eigenvalue } \geq 0\}, \qquad (2.4)$$

$$E^- = \text{closure of span}\{\text{eigenfunctions with eigenvalue } \leq 0\}. \qquad (2.5)$$

We define the linear projections $P^- : E \to E^-$, $P^+ : E \to E^+$. Then the norm in E is given by

$$\|u\|^2 = \|P^+ u\|^2 + \|P^- u\|^2. \qquad (2.6)$$

Let us define the functional on E corresponding to (1.1)

$$I(u) = \int_Q [\frac{1}{2}(-|u_t|^2 + |u_{xx}|^2) + \frac{b}{2}|u^+|^2 - u - \epsilon h(x,t)u]dtdx. \qquad (2.7)$$

By the following Proposition 2.3, $I(u) \in C^1$ and the weak solutions of (1.1) coincide with the critical points of $I(u)$. We have some propositions which are proved in [4].

PROPOSITION 2.1. *(i) $u_{tt} + u_{xxxx} \in E$ implies $u \in E$.*
(ii) $\|u\| \geq \|u\|_{L^2}$, where $\|u\|_{L^2}$ denotes the L^2 norm of u.
(iii) $\|u\| = 0$ iff $\|u\|_{L^2} = 0$.

PROPOSITION 2.2. *Let $w(x,t) \in E_0$ and δ not an eigenvalue of (1.5) with (1.2) and (1.3). Then all solution in E_0 of*

$$u_{tt} + u_{xxxx} + \delta u^+ = w(x,t) \text{ in } E_0 \qquad (2.8)$$

belong to E.

PROPOSITION 2.3. *The functional $I(u)$ is continuous and Fréchet differentiable at each u in E with*

$$DI(u)v = \int_Q (u_{tt} + u_{xxxx})v + b\int_Q u^+ \cdot v - \int_Q (1 + \epsilon h(x,t))v. \qquad (2.9)$$

Moreover $DI \in C$. That is $I \in C^1$.

By the following Lemma 2.1 and Lemma 2.2, (1.1) with (1.2) and (1.3) has a positive (trivial) solution u_0.

LEMMA 2.1. *For $b > -1$, the boundary value problem*

$$y^{(4)} + by^+ = 1 \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \qquad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0 \qquad (2.10)$$

has a unique solution y, which is even and positive and satisfies

$$y'(-\frac{\pi}{2}) > 0 \text{ and } y'(\frac{\pi}{2}) < 0. \qquad (2.11)$$

For the proof see [11]. From Lemma 2.1 we can obtain the following lemma.

LEMMA 2.2. *Let $b > -1$, with b not an eigenvalue of (1.5) with (1.2) and (1.2). Let $h \in E$, with $\|h\| = 1$, be given. Then there exists $\epsilon_0 > 0$ (depending on b and h) such that if $|\epsilon| < \epsilon_0$ (1.1) with (1.2) and (1.3) has a positive solution u_0.*

Proof. From Lemma 2.1 the problem

$$y^{(4)} + by^+ = 1 \text{ in } (-\frac{\pi}{2}, \frac{\pi}{2}), \qquad y(\pm\frac{\pi}{2}) = y''(\pm\frac{\pi}{2}) = 0$$

has a unique positive solution y_0. We note that if b is not an eigenvalue of (1.5) with (1.2) and (1.3), then the following linear partial differential equation

$$u_{tt} + u_{xxxx} + bu = \epsilon h(x,t) \text{ in } E \tag{2.12}$$

has a unique solution u_ϵ. We can choose sufficiently small $\epsilon_0 > 0$ (depending on b and h) such that if $|\epsilon| < \epsilon_0$ then $u_\epsilon + y_0 > 0$, which is a solution of (1.1) with (1.2) and (1.3). □

Since (1.1) with (1.2) and (1.3) has a positive (trivial) solution, it is convenient to look for solutions in the form $u = u_0 + z$, so that z is a critical point for the functional $J(w) = I(u_0 + w) - I(u_0)$, where

$$J(z) = \frac{1}{2}\int_Q [-|z_t|^2 + |z_{xx}|^2]dtdx + \frac{b}{2}\int_Q |z|^2 dtdx - \frac{b}{2}\int_Q |(u_0+z)^-|^2 dtdx. \tag{2.13}$$

Moreover

$$\nabla J(z)w = \int_Q (z_{tt} - z_{xxxx} + bz + b(u_0+z)^-)wdtdx. \tag{2.14}$$

Thus it is suffices to estimate the number of critical points of the strongly indefinite functional J. To find the critical points of the functional J we will describe the behaviour of J depending on the position of $-b$ with respect to the negative eigenvalues λ_i^-.

3. Existence of four critical points

In this section we will show that the functional $J(z)$ has at least four nontrivial critical points of mountain pass type via the critical point theory induced from the limit relative category of the torus with three holes. We assume that b is any number with $\lambda_4^- < -b < \lambda_3^-$. Let us set

$$X_0 \equiv E^+ \equiv \text{closure of span}\{\text{eigenfunctions with eigenvalue } \lambda > 0\},$$

$$X_1 \equiv \text{closure of span}\{\text{eigenfunctions with eigenvalue } \lambda = \lambda_1^-\},$$

$$X_2 \equiv \text{closure of span}\{\text{eigenfunctions with eigenvalue } \lambda = \lambda_2^-\},$$

$$X_3 \equiv \text{closure of span}\{\text{eigenfunctions with eigenvalue } \lambda = \lambda_3^-\},$$

$$X_4 \equiv \text{closure of span}\{\text{eigenfunctions with eigenvalue } \lambda \le \lambda_4^-\}.$$

Then E is the topological direct sum of the subspaces X_0, X_1, X_2, X_3 and X_4, where X_1, X_2 and X_3 are one dimensional subspaces. Let w_i be fixed elements of X_i, $i = 1, 2, 3$, and let $\rho_i > 0$, $R_1 > 0$ and $R > R_1$, $i = 1, 2, 3$. We also set

$$S_i(\rho_i) = \{z \in X_i |\ \|z\| = \rho_i\},\ i = 1, 2, 3.$$

$$S_i(\rho_i) - w_i = \{z - w_i |\ z \in S_i(\rho_i)\},\ i = 1, 2, 3.$$

$$\begin{aligned}
&\Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4) \\
&= \{z = (z_1 - w_1) + (z_2 - w_2) + (z_3 - w_3) + z_4 |\ z_i \in X_i,\ i = 1, 2, 3, \\
&\ \rho_1 \le \|z_1 - w_1\| \le R,\ \rho_2 \le \|z_2 - w_2\| \le R,\ \rho_3 \le \|z_3 - w_3\| \le R, \\
&\ \|z_4\| \le R_1,\ \|z\| \le R\},
\end{aligned}$$

$$\begin{aligned}
&\Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4) \\
&= \{z = (z_1 - w_1) + (z_2 - w2) + (z_3 - w_3) + z_4 |\ z_i \in X_i,\ i = 1, 2, 3, \\
&\|z_4\| \le R_1,\ \|z_1 - w_1\| = \rho_1,\ \|z_2 - w_2\| = \rho_2,\ \|z_3 - w_3\| = \rho_3, \\
&\|z\| = R\} \\
&\cap\{z = z_1 + z_2 + z_3 + z_4 |\ z_i \in X_i,\ i = 1, 2, 3, 4, \\
&\|z_4\| = R_1, \quad \rho_1 \le \|z_1 - w_1\| \le R,\ \|z\| = R\ , w_1 \in X_1\} \\
&\cap\{z = z_1 + z_2 + z_3 + z_4 |\ \ z_i \in X_i,\ , i = 1, 2, 3, 4, \\
&\quad \|z_4\| = R_1,\ \rho_2 \le \|z_2 - w_2\| \le R,\ \|z\| = R\ , w_2 \in X_2\} \\
&\cap\{z = z_1 + z_2 + z_3 + z_4 |\ z_i \in X_i,\ i = 1, 2, 3, 4,\ \ \|z_4\| = R_1, \\
&\quad \rho_3 \le \|z_3 - w_3\| \le R\ \|z\| = R\ w_3 \in X_3\}.
\end{aligned}$$

We have the following Three holes Torus-Sphere variational linking inequality of J.

LEMMA 3.1. *Let b be any number with $\lambda^-_4 < -b < \lambda^-_3$. Then there exist $r > 0$, $\rho_i > 0$, $i = 1, 2, 3$, $R_1 > 0$, $R > R_1$ such that $R > r$ and*

$$\sup_{z \in \Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} I(z) < 0 < \inf_{z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} I(z), \tag{3.1}$$

$$\inf_{z \in B_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} I(z) > -\infty \tag{3.2}$$

and

$$\sup_{z \in \Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} I(z) < \infty. \tag{3.3}$$

Proof. First we will show that there exists $r > 0$ such that if $z \in S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)$, then $I(z) > 0$. Let $z = z_0 + z_1 + z_2 + z_3 \in X_0 \oplus X_1 \oplus X_2 \oplus X_3$. Then $P^- z_0 = 0$ and $P^+(z_1 + z_2 + z_3) = 0$. We can choose $r_1 > 0$ such that if $\|z\| \le r_1$, then $u_0 + P^-(z_1 + z_2 + z_3) > 0$. let us choose $r > 0$ with $r < r_1$. Then we have, for $z \in X_0 \oplus X_1 \oplus X_2 \oplus X_3$ with $\|z\| \le r$,

$$\begin{aligned}
J(z) &= \frac{1}{2}\int_Q [-|z_t|^2 + |z_{xx}|^2]dtdx + \frac{b}{2}\int_Q |z|^2 dtdx \\
&\quad -\frac{b}{2}\int_Q |(u_0 + z)^-|^2 dtdx \\
&= \frac{1}{2}\|P^+ z\|^2 - \frac{1}{2}\|P^- z\|^2 + \frac{b}{2}\int_Q |z|^2 dtdx - \frac{b}{2}\int_Q |(u_0 + z)^-|^2 dtdx \\
&= \frac{1}{2}\|P^+ z_0\|^2 - \frac{1}{2}\|P^-(z_1 + z_2 + z_3)\|^2 + \frac{b}{2}\int_Q |P^+ z_0|^2 dtdx \\
&\quad + \frac{b}{2}\int_Q |P^-(z_1 + z_2 + z_3)|^2 dtdx \\
&\quad - \frac{b}{2}\int_Q |(u_0 + P^+ z_0 + P^-(z_1 + z_2 + z_3))^-|^2 dtdx \\
&\ge \frac{1}{2}\|P^+ z_0\|^2 + \frac{b}{2}\int_Q |P^+ z_0|^2 + \frac{1}{2}\|P^- z_1\|^2(-1 + \frac{b}{|\lambda_1^-|}) \\
&\quad + \frac{1}{2}\|P^- z_2\|^2(-1 + \frac{b}{|\lambda_2^-|}) + \frac{1}{2}\|P^- z_3\|^2(-1 + \frac{b}{|\lambda_3^-|}) \\
&\quad - \frac{b}{2}\int_Q |(P^+ z_0)^-|^2 > 0
\end{aligned}$$

since $-1 + \frac{b}{|\lambda_1^-|>0}$, $-1 + \frac{b}{|\lambda_2^-|>0}$, $-1 + \frac{b}{|\lambda_3^-|>0}$. Moreover we have that

$$\inf_{z \in B_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)} I(z) > -\frac{b}{2}\int_Q |(P^+ z_0)^-|^2 > -\infty.$$

Now we will show that

$$\sup_{z \in \Sigma_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} I(z) < 0.$$

Let $z = (z_1 - w_1) + (z_2 - w_2) + (z_3 - w_3) + z_4 \in \Sigma_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)$, $z_i \in S_i(\rho_i)$, $i = 1.2.3$. Since $X_1 \oplus X_2 \oplus X_3 \oplus X_4 \subset E^-$,

$P^+((z_1-w_1)+(z_2-w_2)+(z_3-w_3)+z_4)=0$. Then we have

$$\begin{aligned} J(z) &= \frac{1}{2}\|P^+z\|^2-\frac{1}{2}\|P^-z\|^2+\frac{b}{2}\int_Q |z|^2-\frac{b}{2}\int_Q |(u_0+z)^-|^2 \\ &= -\frac{1}{2}\|P^-((z_1-w_1)+(z_2-w_2)+(z_3-w_3)+z_4)\|^2 \\ &\quad +\frac{b}{2}\int_Q |P^-((z_1-w_1)+(z_2-w_2)+(z_3-w_3)+z_4)|^2 \\ &\quad -\frac{b}{2}\int_Q |(u_0-P^-z)^-|^2 \\ &\le \frac{1}{2}(-1+\frac{b}{|\lambda_1^-|})\rho_1^2+\frac{1}{2}(-1+\frac{b}{|\lambda_2^-|})\rho_2^2+\frac{1}{2}(-1+\frac{b}{|\lambda_3^-|})\rho_3^2+ \\ &\quad \frac{1}{2}(-1+\frac{b}{|\lambda_4^-|})\|P^-z_4\|^2 \\ &\le 0 \end{aligned}$$

since $-1+\frac{b}{|\lambda_1^-|}>0$, $-1+\frac{b}{|\lambda_2^-|}>0$, $-1+\frac{b}{|\lambda_3^-|}>0$, $-1+\frac{b}{|\lambda_4^-|}<0$, $-\frac{b}{2}\int_Q |(u_0-P^-z)^-|^2<0$ and ρ_1, ρ_2, ρ_3 is a small number, there exists $R>0$ with $R>r$ such that if $z\in\Sigma_R^3(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)$, then $J(z)<0$. Therefore

$$\sup_{z\in\Sigma_R^3(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)} J(z)<0.$$

Moreover if $z\in\Delta_R^3(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)$, then $J(z)\le\frac{1}{2}(-1+\frac{b}{|\lambda_4^-|})\|P^-z_4\|^2<\infty$. □

Let $(E_n)_n$ be a sequence of closed finite dimensional subspace of E with the following assumptions: $E_n=E_n^-\oplus E_n^+$ where $E_n^+\subset E^+$, $E_n^-\subset E^-$ for all n (E_n^+ and E_n^- are subspaces of E), $\dim E_n<+\infty$, $E_n\subset E_{n+1}$, $\cup_{n\in N}E_n$ is dense in E.

LEMMA 3.2. *Let $\lambda_4^-<b<\lambda_3^-$. Then the functional J satisfies the $(P.S.)_\gamma^*$ condition with respect to $(E_n)_n$, for any $\gamma\in R$.*

Proof. Let $(k_n)_n$ and $(z_n)_n$ be two sequences such that $k_n\to+\infty$ and $z_n\in E_{k_n}$, $\forall n$, $J(z_n)\to\gamma$ and $\nabla J(z_n)\to 0$. We claim that $(z_n)_n$ is bounded. By contradiction, we suppose that $\|z_n\|\to\infty$. If $w_n=\frac{z_n}{\|z_n\|}$, we can suppose that $w_n \rightharpoonup w_0$ weakly for some $w_0 \in E$. We have

$$\langle \frac{P_{E_{k_n}} \nabla J(z_n)}{\|z_n\|}, w_n \rangle = \frac{2J(z_n)}{\|z_n\|^2} + b \int_Q P_{E_{k_n}} (\frac{u_0}{\|z_n\|} + w_n)^- w_n +$$

$$b \int_Q P_{E_{k_n}} (\frac{u_0}{\|z_n\|} + w_n)^- (\frac{u_0}{\|z_n\|} + w_n)^-. \tag{3.4}$$

Passing to the limit to (3.4) we get

$$\lim_{n\to\infty} b \int_Q P_{E_{k_n}} [(\frac{u_0}{\|z_n\|} + w_n)^- w_n + (\frac{u_0}{\|z_n\|} + w_n)^- (\frac{u_0}{\|z_n\|} + w_n)^-]$$

$$= b \int_Q P_{E_{k_n}} [w_0^- w_0 + w_0^- w_0^-] = b \int_Q P_{E_{k_n}} w_0^- w_0^+ = 0. \tag{3.5}$$

Thus $w_0 = 0$. Moreover we consider

$$\langle \frac{P_{E_{k_n}} \nabla J(z_n)}{\|z_n\|}, P^+ w_n - P^- w_n \rangle = \|P_{E_{k_n}} P^+ w_n\|^2 + \|P_{E_{k_n}} P^- w_n\|^2$$

$$+ b \int_Q P_{E_{k_n}} w_n (P^+ w_n - P^- w_n) + b \int_Q P_{E_{k_n}} (\frac{u_0}{\|z_n\|} + w_n)^- (P^+ w_n - P^- w_n). \tag{3.6}$$

Going to the limit we get

$$\|P^+ w_0\|^2 + \|P^- w_0\|^2 = 0. \tag{3.7}$$

Hence w_n converges to 0 strongly, which is a contradiction. Thus $(z_n)_n$ is bounded. We can suppose that $z_n \rightharpoonup z_0$ weakly in E, for some z_0 in E. We claim that z_n converges to z_0 strongly. We have

$$\langle P_{E_{k_n}} \nabla J z_n, P^+ z_n - P^- z_n \rangle = \|P_{E_{k_n}} P^+ z_n\|^2 + \|P_{E_{k_n}} P^- z_n\|^2$$

$$+ b \int_Q P_{E_{k_n}} [|P^+ z_n|^2 + |P^- z_n|^2] + b \int_Q P_{E_{k_n}} (u_0 + z_n)^- (P^+ z_n - P^- z_n) \to 0. \tag{3.8}$$

Thus we have

$$\|P^+ z_0\|^2 + \|P^- z_0\|^2 \longrightarrow -b \int_Q [|P^+ z_0|^2 + |P^- z_0|^2]$$

$$-b \int_Q (u_0 + z_0)^- (P^+ z_0 - P^- z_0). \tag{3.9}$$

Thus $\|P_{E_{k_n}}P^+z_n\|^2+\|P_{E_{k_n}}P^-z_n\|^2 = \|P_{E_{k_n}}z_n\|^2$ converges. Thus z_n converges strongly(passing to a subsequence), hence $P_{E_{k_n}}z_n \to z_0$ strongly. Therefore we have

$$\nabla J(z_0) = \nabla J(\lim_{n\to\infty} P_{E_{k_n}}z_n) = \lim_{n\to\infty} P_{E_{k_n}}\nabla J(z_n) = 0. \tag{3.10}$$

Thus z_0 is the critical point of J. □

LEMMA 3.3. *Let b be any number with $\lambda_4^- < -b < \lambda_3^-$. If z is a critical point for $J|_{X_0\oplus X_4}$, then $J(z) = 0$ and there is no critical point $z \in X_0 \oplus X_4$ such that*

$$0 < \inf_{z\in S_r(X_0\oplus X_1\oplus X_2\oplus X_3)} J(z) \le J(u) \le \sup_{z\in \Delta_R^3(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)} J(z). \tag{3.11}$$

Proof. We note that from Lemma 3.1, for fixed $z_0 \in X_0$, the functional $z_4 \mapsto J(z_0+z_4)$ is strictly concave in X_4, while, for fixed $z_4 \in X_4$, the functional $z_0 \mapsto I(z_0+z_4)$ is weakly convex in X_0. Moreover 0 is the critical point in $X_0 \oplus X_4$ with $J(0) = 0$. So if $z = z_0 + z_4$ is another critical point for $J|_{X_0\oplus X_4}$, then we have

$$0 = J(0) \le I(z_0) \le I(z_0+z_4) \le I(z_4) \le J(0) = 0. \tag{3.12}$$

So we have $J(u) = J(0) = 0$, and the last statement of the lemma follows. □

Now we will show that J has at least four nontrivial critical points of mountain pass type in the subspace $X_1 \oplus X_2 \oplus X_3$ of E.
Let $P_{X_1\oplus X_2\oplus X_3}$ be the orthogonal projection from E onto $X_1 \oplus X_2 \oplus X_3$ and

$$C = \{z \in E|\ \|P_{X_1\oplus X_2\oplus X_3}z\| \ge 1\}. \tag{3.13}$$

Then C is the smooth manifold with boundary. Let $C_n = C \cap E_n$. Let us define a functional $\Psi : E \setminus \{X_0 \oplus X_4\} \to E$ by

$$\Psi(z) = z-\frac{P_{X_1\oplus X_2\oplus X_3}z}{\|P_{X_1\oplus X_2\oplus X_3}z\|} = P_{X_0\oplus X_4}z+(1-\frac{1}{\|P_{X_1\oplus X_2\oplus X_3}z\|})P_{X_1\oplus X_2\oplus X_3}z. \tag{3.14}$$

We have

$$\nabla\Psi(z)(w) = w - \frac{1}{\|P_{X_1\oplus X_2\oplus X_3}z\|}(P_{X_1\oplus X_2\oplus X_3}w - \langle\frac{P_{X_1\oplus X_2\oplus X_3}z}{\|P_{X_1\oplus X_2\oplus X_3}z\|}, w\rangle\frac{P_{X_1\oplus X_2\oplus X_3}z}{\|P_{X_1\oplus X_2\oplus X_3}z\|}). \tag{3.15}$$

Let us define the functional $\tilde{J} : C \to R$ by

$$\tilde{J} = J \circ \Psi. \tag{3.16}$$

Then $\tilde{J} \in C^{1,1}_{loc}$. We note that if $\tilde{z}$ is the critical point of $\tilde{J}$ and lies in the interior of C, then $z = \Psi(\tilde{z})$ is the critical point of J. We also note that

$$\|grad^-_C \tilde{J}(\tilde{z})\| \geq \|P_{X_0 \oplus X_4} \nabla J(\Psi(\tilde{z}))\| \qquad \forall \tilde{z} \in \partial C. \tag{3.17}$$

Let us set

$$\begin{aligned} \tilde{S_r} &= \Psi^{-1}(S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)), \\ \tilde{B_r} &= \Psi^{-1}(B_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3)), \\ \tilde{\Sigma^3_R} &= \Psi^{-1}(\Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)), \\ \tilde{\Delta^3_R} &= \Psi^{-1}(\Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)). \end{aligned}$$

We note that $\tilde{S_r}$, $\tilde{B_r}$, $\tilde{\Sigma^3_R}$ and $\tilde{\Delta^3_R}$ have the same topological structure as S_r, B_r, Σ^3_R and Δ^3_R respectively.

LEMMA 3.4. *Let b be any number with $\lambda_4^- < -b < \lambda_3^-$. Then $\tilde{J}$ satisfies the $(P.S.)^*_c$ condition with respect to $(C_n)_n$ for every real number $\tilde{c}$ such that*

$$0 < \inf_{\tilde{z} \in \Psi^{-1}(S_r(X_0 \oplus X_1 \oplus X_2 \oplus X_3))} \tilde{J}(\tilde{z}) \leq \tilde{c} \leq \sup_{\tilde{z} \in \Psi^{-1}(\Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4))} \tilde{J}(\tilde{z}), \tag{3.18}$$

where ρ_1, ρ_2, ρ_3, r and R are introduced in Lemma 3.1.

Proof. Let $(k_n)_n$ be a sequence such that $k_n \to +\infty$, $(\tilde{z_n})_n$ be a sequence in C such that $\tilde{z_n} \in C_{k_n}$, $\forall n$, $\tilde{J}(\tilde{z_n}) \to \tilde{c}$ and $grad^-_C \tilde{J}|_{E_{k_n}}(\tilde{z_n}) \to 0$. Set $z_n = \Psi(\tilde{z_n})$ (and hence $z_n \in E_{k_n}$) and $J(z_n) \to \tilde{c}$. We first consider the case in which $z_n \notin X_0 \oplus X_4$, $\forall n$. Since for n large $P_{E_n} \circ P_{X_1 \oplus X_2 \oplus X_3} = P_{X_1 \oplus X_2 \oplus X_3} \circ P_{E_n} = P_{X_1 \oplus X_2 \oplus X_3}$, we have

$$P_{E_{k_n}} \nabla \tilde{J}(\tilde{z_n}) = P_{E_{k_n}} \Psi'(\tilde{z_n})(\nabla J(z_n)) = \Psi'(\tilde{z_n})(P_{E_{k_n}} \nabla J(z_n)) \longrightarrow 0. \tag{3.19}$$

By (3.14) and (3.15),

$$P_{E_{k_n}} \nabla J z_n \to 0 \qquad \text{or} \tag{3.20}$$

$$P_{X_0 \oplus X_4} P_{E_{k_n}} \nabla J(z_n) \to 0 \qquad \text{and} \qquad P_{X_1 \oplus X_2 \oplus X_3} z_n \to 0. \tag{3.21}$$

In the first case the claim follows from the limit Palais-Smale condition for J. In the second case $P_{X_0 \oplus X_4} P_{E_{k_n}} \nabla J(z_n) \to 0$. We claim that $(z_n)_n$ is bounded. By contradiction, we suppose that $\|z_n\| \to +\infty$ and set

$w_n = \frac{z_n}{\|z_n\|}$. Up to a subsequence $w_n \rightharpoonup w_0$ weakly for some $w_0 \in X_0 \oplus X_4$. By the asymptotically linearity of $\nabla J(z_n)$ we have

$$\begin{aligned}\langle \frac{\nabla J(z_n)}{\|z_n\|}, w_n\rangle &= \langle P_{X_0\oplus X_4} P_{E_{k_n}} \frac{\nabla J(z_n)}{\|z_n\|}, w_n\rangle \\ &\quad + \langle \frac{\nabla J(z_n)}{\|z_n\|^2}, P_{X_1\oplus X_2 \oplus X_3} z_n\rangle \longrightarrow 0.\end{aligned}$$

We have

$$\langle \frac{\nabla J(z_n)}{\|z_n\|}, w_n\rangle = \frac{2J(z_n)}{\|z_n\|^2} + b\int_Q (\frac{u_0}{\|z_n\|} + w_n) w_n + b\int_Q |(\frac{u_0}{\|z_n\|} + w_n)^-|^2,$$

where $z_n = ((z_n)_1, \ldots, (z_n)_{2n})$. Passing to the limit we get

$$\lim_{n\to\infty} b\int_Q (\frac{u_0}{\|z_n\|} + w_n) w_n + b\int_Q |(\frac{u_0}{\|z_n\|} + w_n)^-|^2 = b\int_Q w_0^- w_0^+ = 0.$$

Thus $w_0 = 0$. On the other hand we have

$$\begin{aligned}&\langle P_{X_0\oplus X_4} P_{E_{k_n}} \frac{\nabla J(z_n)}{\|z_n\|}, P^+ w_n - P^- w_n\rangle \\ &= \|P_{X_0} P^+ w_n\|^2 + \|P_{X_4} P^- w_n\|^2 + b P_{X_0\oplus X_4} P_{E_{k_n}} \int_Q (|P^+ w_n|^2 + |P^- w_n|^2) \\ &+ b P_{X_0\oplus X_4} P_{E_{k_n}} \int_Q (\frac{u_0}{\|z_n\|} + w_n)^- (P^+ w_n - P^- w_n) \longrightarrow 0.\end{aligned}$$

Since w_n converges to 0 weakly, $\|P_{X_0} P^+ w_n\|^2 + \|P_{X_4} P^- w_n\|^2 \to 0$. Since $\|P_{X_1\oplus X_2\oplus X_3} w_n\|^2 \to 0$, w_n converges to 0 strongly, which is a contradiction. Hence $(z_n)_n$ is bounded. Up to a subsequence, we can suppose that z_n converges to z_0 for some $z_0 \in X_0 \oplus X_4$. We claim that z_n converges to z_0 strongly. We have

$$\begin{aligned}&\langle P_{X_0\oplus X_4} P_{E_{k_n}} \nabla J z_n, P^+ z_n - P^- z_n\rangle \\ &= \|P_{X_0} P_{E_{k_n}} P^+ z_n\|^2 + \|P_{X_4} P_{E_{k_n}} P^- z_n\|^2 \\ &+ b P_{X_0\oplus X_4} P_{E_{k_n}} \int_Q (|P^+ z_n|^2 + |P^- z_n|^2) \\ &+ b P_{X_0\oplus X_4} P_{E_{k_n}} \int_Q (u_0 + z_n)^- (P^+ z_n - P^- z_n).\end{aligned}$$

Thus we have

$$\begin{aligned}&\|P_{X_0}P_{E_{k_n}}P^+z_n\|^2+\|P_{X_4}P_{E_{k_n}}P^-z_n\|^2\\ &\longrightarrow -P_{X_0\oplus X_4}[b\int_Q(|P^+z_0|^2+|P^-z_0|^2)+b\int_Q(u_0+z_0)^-(P^+z_0-P^-z_0)].\end{aligned}$$

That is, $\|P_{X_0}P_{E_{k_n}}P^+z_n\|^2+\|P_{X_4}P_{E_{k_n}}P^-z_n\|^2$ converges.
Since $\|P_{X_1\oplus X_2\oplus X_3}z_n\|^2 \to 0$, $\|z_n\|^2$ converges, so z_n converges to z_0 strongly. Therefore we have

$$\begin{aligned}grad_C^-\tilde{J}(\tilde{z})=grad_C^-J(z) &= \lim_{n\to\infty}P_{E_{k_n}}grad_C^-J(z_n)\\ &= \lim_{n\to\infty}P_{E_{k_n}}grad_C^-\tilde{J}(\tilde{z_n})=0.\end{aligned}$$

So we proved the first case. We consider the case $P_{X_1\oplus X_2\oplus X_3}z_n=0$, i.e., $z_n\in X_0\oplus X_4$. Then $\tilde{z_n}\in\partial C$, $\forall n$. In this case $z_n=\Psi(\tilde{z_n})\in X_0\oplus X_4$ and $P_{X_0\oplus X_4}\nabla J(z_n)\to 0$. Thus by the same argument as the first case we obtain the conclusion. So we prove the lemma. □

THEOREM 3.1. *Let b be any number with $\lambda_4^- < -b < \lambda_3^-$. Then there exist at least four nontrivial critical points z_i, $i=1,2,3,4$, in $X_1\oplus X_2\oplus X_3$ of mountain pass type of the functional J such that*

$$0<\inf_{z\in S_r(X_0\oplus X_1\oplus X_2\oplus X_3)}J(z)\le J(z_i)\le \sup_{z\in\Delta_R^3(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)}J(z), \tag{3.22}$$

where ρ_1, ρ_2, ρ_3, r and R are introduced in Lemma 3.1.

Proof. It suffices to show that $\tilde{J}$ has at least four nontrivial critical points of mountain pass type. By Lemma 3.1, $\tilde{J}$ satisfies the Torus-Sphere variational linking inequality, i. e., there exist ρ_1, ρ_2, ρ_3, $r>0$ and $R>0$ such that $r<R$ and

$$\sup_{\tilde{z}\in\tilde{\Sigma}_R^3}\tilde{J}(\tilde{z})=\sup_{z\in\Sigma_R^3(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)}J(z)<$$

$$0<\inf_{z\in S_r(X_0\oplus X_1\oplus X_2\oplus X_3)}J(z)=\inf_{\tilde{z}\in\tilde{S}_r}\tilde{J}(\tilde{z}),$$

$$\sup_{\tilde{z}\in\tilde{\Delta}_R^3}\tilde{J}(\tilde{z})=\sup_{z\in\Delta_R^3(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)}J(z)<\infty$$

and

$$\inf_{\tilde{z}\in\tilde{B}_r}\tilde{J}(\tilde{z})=\inf_{z\in B_r(X_0\oplus X_1\oplus X_2\oplus X_3)}J(z)>-\infty.$$

By Lemma 3.4, $\tilde{J}$ satisfies the $(P.S.)^*_{\tilde{c}}$ condition with respect to $(C_n)_n$ for every real number $\tilde{c}$ such that

$$0 < \inf_{\tilde{z}\in\tilde{S_r}} \tilde{J}(\tilde{z}) \le \tilde{c} \le \sup_{\tilde{z}\in\tilde{\Delta^3_R}} \tilde{J}(\tilde{z}). \tag{3.23}$$

Let

$$\Sigma^3_n = \Sigma^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4) \cap E_n,$$
$$\Delta^3_n = \Delta^3_R(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4) \cap E_n,$$
$$\tilde{\Sigma^3_n} = \tilde{\Sigma^3_R} \cap E_n, \qquad \tilde{\Delta^3_n} = \tilde{\Delta^3_R} \cap E_n.$$

We claim that

$$cat_{(C_n,\tilde{\Sigma^3_n})}(\tilde{\Delta^3_n}) = 4. \tag{3.24}$$

In fact, we consider a continuous deformation $r : \tilde{S_r}\backslash X_0 \times [0,1] \to \tilde{S_r}\backslash X_0$ such that
• $r(x,0) = x,\ \ \forall x \in \tilde{S_r}\backslash X_0,$
• $r(x,t) = x,\ \ \forall x \in \tilde{S_r} \cap (X_1 \oplus X_2 \oplus X_3),\ \ \forall t \in [0,1],$
• $r(x,1) \in \tilde{S_r} \cap (X_1 \oplus X_2 \oplus X_3),\ \ \forall x \in \tilde{S_r}\backslash X_0.$
Now we can define, if $x = x_0 + x_{123} + x_4 \in X_0 \oplus (X_1 \oplus X_2 \oplus X_3) \oplus X_4$, $t \in [0,1]$,

$$r_1(x,t) = x_0 + \|x_{123} + x_4\| r(\frac{x_{123} + x_4}{\|x_{123} + x_4\|}, t). \tag{3.25}$$

Using r_1, it is easy to construct, for all n, a continuous deformation $\eta_n : C_n \times [0,1] \to C_n$ such that
• $\eta_n(x,0) = x, \quad \forall x \in C_n$
• $\eta_n(x,t) = x, \quad \forall x \in \tilde{\Delta^3_n},\ \forall t \in [0,1],$
• $\eta_n(x,1) \in \tilde{\Delta^3_n}, \quad \forall x \in C_n,$
• $\eta_n(x,t) \in C_n\backslash\tilde{S_r}, \quad \forall x \in C_n\backslash\tilde{S_r}, \quad \forall t \in [0,1].$
The existence of η_n implies that

$$cat_{(C_n,\tilde{\Sigma^3_n})}(\tilde{\Delta^3_n}) = cat_{(\tilde{\Delta^3_n},\tilde{\Sigma^3_n})}(\tilde{\Delta^3_n}). \tag{3.26}$$

We note that the pair $(\tilde{\Delta^3_n}, \tilde{\Sigma^3_n})$ is homeomorphic to the pair (Δ^3_n, Σ^3_n) and the pair (Δ^3_n, Σ^3_n) is homeomorphic to the pair $(\mathcal{B}^{p+1} \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\}, \mathcal{S}^p \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\})$, where $p = \dim X_4 \cap E_n$, $q_1 = \dim X_1 \cap E_n = 1$, $q_2 = \dim X_2 \cap E_n = 1$, $q_3 = \dim X_3 \cap E_n = 1$ and $\mathcal{B}^r$, $\mathcal{S}^r$ denote the r-dimensional ball, the r-dimensional sphere, respectively. Thus the pair $(\tilde{\Delta^3_n}, \tilde{\Sigma^3_n})$ is homeomorphic to the pair $(\mathcal{B}^{p+1} \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\}, \mathcal{S}^p \times \{(\mathcal{S}^{q_1-1} - w_1) \cup (\mathcal{S}^{q_2-1} - w_2) \cup (\mathcal{S}^{q_3-1} - w_3)\})$.

This fact and the facts that $q_i = 1$, $i = 1, 2, 3$ and (b) of (3.7) in [7] imply that

$$cat_{(C_n, \tilde{\Sigma_n^3})}(\tilde{\Delta_n^3}) = 4.$$

Thus we have

$$cat^*_{(C, \tilde{\Sigma_R^3})}(\tilde{\Delta_R^3}) = 4. \tag{3.27}$$

Let us set

$$\begin{aligned} \mathcal{A}_1 = \{A \subset C|\ cat^*_{(C, \tilde{\Sigma_R^3})}(A) \geq 1\}, \quad \mathcal{A}_2 = \{A \subset C|\ cat^*_{(C, \tilde{\Sigma_R^3})}(A) \geq 2\}, \\ \mathcal{A}_3 = \{A \subset C|\ cat^*_{(C, \tilde{\Sigma_R^3})}(A) \geq 3\}, \quad \mathcal{A}_4 = \{A \subset C|\ cat^*_{(C, \tilde{\Sigma_R^3})}(A) \geq 4\}. \end{aligned} \tag{3.28}$$

Since $cat^*_{(C, \tilde{\Sigma_R^3})}(\tilde{\Delta_R^3}) = 4$, $\tilde{\Delta_R^3} \in \mathcal{A}_i$, $i = 1, 2, 3$. Let us set

$$\tilde{c_1} = \inf_{A \in \mathcal{A}_1} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}), \quad \tilde{c_2} = \inf_{A \in \mathcal{A}_2} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}), \tag{3.29}$$

$$\tilde{c_3} = \inf_{A \in \mathcal{A}_3} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}), \quad \tilde{c_4} = \inf_{A \in \mathcal{A}_4} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}).$$

We first claim that $\tilde{c_i} < \infty$, $i = 1, 2, 3, 4$. In fact, from the facts that

$$\sup_{z \in \Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) < \infty$$

in Lemma 3.1 and $\tilde{\Delta_R^3} \in \mathcal{A}_i$, $i = 1, 2, 3, 4$, we have that

$$\begin{aligned} \tilde{c_i} &= \inf_{A \in \mathcal{A}_i} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) \\ &\leq \sup_{\tilde{z} \in \tilde{\Delta_R^3}} \tilde{J}(\tilde{z}) = \sup_{z \in \Delta_R^3(S_1(\rho_1) - w_1, S_2(\rho_2) - w_2, S_3(\rho_3) - w_3, X_4)} J(z) < \infty \end{aligned}$$

for $i = 1, 2, 3, 4$. We also claim that $\sup_{\tilde{z} \in \tilde{\Sigma_R^3}} \tilde{J}(\tilde{z}) \leq \tilde{c_i}$, $i = 1, 2, 3, 4$. In fact, for any $A \in \mathcal{A}_i$ with $\tilde{\Sigma_R^3} \subset A$, $i = 1, 2, 3, 4$,

$$\sup_{\tilde{z} \in \tilde{\Sigma_R^3}} \tilde{J}(\tilde{z}) \leq \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}), \tag{3.30}$$

and hence

$$\sup_{\tilde{z} \in \tilde{\Sigma_R^3}} \tilde{J}(\tilde{z}) \leq \inf_{A \in \mathcal{A}_i} \sup_{\tilde{z} \in A} \tilde{J}(\tilde{z}) = \tilde{c_i}, \ i = 1, 2, 3, 4. \tag{3.31}$$

By Lemma 3.4, $\tilde{J}$ satisfies the $(P.S.)^*_{\tilde{c}}$ condition with respect to $(C_n)_n$ for any real number $\tilde{c}$ with $0 < \inf_{\tilde{z} \in \tilde{S_r}} \tilde{J}(\tilde{z}) \leq \tilde{c} \leq \sup_{\tilde{z} \in \tilde{\Delta_R^3}} \tilde{J}(\tilde{z})$. Thus, by Theorem 5.1, there exist four nontrivial critical points $\tilde{z}_1$, $\tilde{z}_2$, $\tilde{z}_3$, $\tilde{z}_4$ of mountain pass type of the functional $\tilde{J}$ such that

$$\tilde{c_1} = \tilde{J}(\tilde{z_1}), \quad \tilde{c_2} = \tilde{J}(\tilde{z_2}), \quad \tilde{c_3} = \tilde{J}(\tilde{z_3}), \quad \tilde{c_4} = \tilde{J}(\tilde{z_4}). \tag{3.32}$$

We claim that

$$\inf_{\tilde{z}\in\tilde{S_r}} \tilde{J}(\tilde{z}) \le \tilde{c_1} \le \tilde{c_2} \le \tilde{c_3} \le \tilde{c_4} \le \sup_{\tilde{z}\in\tilde{\Delta_R^3}} \tilde{J}(\tilde{z}). \tag{3.33}$$

Since $cat^*_{(C,\tilde{\Sigma_R^3})}(\tilde{\Delta_R^3}) = 4$, $\tilde{\Delta_R^3} \in \mathcal{A}_4$ and hence

$$\tilde{c_4} = \inf_{A\in\mathcal{A}_4} \sup_{\tilde{z}\in A} \tilde{J}(\tilde{z}) \le \sup_{\tilde{z}\in\tilde{\Delta_R^3}} \tilde{J}(\tilde{z}), \ \forall A \in \mathcal{A}_4. \tag{3.34}$$

For the proof of $\tilde{c_1} \ge \inf_{\tilde{z}\in\tilde{S_r}} \tilde{J}(\tilde{z})$, we construct a deformation $\eta'_n : C_n\backslash\tilde{S_r} \times [0,1] \to C_n\backslash\tilde{S_r}$, for all n, such that
- $\eta'_n(x,0) = x, \ \ \forall x \in C_n\backslash\tilde{S_r}$,
- $\eta'_n(x,t) = x, \ \ \forall x \in \tilde{\Sigma_n^3}, \forall t \in [0,1]$,
- $\eta'_n(x,1) \in \tilde{\Sigma_n^3}, \quad \forall x \in C_n$.

Actually η'_n can be defined by taking the retraction of η_n on $C_n\backslash\tilde{S_r}$ followed by a retraction of $\tilde{\Delta_n^3}\backslash\tilde{S_r}$ to $\tilde{\Sigma_n^3}$. The existence of η'_n, for all n, implies that any $A \in \mathcal{A}_1$ must intersect $\tilde{S_r}$. So $\sup \tilde{J}(A) \ge \inf_{\tilde{z}\in\tilde{S_r}} \tilde{J}(\tilde{z}) \, \forall A \in \mathcal{A}_1$. So we have $\tilde{c_1} = \inf_{A\in\mathcal{A}_1} \sup_{\tilde{z}\in A} \tilde{J}(\tilde{z}) \ge \inf_{\tilde{z}\in\tilde{S_r}} \tilde{J}(\tilde{z})$. Therefore there exist at least four nontrivial critical points $\tilde{z}_1$, $\tilde{z}_2$, $\tilde{z}_3$, $\tilde{z}_4$ for the functional $\tilde{J}$ such that

$$\inf_{\tilde{z}\in\tilde{S_r}} \tilde{J}(\tilde{z}) \le \tilde{J}(\tilde{z_1}) \le \tilde{J}(\tilde{z_2}) \le \tilde{J}(\tilde{z_3}) \le \tilde{J}(\tilde{z}) \le \sup_{\tilde{z}\in\tilde{\Delta_R^3}} \tilde{J}(\tilde{z}). \tag{3.35}$$

Setting $z_i = \Psi(\tilde{z_i})$, $i = 1,2,3,4$, we have

$$\begin{aligned} 0 < \inf_{z\in S_r} J(z) &= \inf_{\tilde{z}\in\tilde{S_r}} \tilde{J}(\tilde{z}) \le J(z_1) \le J(z_2) \\ &\le J(z_3) \le J(z_4) \le \sup_{\tilde{z}\in\tilde{\Delta_R^3}} \tilde{J}(\tilde{z}) = \sup_{z\in\Delta_R^3} J(z). \end{aligned} \tag{3.36}$$

We claim that $\tilde{z_i} \notin \partial C$, that is $z_i \notin X_0 \oplus X_4$, which implies that z_i are the critical points for J in $X_1 \oplus X_2 \oplus X_3$. For this we assume by contradiction that $z_i \in X_0 \oplus X_4$. From (3.17), $P_{X_0\oplus X_4}\nabla J(z_i) = 0$, namely, z_i, $i = 1,2,3,4$, are the critical points for $J|_{X_0\oplus X_4}$. By Lemma 3.3, $J(z_i) = 0$, which is a contradiction for the fact that

$$0 < \inf_{z\in S_r(X_0\oplus X_1\oplus X_2\oplus X_3)} J(z) \le J(z_i) \le$$

$$\sup_{z\in\Delta^3_R(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)} J(z),\ i=1,2,3,4. \tag{3.37}$$

By Lemma 3.3 there is no critical point $z \in X_0 \oplus X_4$ such that

$$\begin{aligned} 0 < \inf_{z\in S_r(X_0\oplus X_1\oplus X_2\oplus X_3)} J(z) \le J(z) \\ \le \sup_{z\in\Delta^3_R(S_1(\rho_1)-w_1,S_2(\rho_2)-w_2,S_3(\rho_3)-w_3,X_4)} J(z). \end{aligned}$$

Hence $z_i \notin X_0 \oplus X_4$, $i = 1, 2, 3, 4$. This proves Theorem 3.1. □

4. Proof of Theorem 1.1

In this section we will use finite dimensional reduction method to show that J has the fifth and the sixth critical points and prove Theorem 1.1. Let $V = X_1 \oplus X_2 \oplus X_3$ and W be the orthogonal complement of V in E. Let $P : E \to V$ denote the orthogonal projection of E onto V and $I-P : E \to W$ denote that of E onto W and $z = v+w$, $v \in V$, $w \in W$.

Lemma 4.1. *Let b be any number with $\lambda_4^- < -b < \lambda_3^-$ and $v \in V$ be given. Then we have:*
(i) There exists a unique solution $w \in W$ of the equation

$$w_{tt} + w_{xxxx} + (I-P)[b(v+w) + b(u_0+v+w)^-] = 0 \qquad \text{in } W. \tag{4.1}$$

If we put $w = \theta(v)$, then θ is continuous on V and we have

$$\nabla J(v+\theta(v))(w) = 0 \qquad \text{for all } w \in W. \tag{4.2}$$

In particular, θ satisfies a uniform Lipschitz condition in v with respect to the L^2 norm (also the norm $\|\cdot\|$).
(ii) If $F : V \to R$ is defined by $F(v) = J(v+\theta(v))$, then F has a continuous Fr'echet derivative ∇F with respect to V and

$$\nabla F(v)(h) = \nabla J(v+\theta(v))(h) \qquad \text{for all } h \in V. \tag{4.3}$$

If v_0 is a critical point of F, then $v_0 + \theta(v_0)$ is a critical point of J and conversely every critical point of J is of this form.
(iii) If $v_0 + \theta(v_0)$ is a critical point of mountain pass type of J, then v_0 is a critical point of mountain pass type of F.

Proof. The reader is referred to Lemma 2.2 of [4] for the proofs of part (i) and part (ii).
(iii) Suppose that v_0 is not of mountain pass type of F. Let M be an open neighborhood of v_0 in V such that either $F^{-1}(-\infty, F(v_0)) \cap M$ is

empty or path connected. If $F^{-1}(-\infty, F(v_0)) \cap M$ is empty, by part (i) we see that $\{v + w| \ v \in M, \ w \in W\} \cap F^{-1}(-\infty, J(v_0 + \theta(v_0)))$ is also empty. Thus $v_0 + \theta(v_0)$ is not of mountain pass type for J. On the other hand if $F^{-1}(-\infty, F(v_0)) \cap M$ is path connected, letting $N = \{v + w| \ v \in M, \ \|w - \theta(v)\| < 1\}$ and using (i) we have that $N \cap J^{-1}(-\infty, J(v_0 + \theta(v_0)))$ is also path connected. This proves (iii). □

LEMMA 4.2. *Let b be any number with $\lambda_4^- < -b < \lambda_3^-$. Then $F(v) \to -\infty$ as $\|v\| \to \infty$, so F is bounded above and satisfies the Palais-Smale condition: Any sequence $\{v_n\} \subset V$ for which $F(v_n)$ is bounded and $\nabla F(v_n) \to 0$ possesses a convergent subsequence.*

Proof. For the proof refer to Lemma 2.4 and Lemma 2.7 in [4]. □

LEMMA 4.3. *Let b be any number with $\lambda_4^- < -b < \lambda_3^-$. Then there exists a small open neighborhood B of 0 in V such that in B, $v = 0$ with value $F(0) = 0$ is neither a minimum nor degenerate critical point of F.*

Proof. Let $v \in V$ be given and $\theta(v)$ the unique solution of (4.1). Then we have

$$\begin{aligned} F(v) &= J(v + \theta(v)) = \int_Q [\frac{1}{2}(-|v_t + \theta(v)_t|^2 + |v_{xx} + \theta(v)_{xx}|^2) \\ &\quad + \frac{b}{2}|v + \theta(v)|^2 - \frac{b}{2}|(u_0 + v + \theta(v))^-|^2] dtdx \\ &= \int_Q [\frac{1}{2}(-|v_t|^2 + |v_{xx}|^2) \\ &\quad + \frac{b}{2}v^2] + \int_Q [-v_t \cdot \theta(v)_t + v_{xx} \cdot \theta(v)_{xx} + bv \cdot \theta(v)] dtdx \\ &\quad + \int_Q [\frac{1}{2}(-|\theta(v)_t|^2 + |\theta(v)_{xx}|^2) + \frac{b}{2}\theta(v)^2] dtdx \\ &= \int_Q [\frac{1}{2}(-|v_t|^2 + |v_{xx}|^2) + \frac{b}{2}v^2] dtdx + C, \end{aligned}$$

where

$$C = \int_Q [\frac{1}{2}(-|\theta(v)_t|^2 + |\theta(v)_{xx}|^2) + \frac{b}{2}\theta(v)^2] dtdx.$$

Since θ is a continuous function, there exists a small neighborhood B of 0 in V such that if $v, \in B, \to 0$, then $\theta(v) \to \theta(0) = 0$, so $\|\theta(v)\| = o(\|v\|$. Thus we have

$$C = o(\|v\|^2).$$

Thus we obtain

$$\frac{\lambda_3^-}{2}\|v\|_{L^2}^2 + \frac{b}{2}\|v\|_{L^2}^2 + o(\|v\|^2) \le F(v) \le \frac{\lambda_1^-}{2}\|v\|_{L^2}^2 + \frac{b}{2}\|v\|_{L^2}^2 + o(\|v\|^2)$$

as $\|v\| \to 0$ in B. Therefore 0 with $F(0) = 0$ is neither a minimum nor degenerate critical point. Thus the lemma is proved. □

Lemma 4.4. *(Deformation Lemma) Let X be a real Banach space and $I \in C^1(X, R)$. Suppose I satisfies the Palais-Smale condition. Let N be a given neighborhood of the set K_c of the critical points of I at a given level c. Then there exists $\epsilon > 0$, as small as we want, and a deformation $\eta : [0,1] \times X \longrightarrow X$ such that, denoting by A_b the set $\{x \in X : I(x) \le b\}$:*
(i) $\eta(0, x) = x \quad \forall x \in X,$
(ii) $\eta(t, x) = x \quad \forall x \in A_{c-2\epsilon} \cup (X \backslash A_{c+2\epsilon}), \quad \forall t \in [0,1],$
(iii) $\eta(1, \cdot)(A_{c+\epsilon} \backslash N) \subset A_{c-\epsilon}.$

The proof of Lemma 4.4 can be found in [14].

Proof of Theorem 1.1

By Lemma 4.3, 0 with value $F(0) = 0$ is neither a minimum nor degenerate critical point of F. Let B be a small open neighborhood of 0. In section 3 we show that the functional $J(z)$ has at least four nontrivial critical points z_i, $i = 1, 2, 3, 4$ of mountain pass type. Since $z_i \in X_1 \oplus X_2 \oplus X_3 = V$, these points are of the form $z_i = v_i + \theta(v_i)$, $\theta(v_i) = 0$. By (iii) of Lemma 4.1, v_i, $i = 1, 2, 3, 4$, are also critical points of mountain pass type of F with $0 < F(v_1) \le F(v_2) \le F(v_3) \le F(v_4)$. Let C_i, $i = 1, 2, 3, 4$, be the open neighborhoods of v_i, $i = 1, 2, 3, 4$, in V respectively such that $B \cap C_1 \cap C_2 \cap C_3 \cap C_4 = \emptyset$. Since $F \in C^1(V, R)$ is bounded from above, satisfies the Palais-Smale condition and $F(v) \to -\infty$ as $\|v\| \to \infty$ (Lemma 4.2), $\max_{v \in V} F(v)$ exists and is a critical value of F. Hence there exists a critical point v_5 of F such that

$$F(v_5) = \max_{v \in V} F(v). \tag{4.4}$$

Let C_5 be an open neighborhood of v_5 in V such that $B \cap C_1 \cap C_2 \cap C_3 \cap C_4 \cap C_5 = \emptyset$. Since $F(v) \longrightarrow -\infty$ as $\|v\| \to \infty$, we can choose $v_0 \in V \backslash (B \cup C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5)$ such that $F(v_0) < F(v_1)$. Let Γ be the set of all paths in V joining v_0 and v_1. We write

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} F(v). \tag{4.5}$$

Let $\Gamma' = \{\gamma \in \Gamma : \ \gamma \cap C_5 = \emptyset\}$ and

$$c' = \inf_{\gamma \in \Gamma'} \sup_{\gamma} F(v). \tag{4.6}$$

The Mountain Pass Theorem in [14] imply that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} F(v)$$

is a critical value of F. First we will prove that if $F(v_5) = c$, then there exists a critical point v_6 of F at level c such that $v_6 \neq v_5$ (of course $v_6 \neq 0$ since $c \neq 0$ (this follows from the fact that 0 with value $F(0) = 0$ is neither a minimum nor degenerate critical point of F by Lemma 4.3 and $c = \max_{v \in V} F(v) > 0$)). We claim that if $F(v_5) = c$, then $c = c'$. In fact, since $\Gamma' \subset \Gamma$, $c \leq c'$. On the other hand, $c' \leq c$ since c is the maximum value of F. Hence $c = c'$. Suppose by contradiction $K_c = \{v_5\}$, By the above claim $c = c'$. Let us fix ϵ, η as in Lemma 4.4 with $X = V$, $I = F$, $c = c$, $N = C_5$ and taking $\epsilon < \frac{1}{2}(c - F(v_1))$. Taking $\gamma \in \Gamma'$ such that $\sup_\gamma F \leq c$. From Lemma 4.4, $\eta(1, \cdot) \circ \gamma \in \Gamma$ and

$$\sup F(\eta(1, \cdot) \circ \gamma) \leq c - \epsilon < c, \tag{4.7}$$

which is a contradiction. Therefore there exists a critical point v_6 of F at level c such that $v_6 \neq v_5$, v_1, v_2, v_3, c_4, 0, which means that $F(v)$ has at least six nontrivial critical points.

Second, we claim that if $c = F(v_i) < F(v_5)$ for some i, $i = 1, 2, 3, 4$, then there exists a critical point v_6 of F at level c such that $v_6 \neq 0$, v_5, v_i, $i = 1, 2, 3, 4$. Let $\Gamma'' = \{\gamma \in \Gamma : \ \gamma \cap C_i = \emptyset, \text{ for some } i, \ i = 1, 2, 3, 4\}$. Suppose by contradiction $K_c = \{v_i\}$ for some i, $i = 1, 2, 3, 4$. Let us fix ϵ, η as in Lemma 4.4 with $X = V$, $I = F$, $c = c$, $N = C_i$, $v_i \in C_i$, and taking $\epsilon < \frac{1}{2}(c - F(v_i))$. Taking $\gamma \in \Gamma''$ such that $\sup_\gamma F(v) \leq c + \epsilon$. By Deformation Lemma 4.4, $\eta(1, \cdot) \circ \gamma \in Gamma$ and

$$\sup F(\eta(1, \cdot) \circ \gamma) \leq c - \epsilon < c,$$

which is a contradiction. Therefore, there exists a critical point v_6 of F at level c such that $v_6 \neq 0$, v_5, v_i, $i = 1, 2, 3, 4$. which means that $F(v)$ has at least six nontrivial critical points. Finally, if $c \neq F(v_i) < F(v_5)$ for all i, then there exists a critical point v_6 of F at level c such that $v_6 \neq 0$, v_5, v_i for all i, $i = 1, 2, 3, 4$ since $0 < F(v_1) < c < F(v_5)$ and

$c \neq F(v_i)$ for all i. Therefore $F(v)$ has at least six nontrivial critical points. Thus we prove Theorem 1.1.

5. Critical point theory on the manifold induced from the limit relative category

Now, we consider the critical point theory on the manifold with boundary induced from the limit relative category. Let E be a Hilbert space and M be the closure of an open subset of E such that M can be endowed with the structure of C^2 manifold with boundary. Let $f : W \to R$ be a $C^{1,1}$ functional, where W is an open set containing M. For applying the usual topological methods of critical points theory we need a suitable notion of critical point for f on M. We recall the following notions: lower gradient of f on M, $(P.S.)^*_c$ condition and the limit relative category (see [7]).

Definition 5.1. *If $u \in M$, the lower gradient of f on M at u is defined by*

$$grad^-_M f(u) = \begin{cases} \nabla f(u) & \text{if } u \in int(M), \\ \nabla f(u) + [< \nabla f(u), \nu(u) >]^- \nu(u) & \text{if } u \in \partial M, \end{cases}$$

where we denote by $\nu(u)$ the unit normal vector to ∂M at the point u, pointing outwards. We say that u is a lower critical for f on M, if $grad^-_M f(u) = 0$.

Since the functional $I(u)$ is strongly indefinite, the notion of the $(P.S.)^*_c$ condition and limit relative category is a very useful tool for the proof of the main theorems.

Let $M_n = M \cap E_n$, for any n, be the closure of an open subset of E_n and has the structure of a C^2 manifold with boundary in E_n. We assume that for any n there exists a retraction $r_n : M \to M_n$. For given $B \subset E$, we will write $B_n = B \cap E_n$.

Definition 5.2. *Let $c \in R$. We say that f satisfies the $(P.S.)^*_c$ condition with respect to $(M_n)_n$, on the manifold with boundary M, if for any sequence $(k_n)_n$ in N and any sequence $(u_n)_n$ in M such that $k_n \to \infty$, $u_n \in M_{k_n}$, $\forall n$, $f(u_n) \to c$, $grad^-_{M_{k_n}} f(u_n) \to 0$, there exists a subsequence of $(u_n)_n$ which converges to a point $u \in M$ such that $grad^-_M f(u) = 0$.*

Let Y be a closed subspace of M.

Definition 5.3. *Let B be a closed subset of M with $Y \subset B$. We define the relative category $cat_{M,Y}(B)$ of B in (M,Y), as the least integer h such that there exist $h+1$ closed subsets U_0, $U_1, \ldots$, U_h with the following properties:*
$B \subset U_0 \cup U_1 \cup \ldots \cup U_h$;
$U_1, \ldots, U_h$ are contractible in M;
$Y \subset U_0$ and there exists a continuous map $F : U_0 \times [0,1] \to M$ such that

$$\begin{aligned} F(x,0) &= x \qquad \forall x \in U_0, \\ F(x,t) &\in Y \qquad \forall x \in Y,\ \forall t \in [0,1], \\ F(x,1) &\in Y \qquad \forall x \in U_0. \end{aligned}$$

If such an h does not exist, we say that $cat_{M,Y}(B) = +\infty$.

Definition 5.4. *Let (X,Y) be a topological pair and $(X_n)_n$ be a sequence of subsets of X. For any subset B of X we define the limit relative category of B in (X,Y), with respect to $(X_n)_n$, by*

$$cat^*_{(X,Y)}(B) = \lim\sup_{n\to\infty} cat_{(X_n,Y_n)}(B_n). \tag{5.1}$$

Let Y be a fixed subset of M. We set

$$\mathcal{B}_{\mathrm{i}} = \{\mathrm{B} \subset \mathrm{M} |\ \mathrm{cat}^*_{(\mathrm{M,Y})}(\mathrm{B}) \geq \mathrm{i}\}, \tag{5.2}$$

$$c_i = \inf_{B\in\mathcal{B}_{\mathrm{i}}} \sup_{x\in B} f(x). \tag{5.3}$$

We have the following multiplicity theorem.

Theorem 5.1. *Let $i \in N$ and assume that*
(1) $c_i < +\infty$,
(2) $\sup_{x\in Y} f(x) < c_i$,
(3) *the $(P.S.)^*_{c_i}$ condition with respect to $(M_n)_n$ holds.*
Then there exists a lower critical point x such that $f(x) = c_i$. If

$$c_i = c_{i+1} = \ldots = c_{i+k-1} = c, \tag{5.4}$$

then

$$cat_M(\{x \in M | f(x) = c,\ grad^-_M f(x) = 0\}) \geq k. \tag{5.5}$$

Proof. Let $c = c_i$; using the $(P.S.)^*_c$ condition, with respect to $(M_n)_n$, one can prove that, for any neighborhood N of

$$K_c = \{x|\ f(x) = c,\ \ grad^-_M f(x) = 0\}, \tag{5.6}$$

there exist n_0 in N and $\delta > 0$ such that $\|grad_M^-\| \geq \delta$ for all $n \geq n_0$ and all $x \in E_n \backslash N$ with $c - \delta \leq f(x) \leq c + \delta$.Moreover it is not difficult to see that, for all n, the function $\tilde{f}_n : E_n \to R \cup \{+\infty\}$ defined by $\tilde{f}_n = f(x)$, if $x \in M_n$, $\tilde{f}_n(x) = +\infty$, otherwise, is ϕ-convex of order two, according to the definitions of [6]. Then the conclusion follows using the same arguments of [1, 7] and the nonsmooth version of the classical Deformation Lemma. □

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