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A SHAPE OPTIMIZATION METHOD USING COMPLIANT FORMULATION ASSOCIATED WITH THE 2D STOKES CHANNEL FLOWS

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Abstract. We are concerned with a free boundary problem for the 2D Stokes channel flows, which determines the profile of the wing for the channel, so that the given traction force is to be distributed along the wing of the channel. Using the domain embedding technique, the free boundary problem is transferred into the shape optimization problem through the compliant formulation by releasing the traction condition along the variable boundary. The justification of the formulation will be discussed.

1. Introduction

Optimal shape design problems associated with the channel flows have been studied in several articles. In [7] and [8], the bump design problem for the channel aiming to minimize the energy dissipation(or drag minimization) for the stationary Navier-Stokes flows have been dealt with. In [13] and [14], an optimal shape design related to the velocity tracking problem inside the channel have been studied.

In this paper, we are interested in a design problem of a wing profile of the channel for the tracking of the stress distribution along the wing. That is, we want to design a wing profile along which given surface stress is produced. Unlike the previous studies for the channel design problems as listed above, this belongs to a branch of free boundary problem. The free boundary problem is a problem of determining unknown boundary by multiple partial measurements of the system. Some of studies

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have been done for the Bernoulli type free boundary problems in ideal fluid dynamics([9]). Free boundary problems associated with the identification of the boundary shape appear in modeling a variety of physical phenomena in connection with diverse requirements of engineering needs and practices([11]). The core part consists of the shape identification for the free boundary. We will apply the shape optimization technique to resolve the modeling free boundary problem.

In this article, a shape optimization technique using the compliant formulation conjunct with the domain embedding will be presented for the identification of the wing profile of the channel. The main characteristics of the formulation lies in the transition of the functional with negative norm associated with the traction condition into the equivalent functional with the positive norm involving the trace of the velocity field along the variable boundary.

The paper is organized as follows. In section 2, we define the geometric model configuration and some useful results for the problem. In section 3, we present the compliant formulation for the free boundary of the channel. The justification of the formulation for the free boundary will be discussed.

2. Problem configurations

We consider the two-dimensional incompressible viscous fluid passing through the channel Ω as shown in Figure 1. As a whole, the channel representing the fluid domain Ω_α is characterized by the fixed boundary Γ_0 and the variable boundary Γ_α, so that $\partial\Omega_\alpha = \Gamma_0 \cup \Gamma_\alpha$. Here, α denotes an appropriate parametrization of the variable part of the channel boundary, which shall be specified later in this section. Thus, the parameter α characterizes the fluid domain for the channel flows.

For the fluid modeling, we assume the velocity $\mathbf{u}$ and the pressure p satisfy the stationary Stokes equations

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega_\alpha \tag{2.1}$$

and

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega_\alpha. \tag{2.2}$$

In (2.1), $\mathbf{f}$ denotes the given body force and ν represents the kinematic viscosity whenever the variables are appropriately non-dimensionalized.

For the boundary data, we assume the Dirichlet condition along the fixed boundary

(2.3) $$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_0\,,$$

and the no-slip condition along the wing profile of the channel

(2.4) $$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_\alpha.$$

For the boundary data (2.3) and (2.4), we need a specific consideration for the smooth transition between the data. Since the discontinuity in boundary velocity may cause large stress at corner points such as junction points between faces of the channel, we assume the boundary velocity $\mathbf{g}$ satisfy that

$$\text{support } \mathbf{g} \subset \Gamma_0 \quad \text{and} \quad \int_{\Gamma_0} \mathbf{g}\cdot\mathbf{n}\, ds = 0\,,$$

where $\mathbf{n}$ denotes the outward unit normal vector to the boundary. Note that the second expression comprises the compatibility condition for the solenoidal vector fields.

The incompressible Stokes equations are characterized by the Newtonian constitutive laws for the Cauchy stress $\mathcal{S}$ in terms of the velocity $\mathbf{u}$ and the pressure p represented by

$$\mathcal{S}(\mathbf{u}, p) = -pI + 2\nu\mathcal{D}(\mathbf{u})\,.$$

Here, the rate of strain $\mathcal{D}(\mathbf{u})$ is represented by

$$\mathcal{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)\,,$$

where $(\nabla\mathbf{u})^T$ denotes the transpose of the spatial gradient of the velocity vector field. Then, the traction force due to the flow can be represented by the normal component of the stress tensor along the boundary of the fluid domain as in

$$\mathbf{t}(\mathbf{u}, p) = \mathcal{S}(\mathbf{u}, p)\mathbf{n} = -pn_j + \nu\sum_i\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)n_i\,.$$

Now, let us state the problem we are interested in. Let $\boldsymbol{\phi}$ be a vector field of class $\mathcal{C}^{1,1}$ with compact support in the sub-region $U := (0,1)\times(a,b)$ of $[0,1]\times[0,b]$, which denotes the desired distribution of the force affecting the wing profile of the channel. Then, our primitive goal is to identify the wing profile Γ_α to satisfy

(2.5) $$\mathbf{t}(\mathbf{u}, p) = \boldsymbol{\phi} \quad \text{on the variable boundary } \Gamma_\alpha.$$

For à-priori given parameter domain Ω_α, the systems (2.1)–(2.5) are not well-posed since the boundary conditions to be satisfied on the free boundary Γ_α are over-determined. Especially, two conditions are imposed along the free boundary: no-slip boundary condition $\mathbf{u} = \mathbf{0}$ and the matching condition for the traction force $\mathbf{t}(\mathbf{u}, p) = \boldsymbol{\phi}\Big|_{\Gamma_\alpha}$. These multiple conditions may be utilized to identify the unknown free boundary of the channel. For recent advances for the free boundary problems associated with the Bernoulli type under the ideal fluids, one may consult with [5] and the references cited there in.

In this paper, we apply the shape optimization technique. The shape optimization technique related to the free boundary problems start by choosing one of the dual boundary conditions for the free boundary as an appropriate least squares shape functional, while the system with the other boundary condition plays the role of the state constraint. Earlier study in this direction can be found in [3] in connection with the simplified model for a dam design problem. We follow the similar framework by taking the traction condition as a boundary condition, on the while the trace state of the velocity as an objective functional to be minimized. In [15], traditional free boundary problems of the Bernoulli type and Stokes flows have been studied over the exterior domain by using the similar kind of approach.

As a premise, we need to specify the parametrization of the free boundary. As seen in Figure 1, the channel domain is determined by the parameterized free boundary Γ_α. In fact, the variable part of the boundary Γ_α is represented by the graph of a Lipschitz continuous function $\alpha : [0,1] \to [a,b]$, so that the set

$$\Gamma_\alpha = \{(t,z) \in [0,1] \times [a,b] \,|\, z = \alpha(t) \text{ with } \alpha(0) = z_0 \text{ and } \alpha(1) = z_1\}$$

corresponds to the parametrization for the wing profile of the channel. For the suitable choice of allowable channel geometry, we need the following considerations.

- The channel domain should be appropriately regular enough to assume the necessary regularities for the velocity and the pressure.
- For each α, the channel domain Ω_α should be enclosed in a fixed domain $\widehat{\Omega} = [0,1] \times [0,b]$, so that the tracking of the traction force is meaningful.
- For the domain embedding, the complement $\Omega_\alpha^c := \widehat{\Omega} - \overline{\Omega}_\alpha$ of the fluid domain Ω_α should be Lipschitz continuous.

FIGURE 1. The flow domain $\Omega(\alpha)$ and its boundary

For this reasons, all admissible parameter α is restricted to $C^{1,1}$-class. Furthermore, there are some practical constraints that may be taken into account; for example, the first derivative at the points $t=0$ and $t=1$ should be specified so that Γ_α is connected smoothly to the fixed boundary Γ_0 by keeping convex corners at the intersection points. Taking into these considerations all together, one may introduce a set of allowable parameters in the following specific way. Let c_0, c_1, c_2 be given non-negative constants and $(0, z_0)$ and $(1, z_1)$ be the intersection of the free boundary Γ_α and the fixed boundary Γ_0 respectively. Then, the set

$$\begin{aligned}\mathcal{U}_{\text{ad}} := \{\alpha \in C^{1,1}([0,1]) \quad | \quad & \alpha(0) = z_0,\, \alpha(1) = z_1 \\ \alpha'(0+) = c_0,\, \alpha'(1-) = c_1,\, & |\alpha''(t)| \le c_2 \text{ a.e. in } (0,1)\}\end{aligned}$$

may be a suitable set of allowable shapes. Here, the restriction for α'' plays the role of suppressing excessive oscillations of the free boundary, which is essential in the shape optimization([11]).

The proposed free boundary problem can be turned into the shape optimization problem in the following manner. Let $(\mathbf{u}_\alpha, p_\alpha)$ be the solution pair of the velocity and pressure for the following Stokes system with the pre-imposed traction condition along the free boundary Γ_α

$$\tag{2.6} \left\{ \begin{array}{rcll} -\nu\Delta\mathbf{u}_\alpha + \nabla p_\alpha &=& \mathbf{f}_\alpha & \text{in } \Omega_\alpha\,, \\ \nabla\cdot\mathbf{u}_\alpha &=& 0 & \text{in } \Omega_\alpha\,, \\ \mathbf{u}_\alpha &=& \mathbf{g} & \text{on } \Gamma_0\,, \\ -p_\alpha\mathbf{n}_\alpha + 2\nu\mathcal{D}(\mathbf{u}_\alpha)\mathbf{n}_\alpha &=& \boldsymbol{\phi} & \text{on } \Gamma_\alpha\,. \end{array}\right.$$

Here, $\mathbf{f}_\alpha$ is the external force defined on Ω_α and $\mathbf{n}_\alpha$ denotes the outward unit normal vector along Γ_α. Then, the desired free boundary can be sought as a solution for the following shape optimization problem, whenever the system (2.6) has a unique solution which is regular enough:

Find the optimal parameter $\alpha \in \mathcal{U}_{\text{ad}}$ which minimizes the shape functional

$$\tag{2.7} J(\alpha) = \frac{1}{2}\int_{\Gamma_\alpha} |\mathbf{u}_\alpha|^2\, dx\,.$$

In the next section, the problem (2.7) is resolved into the compliant formulation in conjunction with the domain embedding method. Note that the above framework makes sense when the system (2.6) is well-posed. The well-posedness and some regularity results for the system (2.6) will be shown in Theorem 2.1.

Let us define some function spaces and notations that will be used throughout the paper. We denote by $H^s(\Omega)$, $s \in \mathbb{R}$, the standard Sobolev space of order s with respect to the set Ω. When m is nonnegative integer, we naturally associate the norm on $H^m(\Omega)$ with $\|\cdot\|_{m,\Omega} = \sqrt{(\cdot,\cdot)_{m,\Omega}}$, where $(\cdot,\cdot)_{m,\Omega}$ is an inner product on $\mathbf{H}^m(\Omega)$. For vector–valued functions and spaces, we use boldface notation. For example, $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$ denotes the space of $\mathbb{R}^n$–valued functions such that each component belongs to $H^s(\Omega)$. Whenever $\Sigma \subset \partial\Omega = \Gamma$ has positive measure, we shall denote the space with the homogeneous boundary condition along Σ by $\mathbf{H}^1_\Sigma(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \,:\, \mathbf{v} = \mathbf{0} \text{ on } \Sigma\}$, and we let $\mathbf{H}^1_0(\Omega) = \mathbf{H}^1_\Gamma(\Omega)$.

For the space of interest to us, we consider the semi–norm defined on $\mathbf{H}^1(\Omega)$

$$|||\mathbf{v}||| = 2\left(\int_\Omega \mathcal{D}(\mathbf{v}) \,:\, \mathcal{D}(\mathbf{v})\,dx\right)^{1/2}.$$

Note that Korn's inequality leads to

$$|||\mathbf{v}||| \geq C\|\mathbf{v}\|_{1,\Omega}\,, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_\Sigma(\Omega) \tag{2.8}$$

for some positive constant C. This implies that the semi–norm $|||\cdot|||$ is a norm which is equivalent to the norm $\|\cdot\|_{1,\Omega}$ on $\mathbf{H}^1_\Sigma(\Omega)$. By $<\cdot,\cdot>_{-s}$, we shall denote the duality pairing between $\mathbf{H}^s_\Sigma(\Omega)$ and its dual space, $\mathbf{H}^{-s}_\Sigma(\Omega)$. For the given boundary force, we take

$$\mathbf{H}^s_0(\Sigma) = \left\{\boldsymbol{\phi} \in \mathbf{H}^s(\Sigma) \mid \text{ support of } \boldsymbol{\phi} \subset \Sigma \text{ and } \int_\Sigma \boldsymbol{\phi}\cdot\mathbf{n}\,ds = 0\right\}.$$

Finally, the following spaces for the solenoidal vector fields are introduced: $\mathbf{V}(\Omega) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) \mid \nabla\cdot\mathbf{v} = 0\}$, $\mathbf{V}_\Sigma(\Omega) := \mathbf{V}(\Omega) \cap \mathbf{H}^1_\Sigma(\Omega)$ and $\mathbf{V}_0(\Omega) := \mathbf{V}(\Omega) \cap \mathbf{H}^1_0(\Omega)$.

In order to represent the Stokes equations (2.6) into a weak form, we denote the continuous bilinear form over $\mathbf{H}^1(\Omega_\alpha)$ by

$$a_\alpha(\mathbf{u},\mathbf{v}) := 2\nu\int_{\Omega_\alpha} \mathcal{D}(\mathbf{u}) \,:\, \mathcal{D}(\mathbf{v})\,dx\,. \tag{2.9}$$

Using the Korn's inequality, the form is coercive over $\mathbf{H}^1_{\Gamma_0}(\Omega_\alpha)$.

For the weak formulation, we need to notice the following relation

$$2\nabla \cdot \mathcal{D}(\mathbf{u}_\alpha) = \Delta \mathbf{u}_\alpha + \nabla(\nabla \cdot \mathbf{u}_\alpha)\,.$$

If we take a dot product with $\mathbf{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega_\alpha)$ and then integration, we obtain by Green's formula

$$2\int_{\Gamma_\alpha} \mathbf{v} \cdot \mathcal{D}(\mathbf{u}_\alpha)\mathbf{n}_\alpha \, ds = \int_{\Omega_\alpha} \Delta \mathbf{u}_\alpha \cdot \mathbf{v}\, dx + 2\int_{\Omega_\alpha} \mathcal{D}(\mathbf{u}_\alpha) \, : \, \nabla \mathbf{v}\, dx\,.$$

Since $\mathcal{D}(\mathbf{v})$ is a symmetric tensor, we have

$$\int_{\Omega_\alpha} \mathcal{D}(\mathbf{u}_\alpha) \, : \, \nabla \mathbf{v}\, dx = \int_{\Omega_\alpha} \mathcal{D}(\mathbf{u}_\alpha) \, : \, \mathcal{D}(\mathbf{v})\, dx\,.$$

Hence, for $\mathbf{v} \in \mathbf{V}_{\Gamma_0}(\Omega_\alpha)$, it follows that

$$2\int_{\Omega_\alpha} \mathcal{D}(\mathbf{u}_\alpha) \, : \, \mathcal{D}(\mathbf{v})\, dx = -\int_{\Omega_\alpha} \Delta \mathbf{u}_\alpha \cdot \mathbf{v}\, dx + \int_{\Gamma_\alpha} 2\mathcal{D}(\mathbf{u}_\alpha)\mathbf{n}_\alpha \cdot \mathbf{v}\, ds$$

Thus, the velocity field for the system (2.6) can be found by the solution of the following weak formulation:

> Seek $\mathbf{u}_\alpha \in \mathbf{V}(\Omega_\alpha)$ which satisfies $\mathbf{u}_\alpha = \mathbf{g}$ on the fixed boundary Γ_0 and the weak form

$$a_\alpha(\mathbf{u}_\alpha, \mathbf{v}) = \int_{\Omega_\alpha} \mathbf{f}_\alpha \cdot \mathbf{v}\, dx + \int_{\Gamma_\alpha} \boldsymbol{\phi} \cdot \mathbf{v}\, ds\,, \quad \forall\, \mathbf{v} \in \mathbf{V}_{\Gamma_0}(\Omega_\alpha)\,. \tag{2.10}$$

In the following theorem, we show the existence, uniqueness, and regularity results for the velocity field.

THEOREM 2.1 *Let* $\alpha \in \mathcal{U}_{ad}$ *so that* Ω_α *is a fluid domain with a convex* $\mathcal{C}^{1,1}$*-boundary. Suppose the data satisfies* $\mathbf{f}_\alpha \in \mathbf{L}^2(\Omega_\alpha)$ *and* $\mathbf{g} \in \mathbf{H}^{3/2}_0(\Gamma_0)$*. Then, there exists a unique solution* $\mathbf{u}_\alpha \in \mathbf{V}(\Omega_\alpha) \cap \mathbf{H}^2(\Omega_\alpha)$ *which satisfies*

$$\|\mathbf{u}_\alpha\|_{2,\Omega_\alpha} \le C(\|\mathbf{f}_\alpha\|_{0,\Omega_\alpha} + \|\mathbf{g}\|_{3/2,\Gamma_0} + \|\boldsymbol{\phi}\|_{1/2,\Gamma_\alpha}) \tag{2.11}$$

with $C > 0$ *is a constant independent of* α*,* $\mathbf{f}_\alpha$ *and* $\mathbf{g}$*.*
Proof: Note that the trace of $\boldsymbol{\phi} \in \mathcal{C}^{1,1}(U)$ along $\Gamma_\alpha \subset U$ belongs to $\mathbf{H}^{1/2}(\Gamma_\alpha)$, which ensures the regularity for the traction data $\mathbf{t}(\mathbf{u}_\alpha, p_\alpha)$ on Γ_α. By Ladyzhenskaya([10]), for $\mathbf{g} \in \mathbf{H}^{3/2}_0(\Gamma_0)$ one can choose a lifting $\mathbf{w}_g \in \mathbf{V}(\Omega_\alpha) \cap \mathbf{H}^2(\Omega_\alpha)$ of $\mathbf{g}$ such that $\mathbf{w} = \mathbf{g}$ on Γ_0 and $\mathbf{w}_g = \mathbf{0}$ on Γ_α(see

also [6]). One of such a choice for $\mathbf{w}_g$ can be found as a solution of the homogeneous Stokes system

$$\begin{cases} -\nu\Delta\mathbf{w}_g + \nabla r &= \mathbf{0} \quad \text{in } \Omega_\alpha\,, \\ \nabla\cdot\mathbf{w}_g &= 0 \quad \text{in } \Omega_\alpha\,, \\ \mathbf{w}_g &= \mathbf{g} \quad \text{on } \Gamma_0\,, \\ \mathbf{w}_g &= \mathbf{0} \quad \text{on } \Gamma_\alpha\,. \end{cases}$$

Since Ω_α is a domain of $\mathcal{C}^{1,1}$-class with convex corners, one can choose $\mathbf{w}_g$ up to $\mathbf{H}^2(\Omega_\alpha)$-regularity.

Now, taking $\mathbf{u}^* = \mathbf{u}_\alpha - \mathbf{w}_g \in \mathbf{V}_{\Gamma_0}(\Omega_\alpha)$, $\mathbf{u}^*$ is the solution of

$$\begin{aligned} a_\alpha(\mathbf{u}^*,\mathbf{v}) &= \int_{\Omega_\alpha}\mathbf{f}_\alpha\cdot\mathbf{v}\,dx + \int_{\Gamma_\alpha}\boldsymbol{\phi}\cdot\mathbf{v}\,ds - a_\alpha(\mathbf{w}_g,\mathbf{v}) \\ &=: F(\mathbf{v})\,, \quad \forall\,\mathbf{v}\in\mathbf{V}_{\Gamma_0}(\Omega_\alpha)\,. \end{aligned}$$

Here, $F \in \mathbf{V}_{\Gamma_0}^{-1}(\Omega_\alpha)$ by the conditions on $\mathbf{f}_\alpha$, $\boldsymbol{\phi}$ and $\mathbf{w}_g$, and by usual argument for the Sobolev spaces and norms one can show that

$$\|F\|_{-1,\Omega_\alpha} \le C(\|\mathbf{f}_\alpha\|_{-1,\Omega_\alpha} + \|\boldsymbol{\phi}\|_{-1/2,\Gamma_\alpha} + \|\mathbf{w}_g\|_{1,\Omega_\alpha})\,.$$

Since the bilinear form $a_\alpha(\cdot,\cdot)$ is continuous and coercive over $\mathbf{V}_0(\Omega_\alpha)$, by Lax–Milgram Lemma there exists a unique $\mathbf{u}^* = \mathbf{u}_\alpha - \mathbf{w}_g$. Moreover, according to the Korn's inequality applied to

$$a_\alpha(\mathbf{u}_\alpha - \mathbf{w}_g, \mathbf{u}_\alpha - \mathbf{w}_g) = F(\mathbf{u}_\alpha - \mathbf{w}_g)\,,$$

one can get for some positive constants C_1 and C_2

$$\|\mathbf{u}_\alpha - \mathbf{w}_g\|_1 \le C_1|\!|\!|\mathbf{u}_\alpha - \mathbf{w}_g|\!|\!| \le C_2(\|\mathbf{f}_\alpha\|_{-1} + \|\boldsymbol{\phi}\|_{-1/2,\Gamma_\alpha} + \|\mathbf{w}_g\|_1)\,.$$

Hence, from the triangle inequality we have

$$\begin{aligned} \|\mathbf{u}_\alpha\|_1 &\le \|\mathbf{u}_\alpha - \mathbf{w}_g\|_1 + \|\mathbf{w}_g\|_1 \\ &\le C(\|\mathbf{f}_\alpha\|_{-1} + \|\boldsymbol{\phi}\|_{-1/2,\Gamma_\alpha} + \|\mathbf{w}_g\|_1) \end{aligned}$$

for a constant $C > 0$. If we take infimum over all lifting $\mathbf{w}_g \in \mathbf{V}(\Omega_\alpha)$ of $\mathbf{g}$, then the following is derived from the above inequality

$$\|\mathbf{u}_\alpha\|_1 \le C(\|\mathbf{f}_\alpha\|_{-1} + \|\boldsymbol{\phi}\|_{-1/2,\Gamma_\alpha} + \|\mathbf{g}\|_{1/2,\Gamma_0})\,.$$

The regularity stated in (2.11) can be acquired from the standard regularity results for the elliptic system and the regularity for $\mathbf{w}_g$ and $\boldsymbol{\phi}$. $\square$

3. Compliant formulation by the domain embedding

The purpose of this section is to find an equivalent condition for the shape identification to the free boundary by examining the compliance (work done by the load) which is followed by the external force and boundary data such as the presumed boundary stress.

For the identification of the free boundary proposed by (2.7), we extend the formulation to the larger domain $\widehat{\Omega}$ by releasing the no-slip boundary condition for the velocity field, in which all admissible domains can be imbedded, i.e., $\Omega_\alpha \subset \widehat{\Omega}$ for all $\alpha \in \mathcal{U}_{\rm ad}$. Let us consider the extension of the pair $(\mathbf{u}_\alpha, p_\alpha, \mathbf{f}_\alpha)$ in the following manner. Given Ω_α, we set $\widehat{\mathbf{u}}_\alpha = \mathbf{u}_\alpha$ and $\widehat{\mathbf{f}}_\alpha = \mathbf{f}_\alpha$ over Ω_α and zero over $\Omega_\alpha^c = \widehat{\Omega} - \Omega_\alpha$. Then, the extension $\widehat{p}_\alpha$ of the pressure is defined by substituting the extended fields into the Stokes equations. Since the Stokes system is represented by the constitutive relation

$$-\nabla \cdot \mathcal{S}(\widehat{\mathbf{u}}_\alpha, \widehat{p}_\alpha) = \widehat{\mathbf{f}}_\alpha \,, \tag{3.1}$$

this extension is well-justified.

One can show that the solenoidal property can be preserved by the extension.

LEMMA 3.1 *Let $\widehat{\mathbf{u}}_\alpha$ be an extension of the velocity field $\mathbf{u}_\alpha$ as defined above. Then, $\widehat{\mathbf{u}}_\alpha$ preserves the norm and $\widehat{\mathbf{u}}_\alpha$ belongs to the solenoidal space $\mathbf{V}(\widehat{\Omega})$.*

Proof: Since $\widehat{\mathbf{u}}_\alpha = \mathbf{u}_\alpha$ in Ω_α and $\widehat{\mathbf{u}}_\alpha = \mathbf{0}$ in Ω_α^c, obviously we have $\|\widehat{\mathbf{u}}_\alpha\|_{1,\widehat{\Omega}} = \|\mathbf{u}_\alpha\|_{1,\Omega_\alpha}$. To show the solenoidal property is preserved by the extension, it is sufficient to check that

$$\int_{\widehat{\Omega}} (\nabla \cdot \widehat{\mathbf{u}}_\alpha)\, \psi \, dx = 0 \,, \quad \forall\, \psi \in H_0^1(\widehat{\Omega}) \,. \tag{3.2}$$

Let us choose $\psi \in \mathcal{C}_0^1(\widehat{\Omega})$. Then, using $\nabla \cdot \widehat{\mathbf{u}}_\alpha = \nabla \cdot \mathbf{u}_\alpha = 0$ in Ω_α and $\widehat{\mathbf{u}}_\alpha = \mathbf{0}$ along the interface Γ_α, we have

$$\begin{aligned} \int_{\widehat{\Omega}} (\nabla \cdot \widehat{\mathbf{u}}_\alpha)\, \psi \, dx &= \int_{\Omega_\alpha} (\nabla \cdot \mathbf{u}_\alpha)\, \psi \, dx + \int_{\Omega_\alpha^c} (\nabla \cdot \widehat{\mathbf{u}}_\alpha)\, \psi \, dx \\ &= \int_{\partial\Omega_\alpha^c} \widehat{\mathbf{u}}_\alpha \cdot (-\mathbf{n}_\alpha) \psi \, ds - \int_{\Omega_\alpha^c} \widehat{\mathbf{u}}_\alpha \cdot \nabla\psi \, dx \\ &= - \int_{\Omega_\alpha^c} \widehat{\mathbf{u}}_\alpha \cdot \nabla\psi \, dx = 0 \,. \end{aligned}$$

Since this holds true for all $\psi \in \mathcal{C}_0^1(\widehat{\Omega})$ and $\mathcal{C}_0^1(\widehat{\Omega})$ is dense in $H_0^1(\widehat{\Omega})$, (3.2) is followed and we have the result. □

The converse direction for the extension has been shown by Tiba([16]): If $\mathbf{z} \in \mathbf{H}^1(\widehat{\Omega})$ satisfies $\mathbf{z} = \mathbf{0}$ a.e. in $\widehat{\Omega} - \Omega_\alpha$, then $\mathbf{z} \in \mathbf{H}^1(\Omega_\alpha)$. Thus, if $\widehat{\mathbf{u}} \in \mathbf{V}(\widehat{\Omega})$ satisfies $\widehat{\mathbf{u}}\Big|_{\Omega_\alpha^c} = \mathbf{0}$, then we have $\widehat{\mathbf{u}} \in \mathbf{V}(\Omega_\alpha)$. For the uniform extension of $H(\mathrm{div})$, one may consult [2].

The following provides the motivation for the compliant formulation of the problem.

Theorem 3.2 *Let $\alpha \in \mathcal{U}_{ad}$ be given and $\mathbf{u}_\alpha$ be a solution of (2.6). Let $\widehat{\mathbf{u}}_\alpha$ be a zero extension to $\widehat{\Omega}$ of $\mathbf{u}_\alpha$. Then, $\widehat{\mathbf{u}}_\alpha \in \mathbf{V}(\widehat{\Omega})$ satisfy the compliant formulation*

$$
(3.3) \quad \left\{ \begin{array}{rcl} a(\widehat{\mathbf{u}}_\alpha, \widehat{\mathbf{v}}) & = & \displaystyle\int_{\widehat{\Omega}} \widehat{\mathbf{f}}_\alpha \cdot \widehat{\mathbf{v}}\, dx + \delta_\alpha(\widehat{\mathbf{v}})\,, \quad \forall\, \widehat{\mathbf{v}} \in \mathbf{V}_0(\widehat{\Omega}) \\ \widehat{\mathbf{u}}_\alpha & = & \mathbf{g} \quad \textit{on } \Gamma_0\,, \end{array} \right.
$$

where

$$
(3.4) \quad \begin{aligned} \delta_\alpha(\widehat{\mathbf{v}}) : &= < \boldsymbol{\phi}, \widehat{\mathbf{v}} >_{\Gamma_\alpha} \\ &= \int_0^1 \boldsymbol{\phi}(x, \alpha(x)) \cdot \widehat{\mathbf{v}}(x, \alpha(x)) \sqrt{1 + \alpha'(x)^2}\, dx\,. \end{aligned}
$$

Proof: Since $\boldsymbol{\phi} \in \mathcal{C}^{1,1}((0,1) \times (a,b))$, by the trace theorem we have

$$
\begin{aligned} |\delta_\alpha(\widehat{\mathbf{v}})| = |< \boldsymbol{\phi}, \widehat{\mathbf{v}} >_{\Gamma_\alpha}| \;&\le \|\boldsymbol{\phi}\|_{0,\Gamma_\alpha} \|\widehat{\mathbf{v}}\|_{0,\Gamma_\alpha} \\ &\le C \|\widehat{\mathbf{v}}\|_{1,\widehat{\Omega}}\,, \end{aligned}
$$

where $C > 0$ is a constant depending on the supremum norm of $\boldsymbol{\phi}$ and the parameter $\alpha \in \mathcal{U}_{ad}$. Hence δ_α belongs to the dual space $\mathbf{V}^*$ of $\mathbf{V}_0(\widehat{\Omega})$ and may be regarded as a δ-distribution supported on Γ_α. Now, let us invoke the identity on Ω_α

$$
\begin{aligned} & 2\nu \int_{\Omega_\alpha} \mathcal{D}(\mathbf{u}_\alpha) : \mathcal{D}(\widehat{\mathbf{v}})\, dx - \int_{\Omega_\alpha} p_\alpha \nabla \cdot \widehat{\mathbf{v}}\, dx \\ &= \int_{\Omega_\alpha} (-\nu \Delta \mathbf{u}_\alpha + \nabla p_\alpha) \cdot \widehat{\mathbf{v}}\, dx - \int_{\Omega_\alpha} \nabla(\nabla \cdot \mathbf{u}_\alpha) \cdot \widehat{\mathbf{v}}\, dx - \int_{\partial\Omega_\alpha} \mathcal{S}(\mathbf{u}_\alpha, p_\alpha)\mathbf{n}_\alpha \cdot \widehat{\mathbf{v}}\, ds \\ &= \int_{\Omega_\alpha} \mathbf{f}_\alpha \cdot \widehat{\mathbf{v}}\, dx - \int_{\Omega_\alpha} \nabla(\nabla \cdot \mathbf{u}_\alpha) \cdot \widehat{\mathbf{v}}\, dx + \int_{\Gamma_\alpha} \mathbf{t}(\mathbf{u}_\alpha, p_\alpha) \cdot \widehat{\mathbf{v}}\, ds \end{aligned}
$$

Since $\nabla \cdot \mathbf{u}_\alpha = 0$ in Ω_α and $\widehat{\mathbf{v}} \in \mathbf{V}_0(\widehat{\Omega})$, this relation is simplified into

$$a_\alpha(\mathbf{u}_\alpha, \widehat{\mathbf{v}}) = \int_{\Omega_\alpha} \mathbf{f}_\alpha \cdot \widehat{\mathbf{v}}\, dx + \int_{\Gamma_\alpha} \mathbf{t}(\mathbf{u}_\alpha, p_\alpha) \cdot \widehat{\mathbf{v}}\, ds\,. \tag{3.5}$$

Hence, using the fact that $\widehat{\mathbf{u}}_\alpha$ and $\widehat{\mathbf{f}}_\alpha$ are zero-extensions of $\mathbf{u}_\alpha$ and $\mathbf{f}_\alpha$ over $\widehat{\Omega} - \Omega_\alpha$, (3.5) yields that

$$\begin{aligned} a(\widehat{\mathbf{u}}_\alpha, \widehat{\mathbf{v}}) &= a_\alpha(\mathbf{u}_\alpha, \widehat{\mathbf{v}}) \\ &= \int_{\Omega_\alpha} \mathbf{f}_\alpha \cdot \widehat{\mathbf{v}}\, dx + \int_{\Gamma_\alpha} \mathbf{t}(\mathbf{u}_\alpha, p_\alpha) \cdot \widehat{\mathbf{v}}\, ds \\ &= \int_{\widehat{\Omega}} \widehat{\mathbf{f}}_\alpha \cdot \widehat{\mathbf{v}}\, dx + \int_{\Gamma_\alpha} \boldsymbol{\phi} \cdot \widehat{\mathbf{v}}\, ds \\ &= \int_{\widehat{\Omega}} \widehat{\mathbf{f}}_\alpha \cdot \widehat{\mathbf{v}}\, dx + \delta_\alpha(\widehat{\mathbf{v}})\,. \end{aligned}$$

This completes the proof. □

Motivated by Theorem 3.2, we can set up the compliant formulation for the free boundary problem related to the domain embedding as follows.

For $\alpha \in \mathcal{U}_{ad}$, let $\widehat{\mathbf{u}} \in \mathbf{V}(\widehat{\Omega})$ be the solution satisfying

$$\left\{ \begin{aligned} a(\widehat{\mathbf{u}}, \widehat{\mathbf{v}}) &= \int_{\widehat{\Omega}} \widehat{\mathbf{f}} \cdot \widehat{\mathbf{v}}\, dx + \delta_\alpha(\widehat{\mathbf{v}})\,, \quad \forall \widehat{\mathbf{v}} \in \mathbf{V}_0(\widehat{\Omega}) \\ \widehat{\mathbf{u}} &= \mathbf{g} \quad \text{on } \Gamma_0\,, \end{aligned} \right. \tag{3.6}$$

where $\mathbf{g} \in \mathbf{H}_0^{1/2}(\Gamma_0)$. Next, we set up the shape functional on $\mathcal{U}_{ad}$ by

$$\mathcal{J}(\alpha) = \frac{1}{2} \int_{\Gamma_\alpha} |\widehat{\mathbf{u}}|^2\, ds\,, \tag{3.7}$$

where $\widehat{\mathbf{u}}$ is the solution of the compliant formulation (3.6). Then, the free boundary for the wing profile of the channel can be sought by the optimal solution of the problem:

Find $\alpha^* \in \mathcal{U}_{ad}$ such that

$$\mathcal{J}(\alpha^*) \le \mathcal{J}(\alpha), \quad \forall \alpha \in \mathcal{U}_{ad}\,. \tag{3.8}$$

The following characterizes the compliant formulation discussed so far and justifies our method.

Theorem 3.3 *Let $\alpha \in \mathcal{U}_{ad}$ be given and $\widehat{\mathbf{u}} \in \mathbf{V}(\widehat{\Omega})$ be a solution of (3.6). Then, the δ-distribution on Γ_α in the compliant formulation (3.6)* *corresponds to the jump state of the traction force across the wing profile* Γ_α*, in the sense that there exists* $\widehat{p} \in L^2(\widehat{\Omega})$ *such that*

$$\delta_\alpha = [[\mathbf{t}(\widehat{\mathbf{u}}, \widehat{p})]] := \mathbf{t}(\widehat{\mathbf{u}}_{in}, \widehat{p}_{in}) - \mathbf{t}(\widehat{\mathbf{u}}_{out}, \widehat{p}_{out})\,, \tag{3.9}$$

where $(\widehat{\mathbf{u}}, \widehat{p})_{in}$ *denotes the restriction of* $(\widehat{\mathbf{u}}, \widehat{p})$ *in the domain* Ω_α *and* $(\cdot)_{out}$ *in the exterior domain* $\Omega_\alpha^c = \widehat{\Omega} - \overline{\Omega}_\alpha$*.*

Proof: For the proof, we need the following version of de Rham's lemma([6], [12]). Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary. Let $\mathbf{V}_0(\Omega)$ denote the solenoidal vector fields with the homogeneous boundary. Suppose $\mathbf{\Psi} \in \mathbf{H}^{-1}(\Omega)$ belongs to the polar set of $\mathbf{V}_0(\Omega)$, i.e.,

$$< \mathbf{\Psi}, \mathbf{v} >_{-1,\Omega} = 0 \quad \forall\, \mathbf{v} \in \mathbf{V}_0(\Omega)\,,$$

then there exists $p \in L^2(\Omega)$ such that $\mathbf{\Psi} = \nabla p$ in Ω.

For a sake of simplicity, we may assume $\widehat{\mathbf{f}} = \mathbf{0}$. Let $\widehat{\mathbf{v}} \in \mathbf{V}_0(\widehat{\Omega})$ and $\alpha \in \mathcal{U}_{ad}$. Then, from (3.6) we have

$$\begin{aligned} \delta_\alpha(\widehat{\mathbf{v}}) &= < \boldsymbol{\phi}, \widehat{\mathbf{v}} >_{\Gamma_\alpha} \\ &= 2\nu \int_{\widehat{\Omega}} \mathcal{D}(\widehat{\mathbf{u}}) : \mathcal{D}(\widehat{\mathbf{v}})\, dx \\ &= 2\nu \int_{\Omega_\alpha^c} \mathcal{D}(\widehat{\mathbf{u}}) : \mathcal{D}(\widehat{\mathbf{v}})\, dx + \int_{\partial\Omega_\alpha^c} 2\nu \mathcal{D}(\widehat{\mathbf{u}})\mathbf{n}_1 \cdot \widehat{\mathbf{v}}\, ds \\ &\quad + 2\nu \int_{\Omega_\alpha} \mathcal{D}(\widehat{\mathbf{u}}) : \mathcal{D}(\widehat{\mathbf{v}})\, dx + \int_{\partial\Omega_\alpha} 2\nu \mathcal{D}(\widehat{\mathbf{u}})\mathbf{n}_2 \cdot \widehat{\mathbf{v}}\, ds\,, \end{aligned} \tag{3.10}$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are outward unit normal vectors to $\partial\Omega_\alpha$ and $\partial\Omega_\alpha^c$ respectively. Let $\mathbf{n}_\alpha$ denote the outward unit normal vector along Γ_α exterior to Ω_α, so that $\mathbf{n}_1 = \mathbf{n}_\alpha$ and $\mathbf{n}_2 = -\mathbf{n}_\alpha$ along the interface Γ_α. Then, since $\widehat{\mathbf{v}} = \mathbf{0}$ on $\partial\widehat{\Omega}$, (3.10) leads to

$$\begin{aligned} \delta_\alpha(\widehat{\mathbf{v}}) &= \int_{\Omega_\alpha^c} -\nu\Delta\widehat{\mathbf{u}} \cdot \widehat{\mathbf{v}}\, dx + \int_{\Omega_\alpha} -\nu\Delta\widehat{\mathbf{u}} \cdot \widehat{\mathbf{v}}\, dx \\ &\quad + \int_{\Gamma_\alpha} 2\nu[[\mathcal{D}(\widehat{\mathbf{u}})]]\mathbf{n}_\alpha \cdot \widehat{\mathbf{v}}\, ds\,, \end{aligned} \tag{3.11}$$

where

$$[[\mathcal{D}(\widehat{\mathbf{u}})]] := \mathcal{D}(\widehat{\mathbf{u}}_{\rm in})\Big|_{\Gamma_\alpha} - \mathcal{D}(\widehat{\mathbf{u}}_{\rm out})\Big|_{\Gamma_\alpha}$$

represents the jump state of the transmission along Γ_α. Note that (3.11) holds for every $\widehat{\mathbf{v}} \in \mathbf{V}_0(\widehat{\Omega})$. Hence, if we take $\mathbf{v} \in \mathbf{V}_0(\Omega_\alpha^c)$ and then extend it over $\widehat{\Omega}$ by zero, then its extension $\hat{\mathbf{v}}$ belongs to $\mathbf{V}_0(\widehat{\Omega})$ by Lemma 3.1, and from (3.11) it follows that

$$\int_{\Omega_\alpha^c} -\nu\Delta\widehat{\mathbf{u}}\cdot\widehat{\mathbf{v}}\,dx = 0\,.$$

Hence, according to de Rham's Lemma, there exists $p_{\text{out}} \in L^2(\Omega_\alpha^c)$ such that

$$-\nu\Delta\widehat{\mathbf{u}} + \nabla p_{\text{out}} = \mathbf{0} \quad \text{in } \Omega_\alpha^c\,.$$

Likewise, we also have $p_{\text{in}} \in L^2(\Omega_\alpha)$ such that

$$-\nu\Delta\widehat{\mathbf{u}} + \nabla p_{\text{in}} = \mathbf{0} \quad \text{in } \Omega_\alpha\,.$$

Thus, if we choose $\widehat{p} \in L^2(\widehat{\Omega})$ by

$$\widehat{p} = \begin{cases} p_{\text{in}} & \text{in } \Omega_\alpha \\ p_{\text{out}} & \text{in } \Omega_\alpha^c \\ [[p]] & \text{on } \Gamma_\alpha\,, \end{cases} \tag{3.12}$$

(3.11) yields that

$$\begin{aligned} \delta_\alpha(\widehat{\mathbf{v}}) &= \int_{\Omega_\alpha^c}(-\nabla\widehat{p}\cdot\widehat{\mathbf{v}})\,dx + \int_{\Omega_\alpha}(-\nabla\widehat{p}\cdot\widehat{\mathbf{v}})\,dx + \int_{\Gamma_\alpha} 2\nu[[\mathcal{D}(\widehat{\mathbf{u}})]]\mathbf{n}_\alpha\cdot\widehat{\mathbf{v}}\,ds \\ &= \int_{\widehat{\Omega}} \widehat{p}\,\nabla\cdot\widehat{\mathbf{v}}\,dx - \int_{\Gamma_\alpha}[[\widehat{p}]]\widehat{\mathbf{v}}\cdot\mathbf{n}_\alpha\,ds + \int_{\Gamma_\alpha} 2\nu[[\mathcal{D}(\widehat{\mathbf{u}})]]\mathbf{n}_\alpha\cdot\widehat{\mathbf{v}}\,ds \\ &= \int_{\Gamma_\alpha}[[-\widehat{p}I + 2\nu\mathcal{D}(\widehat{\mathbf{u}})]]\mathbf{n}_\alpha\cdot\widehat{\mathbf{v}}\,ds \end{aligned}$$

Since this holds for all $\widehat{\mathbf{v}} \in \mathbf{V}_0(\widehat{\Omega})$, it follows that

$$\delta_\alpha = [[-\widehat{p}\,\mathbf{n}_\alpha + 2\nu\mathcal{D}(\widehat{\mathbf{u}})\mathbf{n}_\alpha]] = [[\mathbf{t}(\widehat{\mathbf{u}},\widehat{p})]]\,,$$

and it completes the proof. □

The following shows the minimizer of the functional $\mathcal{J}$ in (3.7) represents an exact free boundary along with a given stress distribution.

THEOREM 3.4 *Let $\alpha \in \mathcal{U}_{ad}$ be given and $\widehat{\mathbf{u}} \in \mathbf{V}(\widehat{(\Omega)})$ be a solution of (3.6) such that $\widehat{\mathbf{u}}\Big|_{\Gamma_\alpha} = \mathbf{0}$. Then, there exists $\widehat{p} \in L^2(\widehat{\Omega})$ such that $(\widehat{\mathbf{u}},\widehat{p})\Big|_{\Omega_\alpha}$ is the solution of (3.3) and (2.5) over the domain Ω_α.*

Proof: By Theorem 3.3, we have $\widehat{p} \in L^2(\widehat{\Omega})$ such that

$$\delta_\alpha(\widehat{\mathbf{v}}) = <\boldsymbol{\phi},\widehat{\mathbf{v}}>_{\Gamma_\alpha} = \int_{\Gamma_\alpha}[[\mathbf{t}(\widehat{\mathbf{u}},\widehat{p})]]\cdot\widehat{\mathbf{v}}\,ds\,, \quad \forall\widehat{\mathbf{v}} \in \mathbf{V}_0(\widehat{\Omega})\,. \tag{3.13}$$

Since $\widehat{\mathbf{u}}\Big|_{\Gamma_\alpha} = \mathbf{0}$, $\widehat{\mathbf{u}}\Big|_{\Omega_\alpha^c} = \mathbf{0}$. By the choice of $\widehat{p}$ in (3.12), we also have $\widehat{p}\Big|_{\Omega_\alpha^c} = 0$, so that $[[\mathbf{t}(\widehat{\mathbf{u}}, \widehat{p})]] = \mathbf{t}(\widehat{\mathbf{u}}\Big|_{\Omega_\alpha}, \widehat{p}\Big|_{\Omega_\alpha})$. Now taking $\widehat{\mathbf{v}}$ in the space of the restriction of $\mathbf{V}_0(\widehat{\Omega})$ to Ω_α, it is followed that

$$\mathbf{t}(\widehat{\mathbf{u}}\Big|_{\Omega_\alpha}, \widehat{p}\Big|_{\Omega_\alpha}) = \boldsymbol{\phi} \quad \text{along } \Gamma_\alpha\,,$$

which takes the desired traction field along the variable boundary. □

References

[1] D. M. Bedivan, *A boundary function for optimal shape design problems*, Appl. Math. Lett. **9** (1996), no. 1, 9–14.

[2] D. M. Bedivan and G. J. Fix, *An extension theorem for the space H_{div}*, Appl. Math. Lett. **9** (1996), no. 5, 17–20.

[3] D. Begis and R. Glowinski, *Application de la méthode des éléments finis à l'approximation d'un problème de domaine optimale. Méthodes de résolution des problèmes approchés*, Appl. Math. Optim. **2** (1974), 130–169.

[4] C. Börgers, *Domain imbedding methods for the Stokes equations*, Numer. Math. **57** (1990), 435–451.

[5] M. Flusher and M. Rumpf, *Bernoulli's free boundary prolem, Qualitative theory and numerical approximations*, J. Reine Angew Math. **486** (1997), 165–204.

[6] V. Girault and P. A. Raviart, *Finite elements methods for Navier-Stokes equations*, Springer-Verlag, Berlin, 1986.

[7] M. Gunzburger and H. Kim, *Existence of a shape control problem for the stationary Navier-Stokes equations*, SIAM J. Control Optim. **36** (1998), 895–909.

[8] M. Gunzburger, H. Kim and S. Manservisi, *On a shape control problem for the stationary Navier-Stokes equations*, M2AN Math. Modelling and Numer. Anal. **34** (2000), 1233–1258.

[9] J. Haslinger, T. Kozubek, K. Kunisch and G. Peichl, *Shape optimization and fictitious domain approach for solving free boundary problems of Bernoulli type*, Comp. Optim. Appl. **26** (2003), 231–251.

[10] O. Ladyzhenskaya, *The mathematical theory of incompressible viscous flows*, Gordon and Breach, New York, 1963.

[11] O. Pironneau, *Optimal shape design for elliptic systems*, Springer-Verlag, Berlin, 1984.

[12] J. Simon, *Démonstration constructive d'un théorème de G. de Rham*, C. R. Acad. Sci. Paris **t. 316**, Série I (1993), 1167–1172.

[13] T. Slawig, *An explicit formula for the derivative of a class of cost functionals with respect to domain variations in Stokes flow*, SIAM J. Control. Optim. **39** (2000), 141–158.

[14] T. Slawig, *A formula for the derivative with respect to domain variations in Navier-Stokes flow based on an embedding domain method*, SIAM J. Control. Optim. **42** (2003), 495–512.
[15] S. Stojanivic and T. Svobodny, *Free boundary problems for potential and Stokes flows via nonsmooth analysis*, SIAM J. Math. Anal. **26** (1995), 633–658.
[16] D. Tiba, *A property of Sobolev spaces and existence in optimal design*, Appl. Math. Optim. **47** (2003), 45–58.

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