Korean J. Math. **16** (2008), No. 1, pp. 85–90

A FOURIER MULTIPLIER THEOREM ON THE BESOV-LIPSCHITZ SPACES

Yong-Kum Cho* and Dohie Kim

Abstract. We consider Fourier multiplier operators whose symbols satisfy a generalization of Hörmander's condition and establish their Sobolev-type mapping properties on the homogeneous Besov-Lipschitz spaces by making use of a continuous characterization of Besov-Lipschitz spaces. As an application, we derive Sobolev-type imbedding theorem.

1. Introduction

The purpose of this paper is to study Fourier multiplier operators whose symbols satisfy a generalization of Hörmander's condition on the homogeneous Besov-Lipschitz spaces. To be more specific, we shall deal with the operators T_α, defined as $(T_\alpha f)\hat{}= m_\alpha \hat{f}$, where the symbols m_α satisfy the following condition:

Given a positive integer ℓ, $m_\alpha \in C^\ell(\mathbb{R}^n \setminus \{0\})$ and

$$\sup_{R>0}\left[R^{-n+2\alpha+2|\sigma|}\int_{R<|\xi|<2R}\left|\partial_\xi^\sigma m_\alpha(\xi)\right|^2 d\xi\right] \le A_\sigma \quad (|\sigma|\le \ell)\,. \tag{1.1}$$

When $\alpha = 0$, it is known as the Hörmander condition (see [4], [8]). Typical examples are given by the symbols of singular integrals R_j. When $\alpha \neq 0$, a typical example is given by $m_\alpha(\xi) = |\xi|^{-\alpha}$, the symbol

Received January 22 , 2008.
2000 Mathematics Subject Classification: Primary 42B15, 42B35, 46E35.
Key words and phrases: Fourier multiplier, Sobolev imbedding, Besov-Lipschitz spaces.
*Corresponding author.

of the Riesz potential I_α which satisfies the condition (1.1) for every positive integer ℓ. Another example is the symbol of a differential operator ∂^σ of order $|\sigma| = \alpha$ when $\alpha > 0$.

Let $\widehat{O}$ denote the class of Schwartz function φ on $\mathbb{R}^n$ such that its Fourier transform $\widehat{\varphi}$ has support in $\{1/2 \le |\xi| \le 2\}$ and $|\widehat{\varphi}(\xi)| \ge c > 0$ for $3/5 \le |\xi| \le 5/3$. Given any $\varphi \in \widehat{O}$, we recall ([3], [7], [9]) that the homogeneous Besov-Lipschitz spaces $\dot{B}^\alpha_{p,r}$ are the spaces of tempered distribution f on $\mathbb{R}^n$, modulo polynomials, such that the quasi-norms

$$(1.2)\quad \|f\|_{\dot{B}^\alpha_{p,r}} = \left(\sum_{j\in\mathbb{Z}} \left(2^{j\alpha}\,\|f * \varphi_{2^{-j}}\|_p\right)^r\right)^{1/r} \qquad (\alpha \in \mathbb{R}\,,\ 0 < p,\, r \le \infty)$$

are finite.

Our principal result reads as

Theorem 1.1. *Given $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $0 < r \le \infty$, let β be any real with $\beta < \alpha$ and let p_* be determined by*

$$\beta - n/p_* = \alpha - n/p \qquad (0 < p_* \le \infty)\,.$$

If m_α satisfies the condition (1.1) with $\ell > n(1/p + 1/2)$, then

$$(1.3)\qquad \|T_\alpha f\|_{\dot{B}^\beta_{p_*,r}} \le C\,\|f\|_{\dot{B}^0_{p,r}}\,.$$

As an application, upon taking $m_\alpha(\xi) = |\xi|^{-\alpha}$ and using the fact $\dot{B}^\alpha_{p,r} = I_\alpha(\dot{B}^0_{p,r})$, we obtain the Sobolev imbedding result (see [2], [5], [6])

Corollary 1.1. *Given reals $\alpha > \beta$ and $0 < p < \infty$, $0 < r \le \infty$, let $0 < p_* \le \infty$ be determined from $\beta - n/p_* = \alpha - n/p$. Then*

$$\dot{B}^\alpha_{p,r} \hookrightarrow \dot{B}^\beta_{p_*,r}\,.$$

In what follows, the letter C will denote a positive constant which may differ in each occurrence and may depend on the parameters but not on the variable quantities involved. As usual, the Fourier transform of an integrable function ϕ on $\mathbb{R}^n$ will be defined as

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\,\phi(x)\,dx \quad (\xi \in \mathbb{R}^n)\,.$$

2. Preliminaries

The main ingredient in proving Theorem 1.1 is a continuous characterization of Besov-Lipschitz spaces. Established by Bui et. el. [1], it is given as

$$\|f\|_{\dot{B}^{\alpha}_{p,r}} \approx \left(\int_0^\infty \left(t^{-\alpha}\|u^*_\lambda(\cdot,t)\|_p\right)^r \frac{dt}{t}\right)^{1/r} \qquad (\lambda > n/p)\,, \tag{2.1}$$

where $u(x,t) = (f * \varphi_t)(x)$, $\varphi \in \widehat{O}$ and u^*_λ denotes the Peetre maximal function of u defined by

$$u^*_\lambda(x,t) = \sup_{y\in\mathbb{R}^n} |u(y,t)| \left(1 + \frac{|y-x|}{t}\right)^{-\lambda} .$$

We now set up a few basic estimates that will be used later on. Let K_α denote the distribution whose Fourier transform is m_α.

LEMMA 2.1. *Let ψ, ζ be Schwartz functions on $\mathbb{R}^n$ such that $\widehat{\psi}$, $\widehat{\zeta}$ have compact support away from the origin. Assume that m_α satisfies (1.1). If $\lambda > 0$ and $\ell > \lambda + n/2$, then for $t > 0$,*

$$\int_{\mathbb{R}^n} \left(1 + \frac{|z|}{t}\right)^{\lambda} |(K_\alpha * \psi_t)(z)|\, dz \,\le\, C\, t^{\alpha} .$$

Proof. Dilating the functions $\widehat{\psi}$, $\widehat{\zeta}$ appropriately, we may assume both have support in $\{1/2 \le |\xi| \le 2\}$. We choose μ so that $\mu > n/2$ and $\lambda + \mu \le \ell$. By the Cauchy-Schwartz inequality,

$$\begin{aligned}
&\left[\int_{\mathbb{R}^n} \left(1 + \frac{|z|}{t}\right)^{\lambda} |(K_\alpha * \psi_t)(z)|\, dz\right]^2 \\
&\quad\le \int_{\mathbb{R}^n} \left(1 + \frac{|z|}{t}\right)^{-2\mu} dz \int_{\mathbb{R}^n} \left(1 + \frac{|z|}{t}\right)^{2(\lambda+\mu)} |(K_\alpha * \psi_t)(z)|^2\, dz \\
&\quad\le C\, t^n \int_{\mathbb{R}^n} \left(1 + \frac{|z|}{t}\right)^{2\ell} |(K_\alpha * \psi_t)(z)|^2\, dz \\
&\quad= C\, t^{2n} \int_{\mathbb{R}^n} (1+|z|)^{2\ell} |(K_\alpha * \psi_t)(tz)|^2\, dz\,. \qquad (2.2)
\end{aligned}$$

Applying the binomial theorem and the Plancherel theorem, the integral in (2.2) is easily seen to be bounded by

$$\begin{aligned} t^{-2n} \sum_{|\sigma|\le \ell} C_\sigma \int_{\mathbb{R}^n} \left| \partial_\xi^\sigma \left[m_\alpha\left(\frac{\xi}{t}\right) \widehat{\psi}(\xi)\right]\right|^2 d\xi \\ \le C\, t^{-2n} \sum_{|\sigma|\le \ell} t^{n-2|\sigma|} \int_{1/t\le|\xi|\le 2/t} \left|\left(\partial_\xi^\sigma m_\alpha\right)(\xi)\right|^2 d\xi \\ \le\; C\, t^{-2n+2\alpha}, \end{aligned}$$

where the last inequality is due to the hypothesis (1.1) on m_α. Inserting this estimate into (2.2), we obtain the desired estimate. □

Lemma 2.2. *Given $\alpha\in\mathbb{R}$ and a positive integer ℓ, assume that m_α satisfies the condition (1.1). Let $\varphi,\psi\in\widehat{O}$. For a tempered distribution f on $\mathbb{R}^n$, set $u(x,t)=(f*\varphi_t)(x)$. If $\ell>\lambda+n/2$ and $\Phi=\varphi*\psi$, then for all $x,y\in\mathbb{R}^n$, $t>0$,*

$$\left|\left(T_\alpha f*\Phi_t\right)(y)\right| \;\le\; C\, t^\alpha \left(1+\frac{|y-x|}{t}\right)^\lambda u_\lambda^*(x,t) \quad (\lambda>0)\,.$$

Proof. In view of the representation

$$\left(T_\alpha f*\Phi_t\right)(y)=\int_{\mathbb{R}^n} u(y-z)(K_\alpha*\psi_t)(z)\,dz\,,$$

the estimate is an immediate consequence of Lemma 2.1. □

3. Proof of Theorem 1.1

Proof. Choose $\varphi,\,\psi\in\widehat{O}$. Let $\Phi=\varphi*\psi$ and

$$u(x,t)=(f*\varphi_t)(x)\,,\quad U(x,t)=(T_\alpha f*\Phi_t)(x)\,.$$

It follows easily from the estimate of Lemma 2.2 that

$$U_\lambda^*(x,t)\;\le\; C\,t^\alpha\, u_\lambda^*(x,t)\,,\quad U_\lambda^*(x,t)\;\le\; C\,t^{\alpha-n/p}\,\left\|u_\lambda^*(\cdot,t)\right\|_p\,. \tag{3.1}$$

Fix $t > 0$. Normalizing if necessary, we may assume $C = 1$ in both estimates of (3.1). With $A = \|u_\lambda^*(\cdot, t)\|_p$, it follows that for any $q > p$,

$$\begin{aligned}\int_{\mathbb{R}^n} [U_\lambda^*(x,t)]^q \, dx &= q \int_0^{A\,t^{\alpha-n/p}} \left|\left\{U_\lambda^*(\cdot,t) > s\right\}\right| \, s^{q-1}\, ds \\ &\le C\, t^{\alpha q} \int_0^{A\,t^{-n/p}} \left|\left\{u_\lambda^*(\cdot,t) > s\right\}\right| \, s^{q-1}\, ds \\ &\le C\, t^{\alpha q}\, \|u_\lambda^*(\cdot,t)\|_p^p \int_0^{A\,t^{-n/p}} s^{q-p-1}\, ds \\ &= C\, t^{q(\alpha-n/p+n/q)}\, \|u_\lambda^*(\cdot,t)\|_p^q ,\end{aligned}$$

where the second inequality follows from Chebychev's inequality. Thus

$$t^{-(\alpha-n/p+n/q)}\, \|U_\lambda^*(\cdot,t)\|_q \le C\, \|u_\lambda^*(\cdot,t)\|_p . \tag{3.2}$$

Upon setting $\beta = \alpha - n/p + n/q$, $q = p_*$, (3.2) gives

$$\int_0^\infty \left(t^{-\beta}\, \|U_\lambda^*(\cdot,t)\|_{p_*}\right)^r \frac{dt}{t} \le \int_0^\infty \left(\|u_\lambda^*(\cdot,t)\|_p\right)^r \frac{dt}{t} ,$$

which yields the desired result in view of (2.1). □

References

[1] H-Q. Bui, M. Paluszyński and M. Taibleson, *A maximal function characterization of weighted Besov-Lipschitz and Triebel-Lizorkin spaces*, Studia Math. 119, pp. 219–246 (1996)

[2] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distributions*, J. Funct. Anal. 93, pp. 34–170 (1990)

[3] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley theory and the study of function spaces*, CBMS 79, Amer. Math. Soc., 1992

[4] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, NJ, 2004

[5] B. Jawerth, *Some observation on Besov and Lizorkin-Triebel spaces*, Math. Scand. 40, pp. 94–104 (1977)

[6] J. Johnsen and W. Sickel, *A direct proof of Sobolev embeddings for quasi-homogeneous Lizorkin-Triebel spaces with mixed norms*, J. Funct. Spaces Appl. 5, pp. 183–198 (2007)

[7] J. Peetre, *On spaces of Triebel-Lizorkin type*, Ark. Mat. 13, pp. 123–130 (1975)

[8] E. M. Stein, *Harmonic Analysis : Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993

[9] H. Triebel, *Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation*, Ark. Mat. 11, pp. 13–64 (1973)

Department of Mathematics
College of Natural Science
Chung-Ang University
Seoul 156-756, Korea
E-mail: ykcho@cau.ac.kr