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NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN A HILBERT SPACE

RakJoong Kim

Abstract. Let H be a Hilbert space. Assume that $0 \le \alpha, \beta \le 1$ and $r(t) > 0$ in $I = [0, T]$. By means of the fixed point theorem of Leray-Schauder type the existence principles of solutions for two point boundary value problems of the form

$$\big(r(t)x'(t)\big)' + f(t, x(t), r(t)x'(t)) = 0, \quad t \in I$$
$$x(0) = x(T) = 0$$

are established where f satisfies for positive constants a, b and c

$$|f(t, x, y)| \le a|x|^{\alpha} + b|y|^{\beta} + c \quad \text{for all } (t, x, y) \in I \times H \times H.$$

1. Introduction

In this paper, we are concerned with the Dirichlet boundary value problems of the type:

$$\big(r(t)x'(t)\big)' + f(t, x(t), r(t)x'(t)) = 0, \quad t \in I = [0, T] \tag{1}$$
$$x(0) = x(T) = 0 \tag{2}$$

where $r(t) \in C(I,\ (0, \infty))$, $T > 0$ is constant, H is a Hilbert space and $f : I \times H \times H \to H$.

We will use the following notation throughout this paper: $|x|_0 = \sup_{t \in I}\ |x(t)|$ for $x \in C(I, H)$, $|x|_1 = \max_{t \in I}\ \{|x|_0,\ |x'|_0)\}$ for $x \in C^1(I, H)$ where $|\cdot|$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle$

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and $||x||_{L^2}^2 = \int_0^T \langle x(t), x(t)\rangle\, dt$. By a solution of (1), (2), we define $x \in C^1(I, H)$ satisfying (1), (2).

The differential equation $x''(t)+kx'(t)+f(t,x(t),x'(t)) = 0$, $x(0) = x(\pi) = 0$ was studied by Mawhin[3] for the case $H = R^n$, where references to the corresponding literature are also given. Also Mawhin[4] and Hai[2] dealt with the same problem for the case that H is a Hilbert space. The purpose of this paper is to establish some existence results and uniqueness of differential equation (1)-(2) which extend their results.

Definition 1. *A function $f : I \times H \times H \to H$ is called a L^1-Carathéodory function if the following conditions are valid:*

(i) *$t \to f(t, x, y)$ is measurable for each $(x, y) \in H \times H$,*
(ii) *$(x, y) \to f(t, x, y)$ is continuous for a. e. $t \in I$,*
(iii) *for any $\gamma > 0$ there exists $h_\gamma \in L^1(I, \mathbb{R})$ such that*

$$|f(t, x, y)| \le h_\gamma(t) \quad \text{a. e.} \quad t \in I,$$

and for all $x,\ y$ with $\max\{|x|_0,\ |y|_0\} \le \gamma$.

Hereafter we assume that the function f is a L^1-Carathéodory function . Our existence principles will be proved by means of the following fixed point theorem[4] of Leray-Schauder type.

Proposition. [*Nonlinear Alternative*] *Assume Ω is a relatively open subset of a convex set C in a Banach Space E. Let $F : \overline{\Omega} \to C$ be a compact map with $p^* \in \Omega$. Then either*

A1. *F has a fixed point in $\overline{\Omega}$, or*
A2. *there exist a $u \in \partial\Omega$ and a $\lambda \in (0, 1)$ such that $u = (1-\lambda)p^* + \lambda F u$.*

2. Existence Principles

Lemma 1[1]. *Let $X,\ Y$ be positive constants and $\sigma \ge 0$. Then the inequality*

$$(\sigma + 1)XY^\sigma \le X^{\sigma+1} + \sigma Y^{\sigma+1}$$

is valid where the equality holds if and only if $X = Y$.

Put $A = \max_{t\in[0,T]} r(t)$, $B = \min_{t\in[0,T]} r(t)$ and denote I_h by $I_h = \int_0^T h(s)\,ds$ for a integrable function h on I.

THEOREM 2. *Suppose that there exists a constant $M > 0$, independent of λ, with*

$$|x|_1 \lneqq M$$

for any solution of

$$(3_\lambda) \qquad \begin{gathered}\big(r(t)x'(t)\big)' + \lambda f(t, x(t), r(t)x'(t)) = 0,\\ x(0) = x(T) = 0\end{gathered}$$

for each $\lambda \in (0,1)$ and $t \in I = [0,T]$. Then the differential equation (1) *satisfying* (2) *has at least one solution in $C^1(I,\ H)$.*

Proof. We find a priori bounds of solutions of (3_λ). To solve (3_λ) is equivalent to find $x(t) \in C^1(I,\ H)$ such that for $\lambda \in (0,1)$

$$(4) \qquad x(t) = \lambda F x(t),$$

where

$$(5) \qquad \begin{gathered}Fx(t) = \int_0^t \frac{1}{r(s)} \Big[C + \int_s^T f(u, x(u), r(u)x'(u))\ du\Big]\, ds,\\ C = -I_{1/r}^{-1} \int_0^T \frac{1}{r(s)} \int_s^T f(u, x(u), r(u)x'(u))\ du\, ds.\end{gathered}$$

By standard argument we can show that $F : C^1(I,\ H) \to C^1(I,\ H)$ is completely continuous. Assume that there exists a constant $M > 0$, independent of λ, such that

$$|x|_1 \lneqq M$$

is valid for any solution of (3_λ), $\lambda \in (0,1)$. Choose then

$$(6) \qquad \Omega = \{x \in C^1(I, H) : |x|_1 < M\}.$$

We apply Proposition with $p^* = 0$. Then A2 of Proposition cannot be occurred. Therefore F has a fixed point $x \in C^1(I,H)$ in $\overline{\Omega}$ by A1. $\square$

THEOREM 3. *Assume that there exist nonnegative real numbers* $a,\ b,\ c$ *such that for all* $(t,x,y)\in I\times H\times H$

$$|f(t,x,y)|\le a|x|^{\alpha}+b|y|^{\beta}+c \tag{7}$$

where $0\le\alpha,\ \beta<1$. *Then the differential equation* (1) *satisfying* (2) *has at least one solution in* $C^1(I,\ H)$.

Proof. Assume that $0\le\alpha,\ \beta<1$. We find a suitable bounded open set $\Omega\subseteq C^1(I,\ H)$ such that all solutions of (3_λ) belong to Ω but for any $\lambda\in(0,1)$ λF has no fixed point in $\partial\Omega$. Note that the equation (4) is equivalent to $\big(r(t)x'(t)\big)'+\lambda f(t,x(t),r(t)x'(t))=0$. Consider the inner product (3_λ) with $x(t)$.

$$\langle\big(r(t)x'(t)\big)',x(t)\rangle+\langle\lambda f(t,x(t),r(t)x'(t)),x(t)\rangle=0. \tag{8}$$

From this we can immediately deduce

$$\langle x'(t),x(t)\rangle=\frac{1}{r(t)}\int_t^T\{\langle\lambda f(u,x(u),r(u)x'(u)),x(u)\rangle-r(u)|x'(u)|^2\}\,du,$$

from which we get

$$|x(t)|^2\le \int_0^t\frac{2}{r(s)}\int_s^T\left\{\langle\lambda f(u,x(u),r(u)x'(u)),x(u)\rangle-\frac{1}{A}\,|r(u)x'(u)|^2\right\}du\,ds. \tag{9}$$

So it follows that for all $\lambda\in(0,\ 1)$

$$\Big|\langle\lambda f(u,x(u),r(u)x'(u)),x(u)\rangle\Big| \le a|x(u)|^{1+\alpha}+b|x(u)||r(u)x'(u)|^{\beta}+c|x(u)|.$$

Applying Lemma 1 with

$$\sigma=\frac{\beta}{2-\beta},\ X=\frac{b}{\sigma+1}\,(\sigma A)^{\sigma/(\sigma+1)}\,|x(u)|,\ Y=\left(\frac{|r(u)x'(u)|^2}{\sigma A}\right)^{1/(\sigma+1)}$$

to $b|x(u)||r(u)x'(u)|^\beta$ we then obtain for all $\lambda \in (0,\, 1)$

$$\begin{aligned}&\langle \lambda f(u, x(u), r(u)x'(u)), x(u)\rangle \\ &\qquad \le a|x(u)|^{1+\alpha} + C|x(u)|^{\frac{2}{2-\beta}} + c|x(u)| + \frac{1}{A}|r(u)x'(u)|^2\end{aligned}$$

where $C = \frac{2-\beta}{2}\left(\frac{A\beta}{2}\right)^{\beta/(2-\beta)} b^{2/(2-\beta)}$. Thus (9) is reduced to

$$|x(t)|^2 \le 2T\, I_{1/r}\left\{a|x|_0^{1+\alpha} + C|x|_0^{\frac{2}{2-\beta}} + c|x|_0\right\}, \quad t \in I.$$

So there exists a $R > 0$ such that

$$|x|_0 \le R. \tag{10}$$

From the fact that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|r(t)x'(t)\right|^2 &= \langle (r(t)x'(t))',\, r(t)x'(t)\rangle \\ &= -\langle \lambda f(t, x(t), r(t)x'(t)), r(t)x'(t)\rangle\end{aligned}$$

it is clear that by means of (10)

$$\begin{aligned}\left|\frac{1}{2}\frac{d}{dt}\left|r(t)x'(t)\right|^2\right| &= \left|\left|r(t)x'(t)\right|\frac{d}{dt}\left|r(t)x'(t)\right|\right| \\ &\le \left(aR^\alpha + b\,|r(t)x'(t)|^\beta + c\right)|r(t)x'(t)|.\end{aligned}$$

Dividing both sides by $\left(aR^\alpha + b\,|r(t)x'(t)|^\beta + c\right)|r(t)x'(t)|$ we obtain

$$\left|\frac{d}{dt}\int_0^{\left|r(t)x'(t)\right|}\frac{du}{aR^\alpha + bu^\beta + c}\right| \le 1.$$

By means of the condition (2) there exists $t_0 \in (0,\, T)$ such that $x'(t_0) = 0$. Integrating $\frac{d}{dt}\int_0^{\left|r(t)x'(t)\right|}\frac{du}{aR^\alpha + bu^\beta + c}$ over $[t_0,\, t]$ for $0 \le t_0 < t \le T$ we have

$$\int_0^{\left|r(t)x'(t)\right|}\frac{du}{aR^\alpha + bu^\beta + c} < T. \tag{11}$$

On the other hand, since

$$\int_0^\infty \frac{du}{aR^\alpha + bu^\beta + c} = \infty$$

there exists a $R_1 > 0$ such that

$$T \le \int_0^{R_1} \frac{du}{aR^\alpha + bu^\beta + c}. \tag{12}$$

From (11) and (12) we obtain $|x'|_0 \le R_1/B$. Put $M = \max\{R,\ R_1/B\}$. Then the inequality $|x|_1 \le M$ is valid for each solution of (3_λ), $\lambda \in (0,\ 1)$ satisfying (2). If therefore we take

$$\Omega = B_{M+1}(0) = \{x \in C^1(I, H) : |x|_1 < M + 1\}$$

our theorem is proved by Theorem 2. □

Remark. Even though $a = c = 0$ Theorem 3 is valid.

Theorem 4. *Assume that there exist nonnegative real numbers* $a,\ b,\ c$ *such that for all* $(t, x, y) \in I \times H \times H$

$$|f(t, x, y)| \le a|x| + b|y| + c, \tag{13}$$

$$T(4a + b^2 A)I_{1/r} < 2. \tag{14}$$

are valid. Then the differential equation (1) *satisfying* (2) *has at least one solution in* $C^1(I,\ H)$.

Proof. The most part of proof is similar to that of Theorem 3. We sketch briefly the process of proof. It is obvious that for all $\lambda \in (0, 1)$

(15)

$$\langle \lambda f(u, x(u), r(u)x'(u)), x(u)\rangle \le \left(a + \frac{b^2 A}{4}\right)|x(u)|^2 + c|x(u)| + \frac{1}{A}|r(u)x'(u)|^2.$$

Thus from (9) we have

$$|x(t)|^2 \le 2T\, I_{1/r} \left\{\left(a + \frac{b^2 A}{4}\right)|x|_0^2 + c|x|_0\right\}, \quad t \in I \tag{16}$$

for all $\lambda \in (0,1)$. So we obtain

$$|x|_0 \le R \tag{17}$$

where

$$R = \frac{4cTI_{1/r}}{2 - T(4a + b^2 A)I_{1/r}}. \tag{18}$$

Note that $R > 0$ by (14). The rest part of proof is the same as that of Theorem 3. □

REMARK. In the case of $|f(t,x,y)| \le a|x|^\alpha + b|y|^\beta + c$, where $0 \le \alpha,\ \beta \le 1$, it is not difficult to show that by Theorem 3 and Theorem 4 the existence property of (1)-(2) is also valid under suitable conditions.

DEFINITION 2. *A function $p : I \times H \to [0,\infty)$ is called a L^1-Carathéodory function such that*

- (i') $t \to p(t,x)$ *is measurable for each* $x \in H$,
- (ii') $x \to p(t,x)$ *is continuous for a. e.* $t \in I$,
- (iii') *for any* $\gamma > 0$ *there exists* $h_\gamma \in L^1(I,\mathbb{R})$ *such that* $|x|_0 \le \gamma$ *implies* $|p(t,x)| \le h_\gamma(t)$ *a. e.* $t \in I$.

THEOREM 5. *Assume that $p : I \times H \to [0,\infty)$ is called a L^1-Carathéodory function and that for all $(t,x,y) \in I \times H \times H$*

(i) *there exist nonnegative real numbers $a,\ b,\ c$ such that*

$$|\langle x, f(t,x,y)\rangle| \le a|x|^2 + b|x||y| + c|x|, \tag{19}$$

and (14) are valid.

(ii) *there exist a continuous function $g : [0,\infty) \to (0,\infty)$ and positive numbers R, R_1 such that*

$$|\langle y,\ f(t,x,y)\rangle| \le p(t,x)g(|y|), \text{ for a. e. } t \in I \text{ and all } y \in H, \tag{20}$$

$$\int_{\sqrt{R_1}}^{\infty} \frac{u}{g(u)}\,du = \infty, \tag{21}$$

where

$$R_1 = 2A\left\{\left(a + \frac{b^2 A}{2}\right) R^2 + c\,R\right\} \tag{22}$$

and R is a number given by (18).

Then the differential equation (1) *satisfying* (2) *has at least one solution in* $C^1(I,\ H)$.

Proof. We sketch briefly the process of proof. It follows that for all $\lambda \in (0,1)$ there exists a $R > 0$ satisfying (17). Here R is a constant number (18). Multiplying (3_λ) by $-x(t)$ and integrating over I we have

$$\begin{aligned}\frac{1}{A}\,||rx'||^2_{L^2} &\le \int_0^T r(t)|x'(t)|^2\,dt \\ &\le \int_0^T \big|\langle f(u,x(u),r(u)x'(u)),x(u)\rangle\big|\,du \\ &\le\ aT|x|_0^2 + cT|x|_0 + b\sqrt{T}|x|_0\,||rx'||_{L^2}.\end{aligned}$$

Since

$$b\sqrt{T}|x|_0\,||rx'||_{L^2} \le \frac{b^2AT}{2}|x|_0^2 + \frac{1}{2A}||rx'||^2_{L^2}$$

the inequality $||rx'||^2_{L^2} \le TR_1$ is valid. There exists a $\xi \in [0,T]$ such that $|r(\xi)x'(\xi)| \le \sqrt{R_1}$. It is clear that by (20)

$$\begin{aligned}\left|\frac{1}{2}\,\frac{d}{dt}\big|r(t)x'(t)\big|^2\right| &= \left|\big|r(t)x'(t)\big|\,\frac{d}{dt}\big|r(t)x'(t)\big|\right| \\ &\le p(t,x)g(|r(t)x'(t)|).\end{aligned}$$

Dividing both sides by $g(|r(t)x'(t)|)$ we obtain

$$\left|\frac{d}{dt}\int_0^{|r(t)x'(t)|}\frac{u}{g(u)}\,du\right| \le p(t,x).$$

By (iii') there exists a $h_R \in L^1(I)$ satisfying $|p(t,x)| \le h_R(t)$ for all I. From (21) it follows that there exists a real number $R_2 > 0$ such that

$$\int_0^T h_R(u)\,du = \int_{\sqrt{R_1}}^{R_2}\frac{u}{g(u)}\,du.$$

Therefore we have for $0 \le \xi < t \le T$

$$\begin{aligned}\int_0^{|r(t)x'(t)|}\frac{u}{g(u)}\,du &\le \int_0^{\sqrt{R_1}}\frac{u}{g(u)}\,du + \int_0^t h_R(u)\,du \\ &\le \int_0^{R_2}\frac{u}{g(u)}\,du.\end{aligned}$$

By means of the proof of Theorem 3 our theorem is proved. □

REMARK. In Theorem 5 even if the condition (20) and (21) are replaced with

(20')
$$|\langle y,\ f(t,x,y)\rangle| \le p(t,x)g(|y|^2),\ \text{for a. e. } t \in I \text{ and all } x,\ y \in H,$$

$$\int_{\sqrt{R_1}}^{\infty} \frac{du}{g(u)} = \infty, \tag{21'}$$

we get the same result.

THEOREM 6. *Assume that there exist positive numbers a, b such that for all $t \in I$*

$$|f(t,x,y) - f(t,u,v)| \le a|x-u| + b|y-v| \tag{23}$$

$$I_{1/r} < \frac{2B}{T(4aB + b^2A^2)} \tag{24}$$

for all $x, y, u, v \in H$. Then the differential equation (1) satisfying

$$x(0) = x_0,\ x(T) = x_T \text{ for } x_0,\ x_T \in \mathbb{R} \tag{25}$$

has at most one solution.

Proof. Assume $x(t)$, $u(t)$ are solutions of (1) satisfying (25). If we put $w(t) = x(t) - u(t)$ we obtain

$$\begin{gathered}\big(r(t)w'(t)\big)' + f(t,x(t),r(t)x'(t)) - f(t,u(t),r(t)u'(t)) = 0,\\ w(0) = w(T) = 0\end{gathered} \tag{26}$$

Consider the inner product of (26) with $w(t)$:

$$\langle \big(r(t)w'(t)\big)', w(t)\rangle + \langle f(t,x(t),r(t)x'(t)) - f(t,u(t),r(t)u'(t)), w(t)\rangle = 0.$$

From (17) it follows that

$$\begin{aligned}|w(t)|^2 \le \int_0^t \frac{2}{r(s)} \int_s^T \big\{&\langle f(\tau,x(\tau),r(\tau)x'(\tau))\\ &- f(\tau,u(\tau),r(\tau)u'(\tau)), w(\tau)\rangle - B\,|w'(\tau)|^2\big\}\,d\tau\,ds.\end{aligned} \tag{27}$$

Thus using (23) and (24) we obtain

$$|w(t)|^2 \leq 2TI_{1/r}\left(a + \frac{b^2A^2}{4B}\right)|w|_0^2.$$

Taking (24) into account we get

$$|w|_0^2 \leq 0$$

which implies $x(t) = u(t)$ for all $t \in I$. □

REMARK. By Theorem 4 and Theorem 5 the differential equation (1)-(2) has a unique solution in $C^1(I, H)$ under the assumptions (14), (15) and (25).

EXAMPLE 1. Let $a(t)$ and $b(t)$ be continuous functions on I. By Theorem 3 the differential equation

$$\begin{aligned}&\left((1+t^{1/3})x'(t)\right)'\\ &\qquad + a(t)\sqrt[3]{x'(t)}\ln\left[1+\{r(t)x'(t)\}^2\right]\sin(t\,x(t)) + b(t)e^{-t^2} = 0,\end{aligned}$$

$$x(0) = x(T) = 0$$

has at least one solution in $C^1(I,\ H)$.

EXAMPLE 2. Let $a(t)$, $b(t)$ and $c(t)$ be continuous functions on I. By Theorem 4 the differential equation

$$\left((1+|\sin t|)\,x'(t)\right)' + a(t)x(t) - b(t)\frac{\{r(t)x'(t)\}^3}{1+|r(t)x'(t)|^2} + c(t)\cos t = 0,$$
$$x(0) = x(T) = 0$$

where $0 \leq \alpha < 1$ has at least one solution in $C^1(I,\ H)$.

REMARK. For the case $\alpha = \beta = 1$ in Theorem 3 and for a completely continuous function f satisfying (27) Mawhin[4] proved the existence result under the assumption $\int_0^\infty \frac{ds}{h(s)+|k|} = \infty$. Hai[2] assumed that $\int_{M/\pi}^K \frac{ds}{h(s)+|k|} \geq 2M$. Then they proved the existence results of the differential equation $(1)-(2)$ with $r(t) = 1$ and $J = [0,\ \pi]$.

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Department of Mathematics
Hallym University,
Chuncheon, Gangwon-Do
200-702, Korea.
E-mail: rjkim@hallym.ac.kr