NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN A HILBERT SPACE

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Abstract. Let $H$ be a Hilbert space. Assume that $0 \leq \alpha, \beta \leq 1$ and $r(t) > 0$ in $I = [0, T]$. By means of the fixed point theorem of Leray-Schauder type the existence principles of solutions for two point boundary value problems of the form

$$(r(t)x'(t))' + f(t, x(t), r(t)x'(t)) = 0, \quad t \in I$$

$$x(0) = x(T) = 0$$

are established where $f$ satisfies for positive constants $a, b$ and $c$

$$|f(t, x, y)| \leq a|x|^{\alpha} + b|y|^{\beta} + c \quad \text{for all } (t, x, y) \in I \times H \times H.$$

1. Introduction

In this paper, we are concerned with the Dirichlet boundary value problems of the type:

$$(r(t)x'(t))' + f(t, x(t), r(t)x'(t)) = 0, \quad t \in I = [0, T]$$

$$x(0) = x(T) = 0$$

where $r(t) \in C(I, (0, \infty))$, $T > 0$ is constant, $H$ is a Hilbert space and $f : I \times H \times H \to H$.

We will use the following notation throughout this paper: $|x|_0 = \sup_{t \in I} |x(t)|$ for $x \in C(I, H)$, $|x|_1 = \max_{t \in I} \{|x|_0, |x'\|^2_{0}\}$ for $x \in C^1(I, H)$ where $|\cdot|$ is the norm induced by the inner product $\langle \cdot, \cdot \rangle$.
and \( \|x\|_{L^2}^2 = \int_0^T \langle x(t), x(t) \rangle \, dt \). By a solution of (1), (2), we define \( x \in C^1(I, H) \) satisfying (1), (2).

The differential equation \( x''(t) + kx'(t) + f(t, x(t), x'(t)) = 0, x(0) = x(\pi) = 0 \) was studied by Mawhin[3] for the case \( H = R^n \), where references to the corresponding literature are also given. Also Mawhin[4] and Hai[2] dealt with the same problem for the case that \( H \) is a Hilbert space. The purpose of this paper is to establish some existence results and uniqueness of differential equation (1)-(2) which extend their results.

**Definition 1.** A function \( f : I \times H \times H \to H \) is called a \( L^1 \)-Carathéodory function if the following conditions are valid:

(i) \( t \to f(t, x, y) \) is measurable for each \( (x, y) \in H \times H \),

(ii) \( (x, y) \to f(t, x, y) \) is continuous for a.e. \( t \in I \),

(iii) for any \( \gamma > 0 \) there exists \( h_\gamma \in L^1(I, R) \) such that

\[
|f(t, x, y)| \leq h_\gamma(t) \quad \text{a.e. } t \in I,
\]

and for all \( x, y \) with \( \max\{|x|_0, |y|_0\} \leq \gamma \).

Hereafter we assume that the function \( f \) is a \( L^1 \)-Carathéodory function. Our existence principles will be proved by means of the following fixed point theorem[4] of Leray-Schauder type.

**Proposition.** [Nonlinear Alternative] Assume \( \Omega \) is a relatively open subset of a convex set \( C \) in a Banach Space \( E \). Let \( F : \overline{\Omega} \to C \) be a compact map with \( p^* \in \Omega \). Then either

A1. \( F \) has a fixed point in \( \overline{\Omega} \), or
A2. there exist a \( u \in \partial \Omega \) and a \( \lambda \in (0, 1) \) such that \( u = (1 - \lambda)p^* + \lambda Fu \).

**2. Existence Principles**

**Lemma 1[1].** Let \( X, Y \) be positive constants and \( \sigma \geq 0 \). Then the inequality

\[
(\sigma + 1)XY^\sigma \leq X^{\sigma + 1} + \sigma Y^{\sigma + 1}
\]
is valid where the equality holds if and only if \( X = Y \).

Put \( A = \max_{t \in [0, T]} r(t) \), \( B = \min_{t \in [0, T]} r(t) \) and denote \( I_h \) by \( I_h = \int_0^T h(s) \, ds \) for a integrable function \( h \) on \( I \).

**Theorem 2.** Suppose that there exists a constant \( M > 0 \), independent of \( \lambda \), with

\[
|x|_1 \leq M
\]

for any solution of

\[
(3_\lambda) \quad (r(t)x'(t))' + \lambda f(t, x(t), r(t)x'(t)) = 0, \\
x(0) = x(T) = 0
\]

for each \( \lambda \in (0, 1) \) and \( t \in I = [0, T] \). Then the differential equation (1) satisfying (2) has at least one solution in \( C^1(I, H) \).

**Proof.** We find a priori bounds of solutions of (3_\lambda). To solve (3_\lambda) is equivalent to find \( x(t) \in C^1(I, H) \) such that for \( \lambda \in (0, 1) \)

\[
(4) \quad x(t) = \lambda Fx(t),
\]

where

\[
(5) \quad Fx(t) = \int_0^t \frac{1}{r(s)} \left[ C + \int_s^T f(u, x(u), r(u)x'(u)) \, du \right] \, ds,
\]

\[
C = -I_1^{-1/r} \int_0^T \frac{1}{r(s)} \int_s^T f(u, x(u), r(u)x'(u)) \, du \, ds.
\]

By standard argument we can show that \( F : C^1(I, H) \to C^1(I, H) \) is completely continuous. Assume that there exists a constant \( M > 0 \), independent of \( \lambda \), such that

\[
|x|_1 \leq M
\]

is valid for any solution of (3_\lambda), \( \lambda \in (0, 1) \). Choose then

\[
(6) \quad \Omega = \{ x \in C^1(I, H) : |x|_1 < M \}.
\]

We apply Proposition with \( p^* = 0 \). Then A2 of Proposition cannot be occurred. Therefore \( F \) has a fixed point \( x \in C^1(I, H) \) in \( \overline{\Omega} \) by A1. \( \square \)
**Theorem 3.** Assume that there exist nonnegative real numbers $a$, $b$, $c$ such that for all $(t, x, y) \in I \times H \times H$

\[(7) \quad |f(t,x,y)| \leq a|x|^\alpha + b|y|^{\beta} + c\]

where $0 \leq \alpha, \beta < 1$. Then the differential equation (1) satisfying (2) has at least one solution in $C^1(I, H)$.

**Proof.** Assume that $0 \leq \alpha, \beta < 1$. We find a suitable bounded open set $\Omega \subseteq C^1(I, H)$ such that all solutions of $(3\lambda)$ belong to $\Omega$ but for any $\lambda \in (0, 1)$ $\lambda F$ has no fixed point in $\partial \Omega$. Note that the equation (4) is equivalent to $(r(t)x'(t))' + \lambda f(t, x(t), r(t)x'(t)) = 0$. Consider the inner product $(3\lambda)$ with $x(t)$.

\[(8) \quad \langle (r(t)x'(t))', x(t) \rangle + \langle \lambda f(t, x(t), r(t)x'(t)), x(t) \rangle = 0.\]

From this we can immediately deduce

\[
\langle x'(t), x(t) \rangle = \frac{1}{r(t)} \int_t^T \{\langle \lambda f(u, x(u), r(u)x'(u)), x(u) \rangle - r(u)|x'(u)|^2 \} \, du,
\]

from which we get

\[(9) \quad |x(t)|^2 \leq \int_0^t \frac{2}{r(s)} \int_s^T \{\langle \lambda f(u, x(u), r(u)x'(u)), x(u) \rangle - \frac{1}{A} |r(u)x'(u)|^2 \} \, du \, ds.\]

So it follows that for all $\lambda \in (0, 1)$

\[
|\langle \lambda f(u, x(u), r(u)x'(u)), x(u) \rangle| \leq a|x(u)|^{1+\alpha} + b|x(u)||r(u)x'(u)|^{\beta} + c|x(u)|.
\]

Applying Lemma 1 with

\[
\sigma = \frac{\beta}{2 - \beta}, \quad X = \frac{b}{\sigma + 1} (\sigma A)^{\sigma/(\sigma+1)} |x(u)|, \quad Y = \left(\frac{|r(u)x'(u)|^2}{\sigma A}\right)^{1/(\sigma+1)}
\]
to $b|x(u)||r(u)x'(u)|^\beta$ we then obtain for all $\lambda \in (0, 1)$
\[
(\lambda f(u, x(u), r(u)x'(u)), x(u)) \\
\leq a|x(u)|^{1+\alpha} + C|x(u)|^{2-\beta} + c|x(u)| + \frac{1}{A}|r(u)x'(u)|^2
\]
where $C = \frac{2-\beta}{2} \left( \frac{4\beta}{2} \right)^{\beta/(2-\beta)} b^{2/(2-\beta)}$. Thus (9) is reduced to
\[
|x(t)|^2 \leq 2T \frac{1}{r} \left\{ a|x|^{1+\alpha} + C|x|^\frac{2-\beta}{2} + c|x| \right\}, \quad t \in I.
\]
So there exists a $R > 0$ such that
\[
(10) \quad |x|_0 \leq R.
\]
From the fact that
\[
\frac{1}{2} \frac{d}{dt} |r(t)x'(t)|^2 = \langle (r(t)x'(t))', r(t)x'(t) \rangle \\
= -\langle \lambda f(t, x(t), r(t)x'(t)), r(t)x'(t) \rangle
\]
it is clear that by means of (10)
\[
\left| \frac{d}{dt} \int_0^{r(t)x'(t)} du \right| \leq (aR^\alpha + b|r(t)x'(t)|^\beta + c) |r(t)x'(t)|.
\]
Dividing both sides by $(aR^\alpha + b|r(t)x'(t)|^\beta + c) |r(t)x'(t)|$ we obtain
\[
\left| \frac{d}{dt} \int_0^{r(t)x'(t)} du \right| \frac{du}{aR^\alpha + bu^\beta + c} \leq 1.
\]
By means of the condition (2) there exists $t_0 \in (0, T)$ such that $x'(t_0) = 0$. Integrating
\[
\frac{d}{dt} \int_0^{r(t)x'(t)} du \frac{du}{aR^\alpha + bu^\beta + c} \quad \text{over } [t_0, t] \quad \text{for } 0 \leq t_0 < t \leq T
\]
we have
\[
(11) \quad \int_0^{r(t)x'(t)} \frac{du}{aR^\alpha + bu^\beta + c} < T.
\]
On the other hand, since
\[ \int_{0}^{\infty} \frac{du}{aR^\alpha + bu^\beta + c} = \infty \]
there exists a \( R_1 > 0 \) such that
\[ T \leq \int_{0}^{R_1} \frac{du}{aR^\alpha + bu^\beta + c}. \]  

From (11) and (12) we obtain \( |x'|_0 \leq R_1/B \). Put \( M = \max \{R, R_1/B\} \). Then the inequality \( |x|_1 \leq M \) is valid for each solution of \( (3\lambda) \), \( \lambda \in (0, 1) \) satisfying (2). If therefore we take
\[ \Omega = B_{M+1}(0) = \{x \in C^1(I, H) : |x|_1 < M + 1\} \]
our theorem is proved by Theorem 2.

Remark. Even though \( a = c = 0 \) Theorem 3 is valid.

Theorem 4. Assume that there exist nonnegative real numbers \( a, b, c \) such that for all \( (t, x, y) \in I \times H \times H \)
\[ |f(t, x, y)| \leq a|x| + b|y| + c, \]
\[ T(4a + b^2A)I_{1/r} < 2. \]

are valid. Then the differential equation (1) satisfying (2) has at least one solution in \( C^1(I, H) \).

Proof. The most part of proof is similar to that of Theorem 3. We sketch briefly the process of proof. It is obvious that for all \( \lambda \in (0, 1) \)
\[ \langle \lambda f(u, x(u), r(u)x'(u)), x(u) \rangle \leq \left( a + \frac{b^2A}{4} \right) |x(u)|^2 + c|x(u)| + \frac{1}{A} |r(u)x'(u)|^2. \]
Thus from (9) we have
\[ |x(t)|^2 \leq 2T I_{1/r} \left\{ \left( a + \frac{b^2A}{4} \right) |x|^2_0 + c|x|_0 \right\}, \quad t \in I \]
for all \( \lambda \in (0, 1) \). So we obtain
\[
|x|_0 \leq R
\]
where
\[
R = \frac{4cT_1}{2 - T(4a + b^2A)I_{1/r}}.
\]
Note that \( R > 0 \) by (14). The rest part of proof is the same as that of Theorem 3. □

**Remark.** In the case of \(|f(t, x, y)| \leq a|x|^{\alpha} + b|y|^\beta + c\), where \( 0 \leq \alpha, \beta \leq 1 \), it is not difficult to show that by Theorem 3 and Theorem 4 the existence property of (1)-(2) is also valid under suitable conditions.

**Definition 2.** A function \( p : I \times H \to [0, \infty) \) is called a \( L^1 \)-Carathéodory function such that

(i') \( t \to p(t, x) \) is measurable for each \( x \in H \),

(ii') \( x \to p(t, x) \) is continuous for a. e. \( t \in I \),

(iii') for any \( \gamma > 0 \) there exists \( h_\gamma \in L^1(I, \mathbb{R}) \) such that \( |x|_0 \leq \gamma \) implies \(|p(t, x)| \leq h_\gamma(t) \) a. e. \( t \in I \).

**Theorem 5.** Assume that \( p : I \times H \to [0, \infty) \) is called a \( L^1 \)-Carathéodory function and that for all \((t, x, y) \in I \times H \times H\)

(i) there exist nonnegative real numbers \( a, b, c \) such that
\[
|\langle x, f(t, x, y) \rangle| \leq a|x|^2 + b|x||y| + c|x|,
\]
and (14) are valid.

(ii) there exist a continuous function \( g : [0, \infty) \to (0, \infty) \) and positive numbers \( R, R_1 \) such that
\[
|\langle y, f(t, x, y) \rangle| \leq p(t, x)g(|y|), \text{ for a. e. } t \in I \text{ and all } y \in H,
\]

\[
\int_{\sqrt{\pi}g(u)}^{\infty} du = \infty,
\]
where
\[
R_1 = 2A \left\{ \left( a + \frac{b^2A}{2} \right) R^2 + cR \right\}
\]
and \( R \) is a number given by (18).
Then the differential equation (1) satisfying (2) has at least one solution in \( C^1(I, H) \).

Proof. We sketch briefly the process of proof. It follows that for all \( \lambda \in (0, 1) \) there exists a \( R > 0 \) satisfying (17). Here \( R \) is a constant number (18). Multiplying (3\( \lambda \)) by \(-x(t)\) and integrating over \( I \) we have

\[
\frac{1}{A} ||rx'||^2_{L^2} \leq \int_0^T r(t)||x'(t)||^2 dt
\]

\[
\leq \int_0^T |\langle f(u, x(u), r(u)x'(u)), x(u) \rangle| du
\]

\[
\leq aT|x|_0^2 + cT|x|_0 + b\sqrt{T}||r||_{L^2}.
\]

Since

\[
 b\sqrt{T}|x|_0 ||rx'||_{L^2} \leq \frac{b^2AT}{2} |x|_0^2 + \frac{1}{2A} ||rx'||_{L^2}^2.
\]

the inequality \( ||rx'||_{L^2}^2 \leq TR_1 \) is valid. There exists a \( \xi \in [0, T] \) such that \( |r(\xi)x'(\xi)| \leq \sqrt{R_1} \). It is clear that by (20)

\[
\left| \frac{1}{2} \frac{d}{dt} |r(t)x'(t)|^2 \right| = \left| |r(t)x'(t)| \frac{d}{dt} |r(t)x'(t)| \right|
\]

\[
\leq p(t, x)|r(t)x'(t)|.
\]

Dividing both sides by \( g(|r(t)x'(t)|) \) we obtain

\[
\frac{d}{dt} \int_0^t |r(t)x'(t)| \frac{u}{g(u)} du \leq p(t, x).
\]

By (iii\( \prime \)) there exists a \( h_R \in L^1(I) \) satisfying \( |p(t, x)| \leq h_R(t) \) for all \( I \). From (21) it follows that there exists a real number \( R_2 > 0 \) such that

\[
\int_0^T h_R(u) du = \int_{\sqrt{R_1}}^{R_2} \frac{u}{g(u)} du.
\]

Therefore we have for \( 0 \leq \xi < t \leq T \)

\[
\int_0^t |r(t)x'(t)| \frac{u}{g(u)} du \leq \int_{\sqrt{R_1}}^{R_2} \frac{u}{g(u)} du + \int_0^t h_R(u) du
\]

\[
\leq \int_0^{R_2} \frac{u}{g(u)} du.
\]
By means of the proof of Theorem 3 our theorem is proved. □

**Remark.** In Theorem 5 even if the condition (20) and (21) are replaced with

\[(20') \quad |\langle y, f(t, x, y) \rangle| \leq p(t, x)g(|y|^2), \text{ for a. e. } t \in I \text{ and all } x, y \in H,\]

\[\int_\infty^\infty \frac{du}{\sqrt{\pi}g(u)} = \infty,\]

we get the same result.

**Theorem 6.** Assume that there exist positive numbers \(a, b\) such that for all \(t \in I\)

\[|f(t, x, y) - f(t, u, v)| \leq a|x - u| + b|y - v|\]

\[I_{1/r} < \frac{2B}{T(4aB + b^2A^2)}\]

for all \(x, y, u, v \in H\). Then the differential equation (1) satisfying

\[x(0) = x_0, \quad x(T) = x_T \text{ for } x_0, x_T \in \mathbb{R}\]

has at most one solution.

**Proof.** Assume \(x(t), u(t)\) are solutions of (1) satisfying (25). If we put \(w(t) = x(t) - u(t)\) we obtain

\[\left((r(t)w'(t))' + f(t, x(t), r(t)x'(t)) - f(t, u(t), r(t)u'(t)) = 0, \right.\]

\[\left. w(0) = w(T) = 0 \right\}

Consider the inner product of (26) with \(w(t)\):

\[\langle (r(t)w'(t))', w(t) \rangle + \langle f(t, x(t), r(t)x'(t)) - f(t, u(t), r(t)u'(t)), w(t) \rangle = 0.\]

From (17) it follows that

\[|w(t)|^2 \leq \int_0^t \int_s^T \left\{ \langle f(\tau, x(\tau), r(\tau)x'(\tau)) - f(\tau, u(\tau), r(\tau)u'(\tau)), w(\tau) \rangle - B |w'(\tau)|^2 \right\} d\tau d s.\]
Thus using (23) and (24) we obtain
\[ |w(t)|^2 \leq 2TI_{1/r} \left( a + \frac{b^2A^2}{4B} \right) |w|_0^2. \]
Taking (24) into account we get
\[ |w|_0^2 \leq 0 \]
which implies \( x(t) = u(t) \) for all \( t \in I \).

**Remark.** By Theorem 4 and Theorem 5 the differential equation (1)-(2) has a unique solution in \( C^1(I, H) \) under the assumptions (14), (15) and (25).

**Example 1.** Let \( a(t) \) and \( b(t) \) be continuous functions on \( I \). By Theorem 3 the differential equation
\[
\left( (1 + t^{1/3})x'(t) \right)' + a(t)\sqrt{x'(t)} \ln \left[ 1 + \{r(t)x'(t)\}^2 \right] \sin(tx(t)) + b(t)e^{-t^2} = 0,
\]
\[ x(0) = x(T) = 0 \]
has at least one solution in \( C^1(I, H) \).

**Example 2.** Let \( a(t) \), \( b(t) \) and \( c(t) \) be continuous functions on \( I \). By Theorem 4 the differential equation
\[
\left( (1 + |\sin t|)x'(t) \right)' + a(t)x(t) - b(t)\frac{(r(t)x'(t))^3}{1 + |r(t)x'(t)|^2} + c(t)\cos t = 0,
\]
\[ x(0) = x(T) = 0 \]
where \( 0 \leq \alpha < 1 \) has at least one solution in \( C^1(I, H) \).

**Remark.** For the case \( \alpha = \beta = 1 \) in Theorem 3 and for a completely continuous function \( f \) satisfying (27) Mawhin[4] proved the existence result under the assumption \( \int_0^\infty \frac{ds}{h(s)+|k|} = \infty \). Hai[2] assumed that \( \int_{M/\pi}^{K/\pi} \frac{ds}{h(s)+|k|} \geq 2M \). Then they proved the existence results of the differential equation (1) – (2) with \( r(t) = 1 \) and \( J = [0, \pi] \).
Nonlinear differential equations of second order in a Hilbert Space

References


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