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THE STABILITY OF PEXIDERIZED COSINE FUNCTIONAL EQUATIONS

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Abstract. In this paper, we investigate the superstability problem for the pexiderized cosine functional equations $f(x+y)+f(x-y) = 2g(x)h(y)$, $f(x+y)+g(x-y) = 2f(x)g(y)$, $f(x+y)+g(x-y) = 2g(x)f(y)$. Consequently, we have generalized the results of stability for the cosine(d'Alembert) and the Wilson functional equations by J. Baker, P. Găvruta, R. Badora and R. Ger, and G.H. Kim.

1. Introduction

Let f be a map from a vector space G to the field $\mathbb{C}$ of complex numbers satisfying the inequality $|f(x+y) - f(x)f(y)| \leq \varepsilon$ for all $x, y \in G$.

Then f is either a bounded or exponential (see Baker [1], Baker-Larrence-Zorzitti [4], Bourgin [5]). The above result can be considered as a special case of the Hyers-Ulam stability problem, which is called the superstability.

The superstability of the cosine functional equation (also called the d'Alembert equation)

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{A}$$

is investigated by J. Baker [1] and P. Găvruta [8], respectively, and that of the sine functional equation

$$f(x)f(y) = f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 \tag{S}$$

is investigated by P.W. Cholewa [6].

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The cosine functional equation (A) is generalized to the following functional equations

$$(A_{fg}) \qquad f(x+y)+f(x-y)=2f(x)g(y),$$

$$(A_{gf}) \qquad f(x+y)+f(x-y)=2g(x)f(y),$$

$$(A_{gg}) \qquad f(x+y)+f(x-y)=2g(x)g(y).$$

The equation (A_{fg}), introduced by Wilson, is sometimes referred to as the Wilson equation.

Let us consider the pexiderized functional equations of the d'Alembert equation (A) as follows:

$$(A_{gh}) \qquad f(x+y)+f(x-y)=2g(x)h(y),$$

$$(A_{fgfg}) \qquad f(x+y)+g(x-y)=2f(x)g(y),$$

$$(A_{fggf}) \qquad f(x+y)+g(x-y)=2g(x)f(y).$$

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers. Whenever we deal with (S), we need to assume additionally that $(G,+)$ is a uniquely 2-divisible group. For brevity, we will write it "under 2-divisibility" hereafter. We may assume that f, g and h are non-zero functions and ε is a nonnegative real constant.

Given mappings $f, g, h : G \to \mathbb{C}$, for the above equations, we will denote their difference by an operator $DA_{gh} : G \times G \to \mathbb{C}$ as

$$DA_{gh}(x,y) := f(x+y)+f(x-y)-2g(x)h(y).$$

The aim of this paper is to investigate the superstability problem for the pexiderized cosine type functional equations (A_{gh}), (A_{fgfg}), (A_{fggf}) under conditions : $|DA_{gh}(x,y)| \le \varepsilon$, $|DA_{fgfg}(x,y)| \le \varepsilon$, $|DA_{fggf}(x,y)| \le \varepsilon$. We also extend the obtained results to the Banach algebra. As a consequence, we obtain the superstability of the cosine and the Wilson functional equations, which proved by Baker, Badora and Ger, Găvruta, Kannappan, Kim, etc ([1], [2], [8], [9], [11],[12]).

2. Stability on the equation (A_{gh})

In this section, we investigate the superstability of (A_{gh}) related with the d'Alembert (A), the Wilson type (A_{fg}) and (A_{gf}), the sine (S) functional equations.

THEOREM 1. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + f(x-y) - 2g(x)h(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.1}$$

If h fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and one of the cases $g(0) = 0$ or $f(-x) = -f(x)$,

(ii) if, additionally, h satisfies (A), then g and h are solutions of the equation $g(x+y) + g(x-y) = 2g(x)h(y)$.

Proof. Let h be the unbounded solution of the inequality (2.1). Then, there exists a sequence $\{y_n\}$ in G such that $0 \ne |h(y_n)| \to \infty$ as $n \to \infty$.

Taking $y = y_n$ in the inequality (2.1), dividing both sides by $|2h(y_n)|$, and passing to the limit as $n \to \infty$ we obtain that

$$g(x) = \lim_{n\to\infty} \frac{f(x+y_n) + f(x-y_n)}{2h(y_n)}, \qquad x \in G. \tag{2.2}$$

Using (2.1), we have

$$\begin{aligned}\big| &f\left(x + (y + y_n)\right) + f\left(x - (y + y_n)\right) - 2g(x)h(y + y_n) \\ &\quad + f\left(x + (y - y_n)\right) + f\left(x - (y - y_n)\right) - 2g(x)h(y - y_n)\big| \le 2\varepsilon\end{aligned}$$

so that

$$\begin{aligned}&\left|\frac{f\left((x+y)+y_n\right) + f\left((x+y)-y_n\right)}{2h(y_n)}\right. \\ &\left. + \frac{f\left((x-y)+y_n\right) + f\left((x-y)-y_n\right)}{2h(y_n)} - 2g(x) \cdot \frac{h(y+y_n) + h(y-y_n)}{2h(y_n)}\right| \\ &\le \frac{\varepsilon}{|h(y_n)|}\end{aligned} \tag{2.3}$$

for all $x, y \in G$.

We conclude that, for every $y \in G$, there exists a limit function

$$k_1(y) := \lim_{n\to\infty} \frac{h(y+y_n)+h(y-y_n)}{h(y_n)},$$

where the function $k_1 : G \to \mathbb{C}$ satisfies the equation

$$g(x+y)+g(x-y) = g(x)k_1(y) \quad \forall x, y \in G. \tag{2.4}$$

Applying the case $g(0) = 0$ in (2.4), it implies that g is odd. Keeping this in mind, by means of (2.4), we infer the equality

$$\begin{aligned} g(x+y)^2 - g(x-y)^2 &= [g(x+y)+g(x-y)][g(x+y)-g(x-y)] \\ &= g(x)k_1(y)[g(x+y)-g(x-y)] \\ &= g(x)\big[g(x+2y)-g(x-2y)\big] \\ &= g(x)\big[g(2y+x)+g(2y-x)\big] \\ &= g(x)g(2y)k_1(x). \end{aligned} \tag{2.5}$$

Putting $y = x$ in (2.4) we get the equation

$$g(2x) = g(x)k_1(x), \quad x \in G.$$

This, in return, leads to the equation

$$g(x+y)^2 - g(x-y)^2 = g(2x)g(2y) \tag{2.6}$$

valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of G, states nothing else but (S).

Next, in the particular case $f(-x) = -f(x)$, it is enough to show that $g(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (2.1), due to $g(0) \neq 0$ and $f(-x) = -f(x)$, we obtain the inequality

$$|h(y)| \le \frac{\varepsilon}{2|g(0)|}, \quad y \in G.$$

This inequality means that h is globally bounded – a contradiction. Thus the claimed $g(0) = 0$ holds.

(ii) In case h satisfies (A), the limit k_1 states nothing else but $2h$, so (2.4) validates that g and h satisfy the Wilson equation $g(x+y)+g(x-y) = 2g(x)h(y)$. □

By replacing, respectively, with h by f, g by f, h by g in Theorem 1, we obtain the following corollaries.

Corollary 1. ([10], [11]) *Suppose that* $f, g : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.7}$$

Then either f*(or* g*) is bounded or* g *satisfies (A).*

Proof. Replacing h by f in Theorem 1.

First, let us consider the case that f is the unbounded solution of (2.7).

For a sequence $\{y_n\}$ in G such that $0 \ne |f(y_n)| \to \infty$ as $n \to \infty$, an obvious slight change in the steps of the proof applied in Theorem 1 gives us that g satisfies (A).

Secondly, for the case of g, it is sufficient from the above result to see that the boundedness of f implies that of g.

If f is bounded, choose y_0 such that $f(y_0) \ne 0$ and then use (2.7),

$$\begin{aligned} |g(x)| - \left|\frac{f(x+y_0) + f(x-y_0)}{2f(y_0)}\right| &\le \left|\frac{f(x+y_0) + f(x-y_0)}{2f(y_0)} - g(x)\right| \\ &\le \frac{\varepsilon}{2|f(y_0)|} \end{aligned}$$

to get that g is also bounded on G. □

Corollary 2. ([10], [11]) *Suppose that* $f, h : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) + f(x-y) - 2f(x)h(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.8}$$

Then (i) either f *is bounded or* h *satisfies (A).*

(ii) if h *fails to be bounded, then* h *satisfies (A), and also* f *and* h *are solutions of the equation* $f(x+y) + f(x-y) = 2f(x)h(y)$.

Proof. Replacing g by f in Theorem 1.

(i) Suppose that f is unbounded, then, for a sequence $\{x_n\}$ in G such that $0 \ne |f(x_n)| \to \infty$ as $n \to \infty$, we obtain

$$h(y) = \lim_{n\to\infty} \frac{f(x_n + y) + f(x_n - y)}{2f(x_n)}. \tag{2.9}$$

Replacing x by $x_n + x$ and $x_n - x$ in (2.8). An obvious slight change in the steps of the proof applied after (2.2) of Theorem 1 imply, with an application of (2.9), that h satisfies (A).

(ii) Suppose that h is unbounded, it is enough from (i) to see that if f is bounded then h is bounded. Really, for f is bounded, choose x_0 such that $f(x_0) \neq 0$ and use (2.8) to get

$$\begin{aligned} |h(y)| - \frac{|f(x_0+y)+f(x_0-y)|}{2|f(x_0)|} &\leq \left| \frac{f(x_0+y)+f(x_0-y)}{2f(x_0)} - h(y) \right| \\ &\leq \frac{\varepsilon}{2|f(x_0)|}, \end{aligned}$$

which shows that h is also bounded - contradiction. Hence it is completed that h satisfies (A). Next, for a sequence $\{y_n\}$ in G such that $0 \neq |h(y_n)| \to \infty$ as $n \to \infty$, we obtain

$$f(x) = \lim_{n\to\infty} \frac{f(x+y_n)+f(x-y_n)}{2h(y_n)}. \tag{2.10}$$

Replacing y by $y + y_n$ and $y - y_n$ in (2.8), since h satisfies (A) and with an application of (2.10), we have that f and h are solutions of the equation $f(x + y) + f(x - y) = 2f(x)h(y)$. □

Corollary 3. ([1], [8]) *Suppose that $f : G \to \mathbb{C}$ satisfies the inequality*

$$|f(x+y)+f(x-y)-2f(x)f(y)| \leq \varepsilon \qquad \forall\, x, y \in G.$$

Then either f is bounded or f satisfies (A).

Theorem 2. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)+f(x-y)-2g(x)h(y)| \leq \varepsilon \quad \forall\, x, y \in G. \tag{2.11}$$

If g fails to be bounded, then

(i) h satisfies (S) under 2-divisibility and $h(0) = 0$,

(ii) if, additionally, g satisfies (A), then g and h are solutions of the equation $h(x + y) + h(x - y) = 2h(x)g(y)$.

Proof. For g to be the unbounded solution of the inequality (2.11), there exists a sequence $\{x_n\}$ in G such that $0 \neq |g(x_n)| \to \infty$ as $n \to \infty$.

Hence we obtain that

$$h(y) = \lim_{n\to\infty} \frac{f(x_n+y)+f(x_n-y)}{2g(x_n)}, \quad x \in G. \tag{2.12}$$

Replacing x by $x_n + y$ and $x_n - y$, and replacing y by x in (2.11), the next step of the proof runs along that of Theorem 1. Namely, we arrive at the required result throughout the limit function

$$k_2(y) := \lim_{n\to\infty} \frac{g(x_n+y)+g(x_n-y)}{g(x_n)},$$

where the function $k_2 : G \to \mathbb{C}$ satisfies the equation

$$h(x+y)+h(x-y) = h(x)k_2(y) \quad \forall x, y \in G.$$

□

Corollary 4. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)+f(x-y)-2g(x)g(y)| \le \varepsilon \quad \forall\, x, y \in G.$$

Then either g is bounded or g satisfies (S) under 2-divisibility and $g(0) = 0$.

Proof. Replacing h by g in Theorem 2. An obvious slight change in the steps of the proof applied in Theorem 2 runs along the result. □

3. Stability of the equations (A_{fgfg}) and (A_{fggf})

By the same procedure as in section 2, we investigate the stability of (A_{fgfg}) and (A_{fggf}) related with the d'Alembert (A), the Wilson type (A_{fg}) and (A_{gf}), and the sine (S) functional equations.

Theorem 3. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)+g(x-y)-2f(x)g(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{3.1}$$

If g fails to be bounded, then

(i) f satisfies (S) under 2-divisibility and $f(0) = 0$,

(ii) if, additionally, g satisfies (A), then f and g are solutions of the equation $f(x+y)+f(x-y) = 2f(x)g(y)$.

Proof. For the unbounded g of the inequality (3.1), we can choose a sequence $\{y_n\}$ in G such that $0 \ne |g(y_n)| \to \infty$ as $n \to \infty$.

A similar reasoning as in the proof applied in Theorem 1 with $y = y_n$ in (3.1) gives us

$$f(x) = \lim_{n\to\infty} \frac{f(x+y_n)+g(x-y_n)}{2g(y_n)}, \quad x \in G. \tag{3.2}$$

Replacing y by $y+y_n$ and $-y+y_n$ in (3.1), we can, with an application of (3.2), state the existence of a limit function

$$k_3(y) := \lim_{n\to\infty} \frac{g(y+y_n)+g(-y+y_n)}{g(y_n)},$$

where the function $k_3 : G \to \mathbb{C}$ satisfies the equation

$$f(x+y)+f(x-y) = f(x)k_2(y) \quad \forall x,y \in G. \tag{3.3}$$

The rest of the proof runs along a similar method of proof as that applied after (2.4) of Theorem 1. □

Theorem 4. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)+g(x-y)-2f(x)g(y)| \le \varepsilon \quad \forall\, x,y \in G. \tag{3.4}$$

If f fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and $g(0)=0$,

(ii) if, additionally, f satisfies (A), then g satisfies (A).

Proof. For the unbounded f of the inequality (3.4), there exists a sequence $\{x_n\}$ in G such that $0 \neq |f(x_n)| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ in the inequality (3.4), dividing both sides by $|2f(x_n)|$, and passing to the limit as $n \to \infty$ we obtain that

$$g(y) = \lim_{n\to\infty} \frac{f(x_n+y)+g(x_n-y)}{2f(x_n)}, \quad x \in G. \tag{3.5}$$

The rest of the proof runs along a same method of proof as that applied after (2.2) of Theorem 1. Namely, replacing x by $x_n + x$ and $x_n - x$ in (3.4), we obtain that

$$k_4(x) := \lim_{n\to\infty} \frac{f(x_n+x)+f(x_n-x)}{f(x_n)},$$

where the function $k_4 : G \to \mathbb{C}$ satisfies the equation

$$g(x+y)+g(-x+y) = k_4(x)g(y) \quad \forall x,y \in G. \tag{3.6}$$

The rest of proof goes smoothly to the result. □

The proof of the following two Theorems 5 and 6 are similar to that of Theorems 3 and 4, so we will note only outline.

Theorem 5. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y)+g(x-y)-2g(x)f(y)| \le \varepsilon \quad \forall\, x,y \in G. \tag{3.7}$$

If f fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and $g(0)=0$,

(ii) if, additionally, f satisfies (A), then g satisfies (A).

Proof. For the unbounded solution f of the inequality (3.7), there exists a sequence $\{y_n\}$ in G such that $0 \neq |f(y_n)| \to \infty$ as $n \to \infty$.

Taking $y = y_n$, and replacing y by $y + y_n$ and $-y + y_n$ in (3.7), the proof of the Theorem runs along a same method as that applied from (3.2) of Theorem 3. □

Theorem 6. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - 2g(x)f(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{3.8}$$

If g fails to be bounded, then

(i) f satisfies (S) under 2-divisibility and $f(0) = 0$,

(ii) if, additionally, g satisfies (A), then f and g satisfy $f(x+y) + f(x-y) = 2f(x)g(y)$.

Proof. For the unbounded solution g of the inequality (3.8), let us follow the proof of the previous theorem. Taking $x = x_n$, and replacing x by $x_n + x$ and $x_n - x$ in (3.8), the proof of the Theorem runs along a same method as that applied from (3.5) of Theorem 4. □

4. Extension to the Banach algebra

The obtained results of sections 2–3 can be extended to the Banach algebra. For simplicity, we only will represent one of them, and the application to other corollaries will be omitted.

Theorem 7. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : G \to E$ satisfy the inequality*

$$\|f(x+y) + f(x-y) - 2g(x)h(y)\| \le \varepsilon \quad \forall\, x, y \in G. \tag{4.1}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

if the superposition $x^* \circ h$ fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and one of the cases $g(0) = 0$ or $f(-x) = -f(x)$,

(ii) if, additionally, $x^* \circ h$ satisfies (A), then g and h are solutions of the equation $g(x+y) + g(x-y) = 2g(x)h(y)$.

Proof. (i) Fix arbitrarily a linear multiplicative functional $x^* \in E^*$, we have $\|x^*\| = 1$ as known. In (4.1), we have

$$\begin{aligned}\varepsilon &\geq \|f(x+y) + f(x-y) - 2g(x)h(y)\| \\ &= \sup_{\|y^*\|=1} \left|y^*\big(f(x+y) + f(x-y) - 2g(x)h(y)\big)\right| \\ &\geq \left|x^*\big(f(x+y)\big) - x^*\big(f(x-y)\big) - 2x^*\big(g(x)\big)x^*\big(h(y)\big)\right|.\end{aligned}$$

In the above inequality, we know that the superpositions $x^* \circ f$, $x^* \circ g$ and $x^* \circ h$ yield a solution of inequality (2.1) in Theorem 1.

Assume that the superposition $x^* \circ h$ is unbounded, then Theorem 1 forces that the superposition $x^* \circ g$ solves (S). These statements mean, keeping the linear multiplicativity of x^* in mind, that the difference $DS_g(x, y) := g(x)g(y) - g\big(\frac{x+y}{2}\big)^2 + g\big(\frac{x-y}{2}\big)^2$ for all $x, y \in G$ falls into the kernel of x^*.

Since x^* is arbitrary, we deduce that

$$DS_g(x, y) \in \bigcap\{\ker x^* : x^* \in E^*\}$$

for all $x, y \in G$.

Since the Banach algebra E has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$DS_g(x, y) = 0 \quad \forall\, x, y \in G,$$

as claimed.

(ii) Under the assumption that the superposition $x^* \circ h$ satisfies (A), we know from Theorem 1 that the superpositions $x^* \circ g$ and $x^* \circ h$ are solutions of the equation

$$x^*\big(g(x+y)\big) - x^*\big(g(x-y)\big) = 2x^*\big(g(x)\big)x^*\big(h(y)\big).$$

Namely,

$$g(x+y) - g(x-y) - 2g(x)h(y) \in \bigcap\{\ker x^* : x^* \in E^*\}.$$

The other argument is similar. $\square$

THEOREM 8. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\|f(x+y) + g(x-y) - 2f(x)g(y)\| \leq \varepsilon \quad \forall\, x, y \in G. \tag{4.2}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(a) if the superposition $x^* \circ g$ fails to be bounded, then

(i) f satisfies (S) under 2-divisibility and $f(0) = 0$,

(ii) if, additionally, $x^* \circ g$ satisfies (A), then f and g are solutions of the equation $f(x+y) + f(x-y) = 2f(x)g(y)$.

(b) if the superposition $x^* \circ f$ fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and $g(0) = 0$,

(ii) if, additionally, $x^* \circ f$ satisfies (A), then g satisfies (A).

Theorem 9. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g : G \to E$ satisfy the inequality*

$$\|f(x+y) + g(x-y) - 2g(x)f(y)\| \le \varepsilon \quad \forall\, x, y \in G. \tag{4.3}$$

For an arbitrary linear multiplicative functional $x^* \in E^*$,

(a) if the superposition $x^* \circ f$ fails to be bounded, then

(i) g satisfies (S) under 2-divisibility and $g(0) = 0$,

(ii) if, additionally, $x^* \circ f$ satisfies (A), then g satisfies (A).

(b) if the superposition $x^* \circ g$ fails to be bounded, then

(i) f satisfies (S) under 2-divisibility and $f(0) = 0$,

(ii) if, additionally, $x^* \circ g$ satisfies (A), then f and g satisfies $f(x+y) + f(x-y) = 2f(x)g(y)$.

Remark 1. *From the results of Theorem 7–9, we obtain following corollaries.*

(i) *Replacing f for h, f for g, g for h in* (4.1), *then we obtain same types of results for the equations (A_{fg}),(A_{gf}) and (A_{gg}), which are founded in papers* ([2], [10], [11]).

(ii) *Replacing f for g in* (4.2), *then we obtain same types of results for the equation (A), which is founded in papers* ([2], [10], [11]).

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