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DETERMINANTS AND TRACES FOR THE COMMUTING OPERATORS ON A FINITE VECTOR SPACE

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Abstract. In the present article, we give a set of axioms for determinants and traces of the l-tuples of commuting operators on a fixed finite dimensional vector space over a field when $l \geq 2$. We describe them with or without a coherence assumption especially when k is the field of real numbers. Under the coherence assumption, it turns out that there are only a trivial determinant and trace over arbitrary field k. This leads us to formulate a more appropriate definition of the determinants. In this case, the set of determinants can be described in terms of the Milnor's K-theory. As for the traces, it is not clear to us how to correctly formulate a definition except for certain cases.

1. Introduction

In Section 7 of [3], the determinants for the l-tuples of commuting operators over a field k are defined and are shown to be related with the maps from a Milnor's K-group. In [4], a trace map for commuting trace-class self-adjoint operators on Hilbert spaces is defined and described. But, in these two articles, the determinant or trace map is considered as coherent system of maps from the total space of l-tuples of commuting operators on vector spaces of various dimensions. In the present article, we look for possibilities of determinants or traces for commuting operators on a fixed finite dimensional vector space k^n.

For a field k, we denote by $Comm_l(k^n)$ the set of l-tuples $(A_1, \ldots, A_l)$ of commuting operators on the n-dimensional k-vector space k^n for $n \geq 1$. If $k = \mathbb{R}$, it can be quipped with the obvious topology which comes from the usual real metric. We also denote by $Comm_l(k^n)^{\times}$ the set of

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l-tuples of commuting invertible operators on k^n, which is obviously a subset of $Comm_l(k^n)$. The following preliminary definition is similar to the one given in [4], but without a coherence condition.

Definition 1.1. *A trace map T_l for $Comm_l(k^n)$ is defined to be a map from $Comm_l(k^n)$ into the underlying additive group $(k,+)$ of a given field k, which satisfies the following two conditions:*
(i) (Multilinearity) For $l+1$ commuting $n\times n$ matrices $A_1,\dots,A_l$ with entries in k, we have

$$T_l(A_1,\dots,A_i{+}B,\dots,A_l) = T_l(A_1,\dots,A_i,\dots,A_l){+}T_l(A_1,\dots,B,\dots,A_l).$$

(ii) (Compatibility with usual trace) For commuting $n\times n$ matrices $A_1,\dots,A_l\in GL_n(k)$ and commuting $n\times n$ matrices $B_1,\dots,B_l$ such that $\operatorname{Tr} A_i = \operatorname{Tr} B_i$ for $i=1,\dots,l$, we have $T_l(A_1,\dots,A_l)=T_l(B_1,\dots,B_l)$.

By (i) of Definition 1.1, it is immediate that $T_l(A_1,\dots,A_l)=0$ if A_i, for some i, is equal to the zero matrix.

For $l=1$, the usual trace map gives rise to an example of a trace for $Comm_l(k^n)$ for each $n\geq 1$. When $n=1$, the map $T_l(a_1,\dots,a_l)=ca_1\dots a_l$ gives a nontrivial trace for any nonzero constant $c\in k^\times$. It can be easily shown that these are the only possible continuous traces for the case $n=1$ when $k=\mathbb{R}$.

Lemma 1.2. *Any continuous trace T_l from $Comm_l(\mathbb{R}^1)$ into $\mathbb{R}$ is of the form $T_l(a_1,\dots,a_l)=ca_1\dots a_l$ for some $c\in\mathbb{R}$.*

Proof. The lemma follows easily from the fact that any continuous endomorphism for the additive group $(\mathbb{R},+)$ is of the form $x\mapsto cx$ for some $c\in\mathbb{R}$. We remark that, if we drop the continuity assumption, then, once we choose a Hamel basis (a $\mathbb{Q}$-basis of $\mathbb{R}$, c.f. [1]), an endomorphism of $\mathbb{R}$ is be given by a choice of rational numbers for all basis vectors. Accordingly, the trace maps for $Comm_l(\mathbb{R}^1)$ may be described as products of such endomorhpisms. □

For $n\geq 2$, by the condition (ii) of Definition 1.1, we have, for commuting real matrices $A_1,\dots,A_l$,

$$T_l(A_1,\dots,A_l) = T_l\left(\begin{pmatrix}\operatorname{Tr}(A_1) & 0\\ 0 & 0_{n-1}\end{pmatrix},\dots,\begin{pmatrix}\operatorname{Tr}(A_l) & 0\\ 0 & 0_{n-1}\end{pmatrix}\right),$$

where 0_{n-1} is an $(n-1)\times(n-1)$ matrix with all entries equal to 0. Furthermore, it is immediate from Definition 1.1 that T_l composed with the inclusion $Comm_l(\mathbb{R}^1) \hookrightarrow Comm_l(\mathbb{R}^n)$ via

$$(a_1, \dots, a_l) \mapsto \left(\begin{pmatrix} a_1 & 0 \\ 0 & 0_{n-1} \end{pmatrix}, \dots, \begin{pmatrix} a_l & 0 \\ 0 & 0_{n-1} \end{pmatrix} \right)$$

is a trace for $Comm_l(\mathbb{R}^1)$. Therefore, by Lemma 1.2, we have the following description of continuous traces for arbitrary $l, n \geq 1$:

PROPOSITION 1.3. *Any continuous trace T_l from $Comm_l(\mathbb{R}^n)$ into $\mathbb{R}$ can be written as $T_l(A_1, \dots, A_l) = c\mathrm{Tr}\,(A_1) \dots \mathrm{Tr}\,(A_l)$ for some $c \in \mathbb{R}$.*

In Section 2, it is shown in Proposition 2.4 that the trace T_l should be identically 1, under the coherence assumption, when $l, n \geq 2$.

The following ad hoc definition of determinants is similar to the one given in [3], but again without a coherence condition.

DEFINITION 1.4. *A determinant D_l for $Comm_l(k^n)^\times$ is defined to be a map from $Comm_l(k^n)^\times$ into the multiplicative group $k^\times$ of units in a given field, which satisfies the following two conditions:*
(i) (Multilinearity) For $l+1$ commuting matrices $A_1, \dots, A_l$ and B in $GL_n(k)$, we have $D_l(A_1, \dots, A_iB, \dots, A_l) = D_l(A_1, \dots, A_i, \dots, A_l) \cdot D_l(A_1, \dots, B, \dots, A_l)$.
(ii) (Compatibility with usual determinant) For commuting $A_1, \dots, A_l \in GL_n(k)$ and commuting $B_1, \dots, B_l \in GL_n(k)$ such that $\det A_i = \det B_i$ for $i = 1, \dots, l$, we have $D_l(A_1, \dots, A_l) = D_l(B_1, \dots, B_l)$.

Again, by (i) of Definition 1.4, we have $D_l(A_1, \dots, A_l) = 1$ if A_i is the identity matrix for some i. For $l = 1$, the usual determinant gives rise to an example of a determinant for $Comm_l(\mathbb{R}^n)^\times$ for each $n \geq 1$. When $n = 1$, the map $D_l(a_1, \dots, a_l) = c^{\log|a_1| \cdots \log|a_l|}$ gives a nontrivial determinant for any constant $c \in \mathbb{R}_0$. But, there is one notable exceptional determinant, namely the Hilbert symbol $(a_1, \dots, a_l)_{\mathbb{R}}$ which is defined to be 1 if $a_1x_1^2 + \cdots + a_nx_n^2 = z^2$ has a nonzero solution in $x_1, \dots, x_n, z \in \mathbb{R}^{l+1}$ and -1 otherwise. In fact, $(a_1, \dots, a_l)_{\mathbb{R}} = -1$ if and only if all of $a_1, \dots, a_n$ are negative real numbers. The following proposition completely describes the continuous determinant maps for $Comm_l(\mathbb{R}^n)^\times$ for arbitrary $l, n \geq 1$:

PROPOSITION 1.5. *Any continuous determinant D_l from $Comm_l(\mathbb{R}^n)^\times$ into $\mathbb{R}^\times$ is of the form*

$$D_l(A_1, \dots, A_l) = \begin{cases} c^{\log|\det(A_1)| \cdots \log|\det(A_l)|} & \text{or} \\ (\det(A_1), \dots, \det(A_l))_{\mathbb{R}}\, c^{\log|\det(A_1)| \cdots \log|\det(A_l)|} \end{cases}$$

for some $c \in \mathbb{R}_{>0}$, where $(\ ,\dots,\ )_{\mathbb{R}}$ is the Hilbert symbol.

Proof. By (ii) of Definition 1.4, we have for commuting invertible matrices $A_1, \dots, A_l$,

$$D_l(A_1, \dots, A_l) = D_l\left(\begin{pmatrix} \det(A_1) & 0 \\ 0 & I_{n-1} \end{pmatrix}, \dots, \begin{pmatrix} \det(A_l) & 0 \\ 0 & I_{n-1} \end{pmatrix}\right),$$

where I_{n-1} is an $(n-1) \times (n-1)$ identity matrix of rank $n-1$.

Since the inclusion $Comm_l(\mathbb{R}^1)^\times \hookrightarrow Comm_l(\mathbb{R}^n)^\times$ via

$$(a_1, \dots, a_l) \mapsto \left(\begin{pmatrix} a_1 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \dots, \begin{pmatrix} a_l & 0 \\ 0 & I_{n-1} \end{pmatrix}\right),$$

for suitably chosen ϵ for each a_i's, followed by D_l is a determinant for $Comm_l(\mathbb{R}^n)^\times$, we see that D_l is the composite of $(A_1, \dots, A_l) \mapsto (\det A_1, \dots, \det A_l)$ and a determinant for $Comm_l(\mathbb{R}^n)^\times$. So, it suffices to prove the proposition for the case $n = 1$.

Let D_l be any continuous determinant for $Comm_l(\mathbb{R}^1)^\times$. We then note that the map $Comm_l(\mathbb{R}^1) \to Comm_l(\mathbb{R}^1)^\times$ via $(a_1, \dots, a_l) \mapsto (e^{a_1}, \dots, e^{a_l})$ followed by D_l gives a continuous trace for $Comm_l(\mathbb{R}^1)$. By Proposition 1.3, we conclude that, for all positive real numbers $a_1, \dots, a_l$, $D_l(a_1, \dots, a_l) = e^{c_1 \log(a_1) \cdots \log(a_l)}$ for some constant $c_1 \in \mathbb{R}$.

Finally, we have $(a_1, \dots, a_l) = (\operatorname{sgn}(a_1)|a_1|, \dots, \operatorname{sgn}(a_l)|a_l|)$ for any $a_1, \dots, a_l \in \mathbb{R}^\times$, where $\operatorname{sgn}(a_i) = \pm 1$ according to the sign of a_i. By (i) of Definition 1.4, we see that any elements of the form $(\pm 1, \dots, \pm 1)$ maps into the torsion subgroup $\{\pm 1\}$ of $\mathbb{R}^\times$. But, their images are completely determined by the value of $(-1, \dots, -1)$ again by (i) of Definition 1.4. Hence, if $D_l(-1, \dots, -1) = 1$, we have $D_l(a_1, \dots, a_l) = e^{c_1 \log|a_1| \dots \log|a_l|}$ and, if $D_l(-1, \dots, -1) = -1$, we have $D_l(a_1, \dots, a_l) = (a_1, \dots, a_l)_{\mathbb{R}}\, e^{c_1 \log|a_1| \dots \log|a_l|}$. □

Proposition 1.3 and Proposition 1.5 completely describes the traces and determinants for arbitrary $l, n \geq 1$, but a more interesting traces and determinants are the ones with coherence assumptions.

In Section 2, the coherence conditions are defined, but, any trace and determinant over an arbitrary field are immediately shown to be trivial when $l \geq 2$ and $n \geq 2$ with the coherence assumptions. By examining the proof of this fact, it can be realized that the preliminary definition of the determinants given in Definition 1.4 allows too much freedom for manipulation. Therefore, a new set of axioms for the determinants will be given in Definition 2.5 and then the coherent determinants for

$Comm_l(k^n)^\times$ will be described in terms of Milnor's K-theory for a fixed n.

2. Coherence of trace and determinant

We define the coherence assumption for a trace as follows:

Definition 2.1. *Suppose that we are given a trace $T_{l,m}$ from $Comm_l(k^m)$ into k for every $1 \le m < n$. A trace T_l from $Comm_l(k^n)$ into k is said to be coherent with the sub-traces $T_{l,m}$ if it satisfies the following condition:*

$$T_l\left(\begin{pmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{1r} \end{pmatrix}, \dots, \begin{pmatrix} A_{l1} & & 0 \\ & \ddots & \\ 0 & & A_{lr} \end{pmatrix}\right) = \sum_{i=1}^{r} T_{l,n_i}(A_{1i}, \dots, A_{li})$$

whenever $A_{1i}, \dots, A_{li}$, for each $i = 1, \dots, r$, are commuting $n_i \times n_i$ matrices and $\sum n_i = n$.

The determinants with a coherent condition are defined similarly as below:

Definition 2.2. *Suppose that we have a determinant $D_{l,m}$ from $Comm_l(k^m)^\times$ into $k^\times$ for each $1 \le m < n$. A determinant D_l from $Comm_l(k^n)^\times$ into $k^\times$ is said to be coherent with the sub-determinants $D_{l,m}$ if it satisfies the following condition:*

$$D_l\left(\begin{pmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{1r} \end{pmatrix}, \dots, \begin{pmatrix} A_{l1} & & 0 \\ & \ddots & \\ 0 & & A_{lr} \end{pmatrix}\right) = \prod_{i=1}^{r} D_{l,n_i}(A_{1i}, \dots, A_{li})$$

when $A_{1i}, \dots, A_{li}$, for each i, are commuting invertible matrices in $GL_{n_i}(k)$ and $\sum n_i = n$.

Proposition 2.3. *In the sense of Definition 1.4, there is no nontrivial coherent determinant D_l for $Comm_l(k^n)^\times$ when $l, n \ge 2$.*

Proof. First, let us show that D_l vanishes on the following types of l-tuples of commuting invertible matrices in $Comm_l(k^n)$ if $n \ge 2$:

$$\left(\begin{pmatrix} a_1 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \begin{pmatrix} a_2 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \dots, \begin{pmatrix} a_n & 0 \\ 0 & I_{n-1} \end{pmatrix}\right),$$

where $a_i \in k^\times$. The proof of this fact is immediate because, by (ii) of Definition 1.4, we have the same determinant value after we replace the n-th coordinate $\begin{pmatrix} a_n & 0 \\ 0 & I_{n-1} \end{pmatrix}$ by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_n & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$ which has the same determinant. But, after the replacement, the determinant is 1 because of Definition 2.2 and (i) of Definition 1.4 for the sub-determinant $D_{l,1}$.

A consequence of this fact is that $D_{l,1}$ should be trivial as the sub-determinant in Definition 2.2 when $n \geq 2$. Then, by Definition 2.2, we conclude that D_l is trivial on $Comm_l(k^n)^\times$ since

$$D_l(A_1, \ldots, A_l) = D_1\left(\begin{pmatrix} \det A_1 & 0 \\ 0 & I_{n-1} \end{pmatrix}, \ldots, \begin{pmatrix} \det A_n & 0 \\ 0 & I_{n-1} \end{pmatrix}\right).$$

□

The proof of the following proposition is similar.

Proposition 2.4. *There is no nontrivial coherent trace T_l for $Comm_l(k^n)$ in the sense of Definition 1.1 when $l, n \geq 2$.*

To avoid the situation where a determinant D_l depends only on the determinant of each coordinate matrix, it is inevitable to put a restriction on the freedom of individual manipulation of each coordinate matrix and substitute the condition (ii) of Definition 1.4 by a more rigid set of axioms. From now on, a determinant will mean a map which satisfies the following definition:

Definition 2.5. *A determinant D_l for $Comm_l(k^n)^\times$ is defined to be a map from $Comm_l(k^n)^\times$ into the multiplicative group $k^\times$ of units in a given field, which satisfies the following three conditions:*
(i) (Multilinearity) For $l+1$ commuting matrices $A_1, \ldots, A_l$ and B in $GL_n(k)$, we have $D_l(A_1, \ldots, A_iB, \ldots, A_l) = D_l(A_1, \ldots, A_i, \ldots, A_l) \cdot D_l(A_1, \ldots, B, \ldots, A_l)$.
(ii) (Similar Matrices) For commuting matrices $A_1, \ldots, A_l \in GL_n(k)$ and some $S \in GL_n(k)$, we have $D(SA_1S^{-1}, \ldots, SA_lS^{-1}) = D(A_1, \ldots, A_l)$.
(iii) (Polynomial Homotopy) For commuting $A_1(t), \ldots, A_l(t) \in GL_n(k[t])$, we have $D(A_1(0), \ldots, A_l(0)) = D(A_1(1), \ldots, A_l(1))$.

In [3], it is proved that a coherent system of determinants for $Comm_l(k^n)^\times$ for all $n \geq 1$ is in one-to-one correspondence with the group homomorphisms from the Milnor's K-group $K_l^M(k)$ (c.f. [2] or [5]) into the multiplicative group $k^\times$.

It is not clear how we should modify the definition for the traces to allow only rigid simultaneous deformation of the tuples of commuting matrices. But, in case of hermitian operators, it is possible to take advantage of the definition of determinants using the exponential map (See [4]).

We conclude the article with a theorem which describes the coherent determinants for $Comm_l(k^n)^\times$.

THEOREM 2.6. *There exists a one-to-one correspondence between set of coherent determinants for* $Comm_l(k^n)^\times$ *in the sense of Definition 2.5 and the set of group homomorphisms from* $K_l^M(k)$ *into* $k^\times$ *if* $n \geq 3$.

Proof. In Theorem 7.2 of [3], it is shown that there is a one-to-one correspondence between the set of coherent system of determinants for $Comm_l(k^n)^\times$ for all $n \geq 1$ and the set of group homomorphisms from $K_l^M(k) \simeq GW_l(k)$ into the multiplicative group $k^\times$ of units of k. Therefore, it suffices to prove that the generators and relations for the Goodwillie group $GW_l(k)$ in Definition 4.1 of [3] coming only from the sets $Comm_l(k^m)^\times$ $(m \leq n)$ generate the whole set of generators and relations in Definition 4.1. Let $GW_l(k)_n$ be the abelian group which is generated by $Comm_l(k^m)^\times$ $(m \leq n)$ subject to the relations as in Definition 4.1 of [3] which come from $Comm_l(k^m)^\times$ $(m \leq n)$ so that we have $GW_l(k) = \varinjlim_{n} GW_l(k)_n$. If one tracks down the proof of Lemma 5.2 of [3], it can be checked that the multilinearity for commuting diagonal matrices can be obtained using only the relations coming from the sets $Comm_l(k^m)^\times$ $(m \leq 2)$. Also, (iv) of the same lemma can be proved using only the relations coming from the sets $Comm_l(k^m)^\times$ $(m \leq 3)$, so we have a homomorphism $K_l^M(k) \to GW_l(k)_n$, which is similar to the one in Proposition 5.6 of [3]. Furthermore, if one goes through Section 5-6 of [3], it can be verified that $GW_l(k)_n \simeq K_l^M(k)$ when $n \geq 3$. Therefore a group homomorphism from the abelian group $K_l^M(k)$ into $k^\times$, gives rise to a coherent determinant for $Comm_l(k^n)^\times$ in the sense of Definition 2.5 and conversely, each coherent determinant for $Comm_l(k^n)^\times$ determines a group homomorphism from $GW_l(k)_n$ into $k^\times$. $\square$

Corollary 2.7. *If a coherent determinant for $Comm_l(k^3)^\times$ in the sense of Definition 2.5 is given, then we are given a coherent system of determinants for $Comm_l(k^n)^\times$ for all n. The converse is trivially true.*

Remark 2.8. *The theorem is clearly false when $n = 1$ in view of Proposition 1.5 with $n = 1$ as there is no coherence condition required for $n = 1$ and Definition 1.4 and Definition 2.5 coincide if $n = 1$. When $n = 2$, $GW_l(k)_2$ may not be isomorphic to $K_l^M(k)$ in general. But, we remark that it is still true that each group homomorphism $K_l^M(k)$ into $k^\times$ gives rise to a coherent determinant for $Comm_l(k^2)^\times$ since a coherent system of determinants for $Comm_l(k^n)^\times$ for all $n \geq 1$ clearly induces a coherent determinant for each $n \geq 1$.*

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