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ON DOUBLE EXACT SEQUENCES

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Abstract. In the present article, we give a description of a chain complex of a cyclic cohomology type proposed by Grayson, which is constructed from the double exact sequences of an exact category. We also provide some results which relate it with motivic cohomology of a smooth scheme over a field.

1. Introduction

A double exact sequence is a pair of bounded acyclic complexes with the same underlying terms. For an exact category $\mathcal{M}$, let $DES(\mathcal{M})$ be a suitably defined exact category of double exact sequences in $\mathcal{M}$. Let us 'mod it out' by the double exact sequences consisting of isomorphic pairs to obtain $DES^{\wedge t}(\mathcal{M})$. Then a double exact sequence of length 1 can be identified with a pair of automorphism $u : A \to A$ and the identity map of A, for some object A in $\mathcal{M}$.

On the other hand, let $Aut\mathcal{M}$ be the category of pairs (A, θ) where $A \in \mathcal{M}$ and θ is an automorphism of A. $Aut\mathcal{M}$ can be then naturally considered as an exact category. Then we have a functor $Aut\mathcal{M} \to \underline{Func}(\mathbb{Z}, \mathcal{M})$, where (A, θ) is sent to the functor $\mathbb{Z} \to \mathcal{M}$ which sends the object $*$ to A and a morphism $i \in \mathbb{Z}$ to θ^i. This induces a functor $Q(Aut\mathcal{M}) \to Q(\underline{Func}(\mathbb{Z}, \mathcal{M})) \to \underline{Func}(\mathbb{Z}, Q\mathcal{M})$ and thus gives rise to a map of simplicial sets $NQ(Aut\mathcal{M}) \to \underline{Hom}(N\mathbb{Z}, NQ(\mathcal{M}))$. But, the geometric realization $B\mathbb{Z}$ of $N\mathbb{Z}$ is homotopy equivalent to the circle S^1 and thus we have a continuous map $BQ(Aut\mathcal{M}) \to \Omega BQ(\mathcal{M})$. Applying π_1 to both sides of the arrow, we get a homomorphism $K_0(Aut\mathcal{M}) \to K_1(\mathcal{M})$. But the elements of $K_0(Aut\mathcal{M})$ of the form $(A, \theta_1\theta_2) - (A, \theta_1) -$

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(A, θ_2) vanish under this map. Hence, $K_0(Aut\mathcal{M})$ modulo the equivalence relations generated by $(A, \theta_1\theta_2) \sim (A, \theta_1) + (A, \theta_2)$ is mapped naturally to $K_1(\mathcal{M})$ (See §5 of [1]). In fact, it is an isomorphism if $\mathcal{M}$ is semisimple, i.e., when every short exact sequence in $\mathcal{M}$ splits ([9]). For example, it is so when $\mathcal{M}$ is the category of finitely generated projective modules over a ring. In any case, a double exact sequence of length 1 naturally gives rise to an element of $K_1(\mathcal{A})$.

Furthermore, a double exact sequence of length 2 concentrated on degrees 0,1 and 2 form an additive full subcategory $DSES(\mathcal{M})$ of double short exact sequences, which is itself an exact category, whose Grothendieck group $K_0(DSES(\mathcal{M}))$ also maps onto $K_1(\mathcal{M})$ with the kernel generated by relations which can be explicitly described as in [5]. These observations provide us motivations to study double exact sequences.

If we iterate the process to obtain $DES^{\wedge t}(\mathcal{M})$ and take a chain complex which resembles a cyclic cohomology of exactly weight t, then we have a cohomology theory which may extend the motivic cohomology construction of Goodwillie and Lichtenbaum given in [4]. In this article, we give a description of this construction using double exact sequences and relate it with K_1 of a ring and motivic cohomology of a smooth scheme over a field.

2. Exact Categories and Double exact sequences

We begin by recalling a definition of an exact category which is given in [7]. Suppose that we are given an additive category $\mathcal{M}$ and a class $\mathcal{E}$ of sequences of the form

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{j} M'' \longrightarrow 0 , \tag{2.1}$$

called the short exact sequences of $\mathcal{M}$, where $i : M' \to M$ and $j : M \to M''$ are morphisms of $\mathcal{M}$. If i occurs as the first arrow of a sequence in $\mathcal{E}$, then i is called an admissible monomorphism and typically denoted by an arrow $\rightarrowtail$ and if j occurs as the second arrow of a sequence in $\mathcal{E}$, then it is called an admissible epimorphism which is denoted by an arrow of the form $\twoheadrightarrow$

Definition 2.1. *An additive category $\mathcal{M}$ given with a family $\mathcal{E}$ as above is called an exact category if the following properties hold.*
(i) A sequence of the form (2.1) in $\mathcal{M}$ which is isomorphic to a sequence

in $\mathcal{E}$ as a cochain complex is again in $\mathcal{E}$.
(ii) *For any objects M' and M'' of $\mathcal{M}$, the sequence*

$$0 \longrightarrow M' \xrightarrow{(id_{M'},0)} M' \oplus M'' \xrightarrow{pr_2} M'' \longrightarrow 0$$

is in $\mathcal{E}$.
(iii) *For any sequence of the form (2.1) in $\mathcal{E}$, i is a kernel for j and j is a kernel for i in the additive category $\mathcal{M}$.*
(iv) *The family of admissible epimorphisms is closed under composition and under base change (pullback). Dually, the family of admissible monomorphisms is closed under composition and under cobase change (pushout).*
(v) *If a map $M \to M''$ has a kernel in $\mathcal{M}$ and there is a map $N \to M$ in $\mathcal{M}$ such that the composite $N \to M \to M''$ is an admissible epimorphism, then $M \to M''$ is an admissible epimorphism. Dually, if a map $M' \to M$ has a cokernel in $\mathcal{M}$ and a composite $M' \to M \to N$ is an admissible monomorphism, then $M' \to M$ is an admissible monomorphism.*

We also record the following result about an exact category given in [7].

PROPOSITION 2.2. *An additive category $\mathcal{M}$ can be given a class $\mathcal{E}$ such that it is an exact category if and only if $\mathcal{M}$ can be embedded as a full subcategory of an abelian category $\mathcal{A}$ which is essentially closed under extension, i.e., if an object A of $\mathcal{A}$ has a subobject A' such that A' and A/A' are isomorphic to objects of $\mathcal{M}$, then A is also isomorphic to an object of M.*

For a proof of 'only if' part, one may take $\mathcal{A}$ to be the additive category whose objects are additive contravariant left exact functors from $\mathcal{M}$ into the category of abelian groups, which can be actually shown to be an abelian category. Note that the Yoneda functor h which sends an object A of $\mathcal{M}$ to the functor $Hom_{\mathcal{M}}(\ ,A)$ gives an embedding of M as a full subcategory of the abelian category $\mathcal{A}$, which is essentially closed under extension. Note also that a sequence of the form (2.1) is in $\mathcal{E}$ if and only if it is carried to an exact sequence in $\mathcal{A}$ by h.

For a given exact category $\mathcal{M}$, there are several equivalent methods which give rise to the K-theory of $\mathcal{M}$. The first method is called the Q-construction and was invented by Quillen ([7]).

Definition 2.3. *For an exact category $\mathcal{M}$, $Q\mathcal{M}$ is the category whose objects are same as the objects of $\mathcal{M}$ and whose morphisms from M to M' are the diagrams of the form*

$$M \underset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M'$$

The composition of a morphism $M \underset{j}{\twoheadleftarrow} N \overset{i}{\rightarrowtail} M'$ from M to M' and a morphism $M' \underset{j'}{\twoheadleftarrow} N' \overset{i'}{\rightarrowtail} M''$ from M' to M'' in $Q\mathcal{M}$ is given by the following diagram.

$$\begin{tikzcd}
N \times_{M'} N' \arrow[r, tail, "pr_2"] \arrow[d, "pr_1"] & N' \arrow[r, tail, "i'"] \arrow[d, "i"] & M'' \\
N \arrow[r, tail, "i"] \arrow[d, "j"] & M' & \\
M & &
\end{tikzcd}$$

For an arbitrary small category $\mathcal{C}$, we may define a simplicial set $N\mathcal{C}$, called 'the nerve of $\mathcal{C}$', whose n-simplex x is a diagram in $\mathcal{C}$ of the form

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n.$$

The i-th face $\partial_i(x)$ $(i = 0, \dots, n)$ of the above simplex is defined to be the $(p-1)$-simplex

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_{i+1} \circ f_i} A_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} X_n,$$

and its i-th degeneracy $\sigma_i(x)$ $(i = 0, \dots, n)$ is the $(p+1)$-simplex

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} X_i \xrightarrow{id} X_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} X_n.$$

The geometric realization $|N\mathcal{C}| = \coprod_{n \geq 0} N\mathcal{C}_n \times \Delta^n / \sim$, where $\sim$ is the equivalence relation generated by $(\sigma_i(x), y) \sim (x, \sigma^i(y))$ and $(\partial_i(x), y) \sim (x, \partial^i(y))$, is a topological space which is called the classifying space $B\mathcal{C}$ of the small category $\mathcal{C}$.

Definition 2.4. *The n-th K-theory of an exact category $\mathcal{M}$ is defined to be the $(n+1)$-th homotopy group*

$$K_n(\mathcal{M}) = \pi_{n+1}(BQ\mathcal{M}).$$

For example, if X is a scheme, $K_n(X)$ is defined to be $\pi_{n+1}(BQ\mathcal{P}(X))$, where $\mathcal{P}(X)$ is the category of locally free sheaves of finite rank over X.

Another method, called the S-construction by Waldhausen ([10]), is defined for any categories with cofibrations of which an exact category is an example, is a slight generalization of Q-construction of Quillen and uses a similar procedure to define K-theory, but $N(Q\mathcal{M})$ is replaced by a simplicial set $S\mathcal{M}$ which is naturally homotopy equivalent to $N(Q\mathcal{M})$ via edgewise subdivision.

Let $[n]$ denote the ordered set whose elements are the integers $0, 1, \ldots, n$. Then $Ar[n]$ is the arrow category of $[n]$, i.e., the category whose objects are the ordered pair j/i whenever $j \geq i$ in $[n]$. There exists a unique morphism from j/i to j'/i' if and only if $j \leq j'$ and $i \leq i'$. A sequence of the form $j/i \to k/i \to k/j$ is declared to be an exact sequence in $Ar[n]$. For an exact category $\mathcal{M}$, $S\mathcal{M}$ is the simplicial set whose n-simplicies are exact functors from $Ar[n]$ to $\mathcal{M}$, i.e., it is a functor $M : Ar[n] \to \mathcal{M}$ such that
(i) $M_{i/i} = 0$ for every $i = 0, 1, \ldots, n$, and
(ii) $0 \to M_{j/i} \to M_{k/i} \to M_{k/j} \to 0$ is a short exact sequence in $\mathcal{M}$.

An element of $S_n(\mathcal{M})$ can be thought of as a sequence of admissible monomorphisms $0 = M_0 \rightarrowtail M_1 \rightarrowtail \cdots \rightarrowtail M_n$, where $M_i = M_{i/0}$, together with choices $M(j/i)$ for all 'quotients' M_j/M_i, $0 < i \leq j$. The face and degeneracy maps are forgetting and duplicating an M_i, respectively, except that we factor out by M_1 in case M_0 is forgotten.

In [2], Gillet and Grayson introduced a simplicial set ΩX for any simplicial set X and defined the simplicial set $G\mathcal{M}$ by $G\mathcal{M} = \Omega S\mathcal{M}$. More specifically, an n-simplex in $G\mathcal{M}$ is a pair of diagrams in $\mathcal{M}$ of the form ([5])

$$
\begin{tikzcd}
 & & & & P_{n/n-1} \\
 & & & & \vdots \arrow[u] \\
 & & P_{2/1} \arrow[r] & \cdots \arrow[r] & P_{n/1} \arrow[u] \\
 & P_{1/2} \arrow[r] & P_{2/0} \arrow[r] \arrow[u] & \cdots & P_{n/0} \arrow[u] \\
P_0 \arrow[r] & P_1 \arrow[r] \arrow[u] & P_2 \arrow[r] \arrow[u] & \cdots \arrow[r] & P_n \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & & Q_{n/n-1} \\
 & & & & \vdots \arrow[u] \\
 & & Q_{2/1} \arrow[r] & \cdots \arrow[r] & Q_{n/1} \arrow[u] \\
 & Q_{1/2} \arrow[r] & Q_{2/0} \arrow[r] \arrow[u] & \cdots & Q_{n/0} \arrow[u] \\
Q_0 \arrow[r] & Q_1 \arrow[r] \arrow[u] & Q_2 \arrow[r] \arrow[u] & \cdots \arrow[r] & Q_n \arrow[u]
\end{tikzcd}
$$

subject to the following three conditions.
(i) $P(j/i) = Q(j/i)$ for every $j \geq i$ in $[n]$ and the maps $P_{j/i} \to P_{j+1/i}$ and $P_{j/i} \to P_{j/i+1}$ are equal to the maps $Q_{j/i} \to Q_{j+1/i}$ and $Q_{j/i} \to Q_{j/i+1}$, respectively.
(ii) All the squares in both diagrams are commutative.
(iii) All sequences of the form $0 \to P_i \to P_j \to P_{j/i} \to 0$, $0 \to Q_i \to Q_j \to Q_{j/i} \to 0$ and $0 \to P_{j/i} \to P_{k/i} \to P_{k/j} \to 0$ are short exact sequences in $\mathcal{M}$.

The i-th face map is deleting all the objects with indices containing i. Then the geometric realization $|G\mathcal{M}|$ is shown to be homotopy equivalent to the loop space $\Omega|S\mathcal{M}|$ ([2]), thus we have

$$K_n(\mathcal{M}) = \pi_n(|G\mathcal{M}|).$$

Given an exact category $\mathcal{M}$, we consider it as a full subcategory of an abelian category $\mathcal{A}$ via Yoneda embedding in Proposition 2.2 such that a sequence in $\mathcal{M}$ is exact if and only if it is exact as a sequence in $\mathcal{A}$

Definition 2.5. *A double exact sequence in $\mathcal{M}$ is a pair (A^*, A'^*) of bounded acyclic cochain complexes in $Ch_b^*(\mathcal{A})$ with terms in $\mathcal{M}$, where*

$$A^* : \qquad \cdots \longrightarrow 0 \longrightarrow A^k \xrightarrow{d^k} \cdots \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} A^n \xrightarrow{d^n} 0 \longrightarrow \cdots$$

and

$$A'^* : \qquad \cdots \longrightarrow 0 \longrightarrow A^k \xrightarrow{d'^k} \cdots \xrightarrow{d'^{-1}} A^0 \xrightarrow{d'^0} A^1 \xrightarrow{d'^1} \cdots \xrightarrow{d'^{n-1}} A^n \xrightarrow{d'^n} 0 \longrightarrow \cdots$$

have the same i-th term for each index $i \in \mathbb{Z}$.

The category $DES(\mathcal{M})$ has double exact sequences as objects. A morphism f^* from a double exact sequence (A^*, A'^*) to another (B^*, B'^*) is a collection maps $f^i : A^i \to B^i$ such that $f^* : A^* \to B^*$ and $f^* : A'^* \to B'^*$ are both maps of cochain complexes, i.e., $f^{i+1}d^i = d^i f^i$ and $f^{i+1}d'^i = d'^i f^i$ for every index $i \in \mathbb{Z}$.

A sequence $0 \longrightarrow (A^*, A'^*) \xrightarrow{f^*} (B^*, B'^*) \xrightarrow{g^*} (C^*, C'^*) \longrightarrow 0$ of morphisms in $DES(\mathcal{M})$ is declared to be a short exact sequence if

$$0 \longrightarrow A^* \xrightarrow{f^*} B^* \xrightarrow{g^*} C^* \longrightarrow 0$$

is a short exact sequences in $DES(\mathcal{M})$.

If we embed $\mathcal{M}$ as a full subcategory of an abelian category $\mathcal{A}$ as in Proposition 2.2, then $DES(\mathcal{M})$ can be considered as an additive full subcategory of the abelian category $Ch_b^*(\mathcal{A}) \times Ch_b^*(\mathcal{A})$, the product of the categories of the bounded cochain complexes in $\mathcal{A}$, which is closed under taking extensions. (c.f., by long exact sequence of cohomology groups). Hence, $DES(\mathcal{M})$ is an exact category with the short exact sequences prescribed as above.

Therefore, one can iterate this construction and we write $DES^0(\mathcal{M}) = \mathcal{M}$ and $DES^{t+1}(\mathcal{M}) = DES(DES^t(\mathcal{M}))$ for $t \geq 0$. Then we may naturally regard $DES^t(\mathcal{M})$ as an additive full subcategory of the pairs of t-cubic cochain complexes $(i_1, \dots, i_t) \mapsto C^{i_1,\dots,i_t}$ which give rise to acyclic cochain complexes once one vary only one index while fixing the other $t-1$ indices. From this identification and using the diagonal embedding $\mathcal{M} \to DES(\mathcal{M})$, we may construct a cube of exact categories in the sense of [3], and consequently a multisimplicial exact category, the Grothendieck group of which we denote by $K_0(DES^{\wedge t}(\mathcal{M}))$.

In particular, $K_0(DES^{\wedge 1}(\mathcal{M}))$ is a quotient of $K_0(DES(\mathcal{M}))$ by the subgroup generated by the isomorphism classes of pairs (A^*, A^*) of the same acyclic cochain complexes.

There is an exact category $DSES(\mathcal{M})$ which consists of only double short exact sequences in $\mathcal{M}$ ([6]). Clearly, the natural embedding of $DSES(\mathcal{M})$ as a full subcategory of $DES(\mathcal{M})$ is an exact functor. $K_0(DSES^{\wedge 1}(\mathcal{M}))$ is then a quotient of $K_0(DSES(\mathcal{M}))$ by the subgroup generated by short double exact sequences (A^*, A'^*) where A^* and A'^* are the same short exact sequences in $\mathcal{M}$. The following result can be found in [6].

Proposition 2.6. *We have an epimorphism* $K_0(DSES^{\wedge 1}(\mathcal{M})) \twoheadrightarrow K_1(\mathcal{M})$ *for any exact category* $\mathcal{M}$.

Proof. We will only describe the map in our proposition and refer the reader to [6] for a detailed proof.

For a double short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ and $0 \longrightarrow A \xrightarrow{f'} B \xrightarrow{g'} C \longrightarrow 0$ in $\mathcal{M}$, we may associate an edge $e(l)$ from (A, A) to (B, B) in the G-construction $G\mathcal{M}$, which is represented by the following pair of diagrams in $\mathcal{M}$.

$$\begin{array}{ccc} & & C \\ & & \uparrow{\scriptstyle g} \\ A & \xrightarrow{f} & B \end{array} \qquad \begin{array}{ccc} & & C \\ & & \uparrow{\scriptstyle g'} \\ A & \xrightarrow{f'} & B \end{array}$$

Also, for each $A \in \mathcal{M}$, we construct the 'standard' edge $e(A)$ from $(0, 0)$ to (A, A), which is represented by the pair of diagrams.

$$\begin{array}{ccc} & & A \\ & & \| \\ 0 & \longrightarrow & A \end{array} \qquad \begin{array}{ccc} & & A \\ & & \| \\ 0 & \longrightarrow & A \end{array}$$

Then the map in our proposition is obtained by sending a double short exact sequence to the loop around $(0, 0)$ in $G\mathcal{M}$ which is represented by $e(B)^{-1}e(l)e(A)$. $\square$

3. A chain complex associated with double exact sequences

In this section, we introduce a chain complex which is modeling cyclic cohomology of weight exactly t for a local ring.

When R is a (commutative) ring, let $R\Delta^d$ be the R-algebra

$$R\Delta^d = R[T_0, \dots, T_d]/(T_0 + \cdots + T_d - 1),$$

for each $d \geq 0$, which is non-canonically isomorphic to a polynomial ring with d indeterminates over R. We denote by **Ord** the category of finite nonempty ordered sets and by $[d]$ where d is a nonnegative integer the object $\{0 < 1 < \cdots < d\}$. Given a map $\varphi : [d] \to [e]$ in **Ord**, the map $\varphi^* : R\Delta^e \to R\Delta^d$ is defined by $\varphi^*(T_j) = \sum_{\varphi(i)=j} T_i$. The map φ^* gives us a simplicial ring $R\Delta^\bullet$. It is connected since, for every $a \in R$, the faces of the 1-simplex aT_0 are $\partial_0(aT_0) = 0$ and $\partial_1(aT_0) = a$ and thus all the vertices are connected to 0, where ∂_i is the face map induced by the map $[0] \to [1]$ which omits i.

One sees that $R\Delta^\bullet$ is contractible since we have a homotopy $H : \Delta^1 \times R\Delta^\bullet \to R\Delta^\bullet$ from 0 to 1, where Δ^1 is the simplicial set $Hom(\ ,[1])$ and $H_n : \Delta^1_n \times R\Delta^n \to R\Delta^n$, for each $n \geq 0$ is given by $H_n(\varphi, a) = (\varphi^* T_0)a$. The same argument actually shows that any connected simplicial ring is contractible.

By applying the functor $K_0(DES^{\wedge t}(\mathcal{P}(\_))$, we get the simplicial abelian group

$$d \mapsto K_0(DES^{\wedge t}(\mathcal{P}(R\Delta^d)),$$

where $\mathcal{P}(R\Delta^d)$ is the exact category of finitely generated projective $R\Delta^d$-modules.

The associated (normalized) chain complex via Dold-Kan correspondence between the simplicial abelian groups and the nonnegative chain complexes of abelian groups, shifted cohomologically by $-t$, is the DES-chain complex of *weight* t for a regular local ring R.

Definition 3.1. *For a local ring, we define $H^q_{DES}\big(\mathrm{Spec}R, \mathbb{Z}(t)\big)$ to be the $(t-q)$-th homology group of the chain complex $(C(t)_*, d)$, where $C(t)_q = K_0(DES^{\wedge t}(\mathcal{P}(R\Delta^q)))$ and the boundary map $d : C(t)_q \to C(t)_{q-1}$ is given by $d = \sum_{i=0}^{q}(-1)^i\partial_i$. For a smooth scheme X over a field k, we may define $H^q_{DES}\big(X, \mathbb{Z}(t)\big)$ to be the $(t-q)$-th hypercohomology group of the sheaf of complexes $(C(t)_*, d)$ in the big Zariski site Sm/k of smooth schemes over k, where $C(t)$ is the sheaf of complexes associated with the presheaf of complexes $X \mapsto K_0(DES^{\wedge t}(\mathcal{P}(k\Delta^q \times_k X)))$ over Sm/k.*

Note that the category $\mathcal{P}(R)$ of finitely generated projective R-modules is a semi-simple exact category, i.e., every exact sequence splits. For a commutative ring R such that $\mathrm{Spec}(R)$ is connected, let A^* be a bounded acyclic cochain complex in $Ch^*_b(\mathcal{P}(R))$ and let r_i be the rank of the projective module A^i over R (See Proposition 1.3.12 of [8]) and s_i be the rank of $\ker(d^i) \simeq \mathrm{Im}(d^{i-1})$. Then each term A^i can be non-canonically written as a direct sum of two projective modules which are isomorphic to $\ker d^i$ and $\mathrm{Im} d^i$, respectively, and so $r_i = s_i + s_{i+1}$ and we have a unique R-isomorphism between the rank 1 projective R-modules

$$\bigotimes_{i \text{ even}} \left(\wedge^{r_i} A^i\right) \xrightarrow{\sim} \bigotimes_{i \text{ odd}} \left(\wedge^{r_i} A^i\right),$$

such that the element of the form

$$\bigotimes_{i \text{ even}} \Big((d_{i-1}a_1^{i-1} \wedge \cdots \wedge d_{i-1}a_{s_i}^{i-1}) \wedge (b_1^i \wedge \cdots \wedge b_{s_{i+1}}^i) \Big),$$

where $a_1^{i-1}, \ldots, a_{s_i}^{i-1}$ are elements of A^{i-1} and $b_1^i, \ldots, b_{s_{i+1}}^i$ are elements of A^i for each even i, is sent to

$$\bigotimes_{i \text{ even}} (-1)^{s_{i+1}s_{i+2}} \Big(a_1^{i+1} \wedge \cdots \wedge a_{s_i}^{i+1}) \wedge (d_i b_1^i \wedge \cdots \wedge d_i b_{s_{i+1}}^i) \Big).$$

Therefore, an acyclic cochain complex A^* in $Ch_b^*(\mathcal{P}(R))$ gives a unique R-isomorphism which is called the determinant of A^*:

$$\det(A^*): \ R \xrightarrow{\sim} \left(\bigotimes_{i \text{ even}} \big(\wedge^{r_i} A^i\big) \right)^{-1} \otimes \bigotimes_{i \text{ odd}} \big(\wedge^{r_i} A^i\big).$$

This argument, even if $\mathrm{Spec}(R)$ is not connected, can be carried out on each Zariski connected component of $\mathrm{Spec}(R)$ and we may define the determinant of any acyclic cochain complex A^* in $Ch_b^*(\mathcal{P}(R))$ by taking the product of determinants on all connected components of $\mathrm{Spec}(R)$.

Proposition 3.2. *For an arbitrary commutative ring R, we have a homomorphism $H^1_{DES}\big(\mathrm{Spec}R, \mathbb{Z}(1)\big)$ into $K_1(R)$ given by $(A^*, A'^*) \mapsto \det(A'^*)^{-1}\det(A^*)$.*

Proof. For a double exact sequence (A^*, A'^*) in $DES(\mathcal{P}(R))$, both A^* and A'^* give R-isomorphisms from R onto $\left(\bigotimes_{i \text{ even}} \big(\wedge^{r_i} A^i\big) \right)^{-1} \otimes \bigotimes_{i \text{ odd}} \big(\wedge^{r_i} A^i\big)$ since both A^* and A'^* have the same terms in $\mathcal{P}(R)$. So, $\det(A'^*)^{-1}\det(A^*)$ gives rise to an R-isomorphism from R onto R and thus the image of 1 is a unit of R, which we denote also by $\det(A'^*)^{-1}\det(A^*)$. Note also that this map is well-defined on $K_0(DES^{\wedge 1}(\mathcal{P}(R))$.

Since an invertible element of a polynomial ring $R[T]$ must be an invertible element of R, two elements $\partial_0(A^*, A'^*)$ and $\partial_1(A^*, A'^*)$ of $K_0(DES^{\wedge 1}(\mathcal{P}(R))$ whenever $(A^*, A'^*) \in K_0(DES^{\wedge 1}(\mathcal{P}(R\Delta^1))$ give rise to the same element of $R^\times$ and this proves the proposition. □

Corollary 3.3. *When R is a local ring, we have an epimorphism*

$$H^1_{DES}\big(\mathrm{Spec}R, \mathbb{Z}(1)\big) \twoheadrightarrow K_1(R).$$

Proof. For a local ring R, we have $K_1(R) \simeq R^\times$. The map
$$R^\times \to H^1_{DES}\big(\mathrm{Spec}R,\ \mathbb{Z}(1)\big),$$
where $a \in R^\times$ is sent to an element represented by the double complex $R \xrightarrow{a} R$ and $R = R$ which is concentrated on degree 0 and 1, gives a right inverse to the map stated in Proposition 3.2. □

THEOREM 3.4. *For a scheme $X \in Sm/k$, we have a homomorphism from the motivic cohomology $H^q_{\mathcal{M}}\big(X,\ \mathbb{Z}(t)\big)$ to $H^q_{DES}\big(X,\ \mathbb{Z}(t)\big)$.*

Proof. We may assume that $X = \mathrm{Spec}\, R$, where R is a localization of the scheme X at a point. In [11], it is shown that the motivic cohomology is isomorphic to the $(t-q)$-th homology group of the chain complex associated, via the Dold-Kan correspondence, with the simplicial abelian group
$$d \mapsto K_0(R\Delta^d, \mathbb{G}_m^{\wedge t}),$$
where $K_0(R\Delta^d, \mathbb{G}_m^{\wedge t})$ is a quotient of the Grothedieck group of the exact category $\mathcal{P}(R, \mathbb{G}_m^t)$, whose objects $(P, \theta_1, \dots, \theta_t)$ consists of a finitely generated projective R-module P and commuting automorphisms $\theta_1, \dots, \theta_t$ on P, by the subgroup generated by those objects $(P, \theta_1, \dots, \theta_t)$ where $\theta_i = id_P$ for some i. A t-tuple $(\theta_1 \dots, \theta_t)$ of commuting automorphisms of a projective module P over $R\Delta^d$ may be considered as the pair of the t-cubic acyclic complex which is given by the product of the acyclic complexes $0 \longrightarrow P \xrightarrow{\theta_i} P \longrightarrow 0$ with P as 0-th and 1-st terms for $i = 1, \dots, t$ and the t-cubic acyclic complex which is the t-fold product of $0 \longrightarrow P = P \longrightarrow 0$. Therefore, it produces an element of $K_0(DES^{\wedge t}(\mathcal{P}(R\Delta^d)))$ and this construction gives rise to a simplicial map from the simplicial set $d \mapsto K_0(R\Delta^d, \mathbb{G}_m^{\wedge t})$ to the simplicial set $d \mapsto K_0(DES^{\wedge t}(\mathcal{P}(R\Delta^d))$. Therefore, we obtain the map stated in the theorem. □

References

[1] S. M. Gersten. Higher K-theory of rings. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf. Seattle Res. Center, Battelle Memorial Inst., 1972)*, pages 3–42. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[2] Henri Gillet and Daniel R. Grayson. The loop space of the Q-construction. *Illinois J. Math.*, 31:574–597, 1987.

[3] Daniel R. Grayson. Adams operations on higher K-theory. *K-Theory*, 6:97–111, 1992.

[4] Daniel R. Grayson. Weight filtrations via commuting automorphisms. *K-Theory*, 9:139–172, 1995.

[5] A. Nenashev. K_1 by generators and relations. *J. Pure Appl. Algebra*, 131(2):195–212, 1998.

[6] Alexander Nenashev. Double short exact sequences and K_1 of an exact category. *K-Theory*, 14(1):23–41, 1998.

[7] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[8] Jonathan Rosenberg. *Algebraic K-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[9] Clayton Sherman. Group representations and algebraic K-theory. In *Algebraic K-theory, Part I (Oberwolfach, 1980)*, volume 966 of *Lecture Notes in Math.*, pages 208–243. Springer, Berlin, 1982.

[10] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.

[11] Mark Walker. *Motivic complexes and the K-theory of automorphisms*. Thesis, University of Illinois, 1996.

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