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EXISTENCE OF NONTRIVIAL SOLUTIONS OF THE NONLINEAR BIHARMONIC SYSTEM

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Abstract. We investigate the existence of nontrivial solutions of the nonlinear biharmonic system with Dirichlet boundary condition

$$\begin{aligned} &\Delta^2\xi + c\Delta\xi = \mu h(\xi+\eta) \quad \text{in } \Omega,\\ &\Delta^2\eta + c\Delta\eta = \nu h(\xi+\eta) \quad \text{in } \Omega, \end{aligned} \tag{0.1}$$

where $c \in R$ and Δ^2 denote the biharmonic operator.

1. Introduction

Let Ω be a smooth bounded region in R^n with smooth boundary $\partial\Omega$. We investigate the existence of nontrivial solutions of the nonlinear biharmonic system with Dirichlet boundary condition

$$\begin{aligned} &\Delta^2\xi + c\Delta\xi = \mu h(\xi+\eta) \quad \text{in } \Omega,\\ &\Delta^2\eta + c\Delta\eta = \nu h(\xi+\eta) \quad \text{in } \Omega,\\ &\xi = 0, \quad \Delta\xi = 0 \quad \text{on } \partial\Omega,\\ &\eta = 0, \quad \Delta\eta = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $c \in R$ and Δ^2 denote the biharmonic operator. Here we assume that $h : R \to R$ is a differentiable function such that $h(0) = 0$ and

$$h'(\infty) = \lim_{|u|\to\infty} \frac{h(u)}{u} \in R.$$

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Let λ_k, $k \geq 1$ denote the eigenvalues and ϕ_k, $k \geq 1$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

In [3, 4, 5] Choi and Jung study the multiplicity of solutions of the nonlinear biharmonic equation

$$\begin{aligned} &\Delta^2 u + c\Delta u = g(u) \quad \text{in } \Omega, \\ &u = 0, \quad \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $c \in R$ and Δ^2 denote the biharmonic operator. Here we assume that $g : R \to R$ is a differentiable function such that $g(0) = 0$ and

$$g'(\infty) = \lim_{|u|\to\infty} \frac{g(u)}{u} \in R.$$

The authors proved in [3] that problem (1.2) has at least two solutions by the Variation of Linking Theorem under the condition that g is a differentiable function with $g(0) = 0$, $\lambda_i < c < \lambda_{i+1}$, $\lambda_{i+1}(\lambda_{i+1} - c) < \lambda_k(\lambda_k - c) < g'(\infty) < \lambda_{k+1}(\lambda_{k+1} - c)$, $\lambda_{k+m}(\lambda_{k+m} - c) < g'(0) < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$ for $m \geq 1$, and $g'(t) \leq \gamma < \lambda_{k+m+1}(\lambda_{k+m+1} - c)$, $k > i+1$. The nonlinear biharmonic equation with jumping nonlinearity was extensively studied by some authors [4, 13, 15]. Choi and Jung studied the following problem in [4]

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu^+ + f \qquad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

They proved that (1.2) has at least two solutions by variational reduction method when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $f = s > 0$, or $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $(k = 1, 2, \ldots)$ and $f = s < 0$. They also investigate a relation between multiplicity of solutions and source term of (1.2) with the nonlinearity crossing an eigenvalue. Tarantello also considered the nonlinear biharmonic equation with jumping nonlinearity, with Dirichlet boundary condition

$$\begin{aligned} &\Delta^2 u + c\Delta u = b[(u+1)^+ - 1] \qquad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

She showed by degree theory that if $b \geq \lambda_1(\lambda_1 - c)$, then (1.4) has a negative solution in Ω.

In section 2 we introduce the completed normed space spanned by eigenfunctions of the biharmonic operator and the basic theorem which will play a crucial role in our argument. In section 3 we prove the main theorem.

2. Nontrivial solutions of the nonlinear biharmonic system

Let Ω be a smooth bounded region in R^n with smooth boundary $\partial\Omega$. We consider the multiplicity of solutions of the nonlinear biharmonic equation

$$\begin{aligned} &\Delta^2 u + c\Delta u = g(u) \quad \text{in } \Omega, \\ &u = 0, \quad \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where $c < \lambda_1$ and Δ^2 denote the biharmonic operator. Here we assume that $g : R \to R$ is a differentiable function such that $g(0) = 0$ and

$$g'(\infty) = \lim_{|u|\to\infty} \frac{g(u)}{u} \in R.$$

Let λ_k, $k \geq 1$ denote the eigenvalues and ϕ_k, $k \geq 1$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega,$$

where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

We assume that $c < \lambda_1$. Let us denote an element u in $L^2(\Omega)$ as

$$u = \sum h_k \phi_k, \qquad \sum h_k^2 < \infty.$$

Now, we define a subspace H of $L^2(\Omega)$ as follows

$$H = \left\{ u \in L^2(\Omega) : \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty \right\}.$$

Then this is a complete normed space with a norm

$$|||u||| = \left[\sum |\lambda_k(\lambda_k - c_j)| h_k^2 \right]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have

(i) $$\Delta^2 u + c\Delta u \in H \text{ implies } u \in H.$$

(ii) $$||||u||| \geq C\|u\|_{L^2(\Omega)}, \text{ for some } C > 0.$$

(iii) $$\|u\|_{L^2(\Omega)} = 0 \text{ if and only if } |||u||| = 0.$$

We assume that g is differentiable,

$$g'(0) < \lambda_1(\lambda_1 - c),$$

$$g'(\infty) \in (\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1} - c)),$$

and $0 < g'(t) \leq \gamma < \lambda_{k+1}(\lambda_{k+1} - c)$. From the assumptions of g there exists $a > 0$ such that $|g(u)| \leq a(1 + |u|)$.

LEMMA 2.1. *All solutions in $L^2(\Omega)$ of*

$$\Delta^2 u + c\Delta u = g(u) \qquad \text{in } L^2(\Omega)$$

belong to H.

For the proof of the lemma, see[4].

By the contraction mapping principle we have the uniqueness result:

LEMMA 2.2. *Let $c < \lambda_1$. Then the system*

$$\Delta^2 u + c\Delta u = 0$$

has only the trivial solution in H.

Let us define the functional in H,

$$I(u) = \int_\Omega \frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - G(u),$$

where $G(u) = \int_o^s g(\sigma)d\sigma$. Then $I(u)$ is well defined. The solutions of (2.1) coincide with the critical points of $I(u)$.

PROPOSITION 1. *Assume that $g(u)$ satisfies the conditions of Theorem 1.1. Then $I(u)$ is continuous and Frechét differentiable in H and for $h \in H$*

$$DI(u)(h) = \int_{\Omega} \Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - g(u)h.$$

For the proof of Proposition 1, see [4], [5].

For the sake of completeness we recall that if I is a function of class C^1 and u_0 is a critical point of I, then u_0 is called of mountain pass type if for every open neighborhood U of u_0, $I^{-1}(-\infty, I(u_0)) \cap U \neq \phi$ and $I^{-1}(-\infty, I(u_0)) \cap U \backslash \{u_0\}$ is not pass-connected.

Let V be k dimensional subspace of h spanned by $\phi_1, \cdots, \phi_k$ whose eigenvalues are $\lambda_1(\lambda_1 - c), \cdots, \lambda_k(\lambda_k - c)$. Let W be the orthogonal complement of V in H. Let $P : H \to V$ be the orthogonal projection of H onto V and $I - P : H \to W$ denote that of H onto W. Then every element $u \in L^2(\Omega)$ is expressed by $u = v + z$, $v \in Pu$, $z = (I - P)u$.

Hence (2.1) is equivalent to the system with two unknowns v and z:

$$\Delta^2 v + c\Delta v = P(g(v + z)),$$
$$\Delta^2 z + c\Delta z = (I - P)(g(v + z)).$$

LEMMA 2.3. *Let $c < \lambda_1$. Assume that $g'(0) < \lambda_1(\lambda_1 - c)$, $g'(\infty) \in (\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1} - c))$, and $0 < g'(t) \leq \gamma < \lambda_{k+1}(\lambda_{k+1} - c)$, $k \geq 2$. Then we have :*

(i) *For any fixed $v \in V$ there are $m > 0$ and $\alpha > 1$ such that for all $w \in W$, $w_1 \in W$*

$$(DI(v + w) - DI(v + w_1), w - w_1) \geq m\|w - w_1\|^{\alpha}.$$

(ii) *There exists a unique solution $z \in W$ of the equation*

$$\Delta^2 z + c\Delta z = (I - P)(g(v + z)) \qquad \text{in } W$$

If we put $z = \theta(v)$, then θ is continuous on V and satisfies a uniform Lipschitz condition in v which respect to the L^2 norm(also norm $|||\cdot|||$). Moreover

$$DI(v + \theta(v))(w) = 0 \qquad \text{for all } w \in W,$$

and

$$I(v + \theta(v)) = \min_{w \in W} I(v + w).$$

(iii) *If* $\tilde{I} : V \to R$ *is defined by* $\tilde{I}(v) = I(v + \theta(v))$, *then* $\tilde{I}$ *has a continuous Fréchet derivative* DI *with respect to* v, *and*

$$D\tilde{I}(v)(h) = DI(v + \theta(v))(h) \qquad \text{for all } h \in V.$$

(iv) *If* $v_0 \in V$ *is a critical point of* $\tilde{I}$ *if and only if* $v_0 + \theta(v_0)$ *is a critical point of* I.

(v) *Let* $S \subset V$ *and* $\Sigma \subset H$ *be open bounded regions such that*

$$\{v + \theta(v); v \in S\} = \Sigma \cap \{v + \theta(v); v \in V\}.$$

If $D\tilde{I}(v) \neq 0$ *for* $v \in \partial S$, *then*

$$d(D\tilde{I}, S, 0) = d(DI, \Sigma, 0),$$

where d *denote the Leray-Schauder degree.*

(vi) *If* $u_0 = v_0 + \theta(v_0)$ *is a critical point of mountain pass type of* I, *then* v_0 *is a critical point of mountain pass type of* $\tilde{I}$.

With the above lemma, Choi and Jung [5] showed, by degree theory, the existence of nontrivial solutions of (2.1):

THEOREM 2.1. *Let* $c < \lambda_1$. *If* $g'(0) < \lambda_1(\lambda_1 - c)$, $g'(\infty) \in (\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1} - c))$ *with* $k \geq 2$, *and* $0 < g'(t') \leq \gamma < \lambda_{k+1}(\lambda_{k+1} - c)$. *Then (2.1) has at least three solutions, two of which are nontrivial.*

3. Nontrivial solutions for the system

In this section we investigate the existence of multiple nontrivial solutions (ξ, η) for a perturbation $(\mu+\nu)h(\xi+\eta)$ of the biharmonic system with Dirichlet boundary condition

$$\begin{aligned}
&\Delta^2\xi + c\Delta\xi = \mu h(\xi + \eta) \quad \text{in } \Omega,\\
&\Delta^2\eta + c\Delta\eta = \nu h(\xi + \eta) \quad \text{in } \Omega,\\
&\xi = 0, \quad \Delta\xi = 0 \quad \text{on } \partial\Omega,\\
&\eta = 0, \quad \Delta\eta = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{3.1}$$

where $c \in R$ and Δ^2 denote the biharmonic operator. Here we assume that $h : R \to R$ is a differentiable function such that $h(0) = 0$ and

$$h'(\infty) = \lim_{|u| \to \infty} \frac{h(u)}{u} \in R.$$

THEOREM 3.1. *Let μ, ν be nonzero constants and $\frac{\mu}{\nu} \neq -1$. Let $c < \lambda_1$. Assume that $(\mu+\nu)h'(0) < \lambda_1(\lambda_1 - c)$, $(\mu+\nu)h'(\infty) \in (\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1}-c))$ with $k \geq 2$, and $0 < (\mu+\nu)h'(t) \leq \gamma < \lambda_{k+1}(\lambda_{k+1}-c)$. Then biharmonic system (3.1) has at least three solutions (ξ, η), two of which are nontrivial solutions.*

Proof. From problem (3.1) we get that $\Delta^2\xi + c\Delta\xi = \frac{\mu}{\nu}(\Delta^2\eta + c\Delta\eta)$. By Lemma 2.2, the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = 0 \quad \text{in } \Omega, \\ &u = 0, \quad \Delta u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.2}$$

has only the trivial solution. So the solution (ξ, η) of problem (3.1) satisfies $\xi = \frac{\mu}{\nu}\eta$. On the other hand, from problem (3.1) we get the equation

$$\begin{aligned} &(\Delta^2 + c\Delta)(\xi + \eta) = (\mu+\nu)h(\xi+\eta) \qquad \text{in } \Omega, \\ &\xi = 0, \quad \Delta\xi = 0 \quad \text{on } \partial\Omega, \\ &\eta = 0, \quad \Delta\eta = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.3}$$

Put $w = \xi + \eta$. Then the above equation is equivalent to

$$\begin{aligned} &(\Delta^2 + c\Delta)w = (\mu+\nu)h(\xi+\eta) \quad \text{in} \qquad \text{in } \Omega, \\ &w = 0, \quad \Delta w = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.4}$$

Under the condition of the theorem, if we use Theorem 2.1, the above equation has at least three solutions , two of which are nontrivial solutions, say w_1, w_2. Hence we get the solutions (ξ, η) of problem (3.1) from the following systems:

$$\begin{aligned} &\xi + \eta = 0 \quad \text{in} \quad \Omega, \\ &\xi = \frac{\mu}{\nu}\eta \quad \text{in} \quad \Omega, \\ &\xi = 0, \quad \Delta\xi = 0 \quad \text{on } \partial\Omega, \\ &\eta = 0, \quad \Delta\eta = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\xi + \eta = w_1 \quad \text{in} \quad \Omega, \\ &\xi = \frac{\mu}{\nu}\eta \quad \text{in} \quad \Omega, \\ &\xi = 0, \quad \Delta\xi = 0 \quad \text{on } \partial\Omega, \\ &\eta = 0, \quad \Delta\eta = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.6}$$

$$
(3.7) \qquad \begin{aligned}
&\xi + \eta = w_2 \quad \text{in} \quad \Omega, \\
&\xi = \frac{\mu}{\nu}\eta \quad \text{in}\Omega, \\
&\xi = 0, \quad \Delta\xi = 0 \quad \text{on } \partial\Omega, \\
&\eta = 0, \quad \Delta\eta = 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

From (3.5) we get the trivial solution $(\xi, \eta) = (0, 0)$. From (3.6), (3.7) we get the nontrivial solutions (ξ_1, η_1), (ξ_2, η_2).

Therefore system (3.1) has at least three solutions, two of which are nontrivial solutions. ∎

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