THE PROOF OF THE EXISTENCE OF THE THIRD
SOLUTION OF A NONLINEAR BIHARMONIC
EQUATION BY DEGREE THEORY

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Abstract. We investigate the multiplicity of solutions of the nonlinear biharmonic equation with Dirichlet boundary condition, \( \Delta^2 u + c \Delta u = bu^+ + s \), in \( \Omega \), where \( c \in \mathbb{R} \) and \( \Delta^2 \) denotes the biharmonic operator. We show by degree theory that there exist at least three solutions of the problem.

1. Introduction

Let \( \Omega \) be a bounded set in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). In this paper we study the multiplicity of the solutions of the nonlinear biharmonic equation with Dirichlet boundary condition

\[
\Delta^2 u + c \Delta u = bu^+ + s \quad \text{in} \; \Omega,
\]

\[
u = 0, \quad \Delta u = 0 \quad \text{on} \; \partial \Omega,
\]

where \( u^+ = \max\{u, 0\} \), \( c \in \mathbb{R} \), \( s \in \mathbb{R} \) and \( \Delta^2 \) denotes the biharmonic operator. Equations with nonlinearities of this type have been extensively studied for the second order elliptic operators (cf. [7]). Tarantello [12] also studied this type equation. She showed that if \( \lambda_1 > 0 \) is the first eigenvalue of \( -\Delta \) in \( H^1_0(\Omega) \) and \( c < \lambda_1 \), then the problem

\[
\Delta^2 u + c \Delta u = b[(u + 1)^+ - 1] \quad \text{in} \; \Omega,
\]

\[
u = 0, \quad \Delta u = 0 \quad \text{on} \; \partial \Omega,
\]

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has at least one solution, which is negative if and only if \( b \geq \lambda_1(\lambda_1 - c) \).

Choi and Jung [4] proved that if \( c < \lambda_1, \lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c) \) and \( s < 0 \), or if \( \lambda_1 < c < \lambda_2, b < \lambda_1(\lambda_1 - c) \) and \( s > 0 \), then problem (1.1) has at least two solutions by use of the variational reduction method.

In this paper, we prove that if \( c < \lambda_1, \lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c) \) and \( s < 0 \), then problem (1.1) has at least three solutions by use of the degree theory.

In section 2 we state the main result and in section 3 we prove the main theorem.

2. Statement of main result

Let \( \Omega \) be a bounded set in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). Let \( \lambda_k, k = 1, 2, \ldots \), denote the eigenvalues and \( \phi_k, k = 1, 2, \ldots \), the corresponding eigenfunctions, suitably normalized with respect to \( L^2(\Omega) \) inner product, of the eigenvalue problem

\[
\Delta u + \lambda u = 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\]

where each eigenvalue \( \lambda_k \) is repeated as often as its multiplicity. We recall that \( 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to +\infty \), and that \( \phi_1(x) > 0 \) for \( x \in \Omega \). The eigenvalue problem

\[
\Delta^2 u + c \Delta u = \nu u \quad \text{in } \Omega, \\
u = 0, \quad \Delta u = 0 \quad \text{on } \partial \Omega
\]

has infinitely many eigenvalues

\[
\nu_k = \lambda_k(\lambda_k - c), \quad k = 1, 2, \ldots,
\]

and corresponding eigenfunctions \( \phi_k(x) \). The set of functions \( \{\phi_k\} \) is an orthonormal base for \( L^2(\Omega) \). Let us denote an element \( u \), in \( L^2(\Omega) \), as

\[
u = \sum h_k \phi_k, \quad \sum h_k^2 < \infty.
\]

We define a subspace \( H \) of \( L^2(\Omega) \) as follows

\[
H = \{ u \in L^2(\Omega) | \sum |\lambda_k(\lambda_k - c)|h_k^2 < \infty \}.
\]

Then this is a complete normed space with a norm

\[
||u|| = [\sum |\lambda_k(\lambda_k - c)|h_k^2]^\frac{1}{2}.
\]
Since $\lambda_k \to +\infty$ and $c$ is fixed, we have

1. $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.

2. $\|u\| \geq C\|u\|$ for some $C > 0$.

3. $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$.

For the proof, refer to Choi and Jung [4].

In this paper we consider weak solutions of the boundary value problem

$$
\Delta^2 u + c\Delta u = b u^+ + s \quad \text{in } \Omega,
$$

$$
u = 0, \quad \Delta u = 0 \quad \text{on } \partial \Omega.
$$

A weak solution of (2.1), which is called a solution in $H$, is of the form

$$
u = \sum h_k \phi_k, \quad \Delta^2 u + c\Delta u = \sum \lambda_k (\lambda_k - c) h_k \phi_k \in L^2(\Omega).
$$

For simplicity of notation, a weak solution of (2.1) is characterized by

$$
\Delta^2 u + c\Delta u = b u^+ + s \quad \text{in } H.
$$

Now, we state the main result of this paper, which is a sharp result for the multiplicity of solutions of a nonlinear biharmonic equation.

**Theorem 2.1.** Assume that $c < \lambda_1, \lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$. Then problem (2.2) has at least three solutions.

**3. Proof of theorem 2.1**

For the proof of Theorem 2.1 we need some lemmas.

**Lemma 3.1.** Let $c < \lambda_1, b \geq 0$ and $b \neq \lambda_k(\lambda_k - c), k \geq 1$, Then the problem

$$
\Delta^2 u + c\Delta u = bu^+ \quad \text{in } H
$$

has only the trivial solution.

For the proof, refer to Theorem 1.3 (ii) and Lemma 2.9 in [4].

**Lemma 3.2.** Let $c < \lambda_1, s < 0$ and $\alpha > 0$ be given. Then there exists an $R_0 > 0$ (depending on $s$ and $\alpha$) such that for all $b$ with $\lambda_1(\lambda_1 - c) + \alpha \leq b \leq \lambda_2(\lambda_2 - c) - \alpha$, the solutions $u$ of (2.2) satisfy $\|u\| < R_0$.

**Proof.** If not, then there exists a sequence $(b_n, u_n)$ with $\lambda_1(\lambda_1 - c) + \alpha \leq b_n \leq \lambda_2(\lambda_2 - c) - \alpha$, $\|u\| \to \infty$ such that

$$
u_n = (\Delta^2 + c\Delta)^{-1}(b u_n^+ + s).
$$
The functions $w_n = \frac{u_n}{\|u_n\|}$ satisfy the equation
$$w_n = (\Delta^2 + c\Delta)^{-1}(bw_n^+ + \frac{s}{\|u_n\|}).$$

Now $(\Delta^2 + c\Delta)^{-1}$ is a compact operator. Therefore we may assume that $w_n \to w_0$, $b_n \to b_0$ and $0 < \lambda_1(\lambda_1 - c) < b_0 < \lambda_2(\lambda_2 - c)$. Since $\|w_n\| = 1$, it follows that $\|w_0\| = 1$ and
$$w_0 = (\Delta^2 + c\Delta)^{-1}(bw_0^+) \quad \text{in } H. \quad (3.4)$$

This contradicts Lemma 3.1 and proved the lemma.

**Lemma 3.3.** Under the assumptions and the notations of Lemma 3.2
$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_R, 0) = 1$$
for all $R \geq R_0$, where $d_{LS}$ denotes the Leray-Schauder degree.

**Proof.** Let $R_0$ be such that solutions of
$$u - (\Delta^2 + c\Delta)^{-1}(bu^+ + \lambda s) = 0, \quad 0 \leq \lambda \leq 1,$$
satisfy $\|u\| \leq R_0$. Since the degree is invariant under a homotopy, we get
$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + \lambda s), B_R(0), 0) = d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+), B_R(0), 0)$$
for $R \geq R_0$. The equation
$$u - (\Delta^2 + c\Delta)^{-1}(bu^+) = 0$$
has only the trivial solution $u = 0$ in $B_R(0)$. Thus we have
$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+), B_R(0), 0) = d_{LS}(u, B_R(0), 0) = 1,$$
since the map is simply identity. Now, we will show the existence of the negative solution of (2.2).

**Lemma 3.4.** Assume that $c < \lambda_1$ and $s < 0$. Then problem (2.2) has a negative solution $u_0(x)$.

For the proof, refer to [4].

Now, we consider the local Leray-Schauder degree of $u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s)$ at the negative solution $u_0$ with respect to zero.
Lemma 3.5. Assume that \( c < \lambda_1, \lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c) \) and \( s < 0 \). Then there exists \( d > 0 \) such that

\[
d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_d(u_0), 0) = 1,
\]

(3.5)

where \( u_0 \) is the negative solution of (2.2).

Proof. Since every solution of problem (2.2) is discrete and \( u_0 \) is a solution of (2.2), there exists \( d > 0 \) such that there is no the other solution of (2.2) in the neighborhood \( B_d(u_0) \) of \( u_0 \) with radius \( d \). Then we have

\[
d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_d(u_0), 0) = d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(s), B_d(u_0), 0) = 1,
\]

since the map is simply a translation of the identity and since \( \| (\Delta^2 + c\Delta)^{-1}s \| < d \) by Lemma 3.4.

Next, we will consider the local Leray-Schauder degree of \( u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s) \) at the changing sign solution of (2.2) with respect to zero.

Now, we denote that for given \( u, \chi(u) \) is the characteristic function of the positive set of \( u \), i.e.,

\[
[\chi(u)](x) = \begin{cases} 
1, & \text{if } u(x) > 0, \\
0, & \text{if } u(x) \leq 0.
\end{cases}
\]

We consider the following eigenvalue problem

\[
(\Delta^2 + c\Delta)u = \nu b\chi(u)u \quad \text{in } H,
\]

(3.6)

when \( \mu(\{x | u(x) = 0\}) = 0 \), where \( \mu \) is the Lebesgue measure.

We assume that \( c < \lambda_1 \) and \( \lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c) \). Let

\[
v = \sum h_m\phi_m, \quad Lv = \sum \lambda_m(\lambda_m - c)h_m\phi_m.
\]

For eigenvalues \( \lambda_m(\lambda_m - c), m \geq 1 \), the corresponding eigenvalues \( \nu_m(b\chi(u)) \) are nontrivial solutions of (3.6) and

\[
\nu_1(b\chi(u)) < \nu_2(b\chi(u)) < \ldots \to +\infty.
\]

(3.7)

Then we have the following lemma.
Lemma 3.6. Assume that $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$. Then if $u$ is a solution of (2.2) which changes sign, then

$$0 < \nu_1(b\chi(u)) < 1.$$  

Proof. We know that (2.2) has the negative solution $u_0$. Writing (2.2) for $u$ and $u_0$ and subtracting we get

$$(\Delta^2 + c\Delta)(u - u_0) = bu^+.$$  

If we use the notation $\frac{bu^+}{u - u_0}$, then we have

$$0 \leq \frac{bu^+}{u - u_0} < b\chi(u) \leq b.$$  

By (3.6), $\nu_m(\frac{bu^+}{u - u_0}) = 1$ for some $m \geq 1$ and by (3.8), $m = 1$, i.e., $\nu_1(\frac{bu^+}{u - u_0}) = 1$. Since

$$0 < \nu_1(b\chi(u)) < \nu_1(\frac{bu^+}{u - u_0}) = 1,$$

we obtain the desired result.

The final step in the proof of our theorem is described in

Lemma 3.7. Assume that $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$. Then if $u_*$ is a solution of (2.2) which changes sign, then there exists $\epsilon > 0$ such that

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_\epsilon(u_*), 0) = +1 \quad \text{or} \quad -1.$$  

Proof. Let $u_*$ be a solution of (2.2) which changes sign. Since the solutions of (2.2) are discrete, we can choose small $\epsilon' > 0$ such that $B_{\epsilon'}(u_*)$ does not contain the other solutions of (2.2). Let us choose $u \in B_{\epsilon'}(u_*)$ and set $v = u - u_*$. Then there exists $\epsilon_* < \epsilon'$ such that $u_*$ and $u_* + v$ have same sign, so the following holds:

$$u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s) = (u_* + v) - (\Delta^2 + c\Delta)^{-1}(b(u_* + v)^+ + s)$$

$$= v - (\Delta^2 + c\Delta)^{-1}(b(u_* + v)^+ - bu^+)$$

$$= v - (\Delta^2 + c\Delta)^{-1}(b\chi(u_*)v),$$

where $u \in B_{\epsilon_*}(u_*)$. Thus we have

$$d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_{\epsilon_*}(u_*), 0) = d_{LS}(v - (\Delta^2 + c\Delta)^{-1}(b\chi(u_*)v), B_0(0), 0).$$
The proof of the existence of the third solution

The eigenvalues of the operator \( v - (\Delta^2 + c\Delta)^{-1}(b\chi(u_*)v) \) are connected with the eigenvalues \( \nu \) of the eigenvalue problem \( (\Delta^2 + c\Delta)v = \nu b\chi(u_*)v \) by

\[
v - (\Delta^2 + c\Delta)^{-1}(b\chi(u_*)v) = \rho v \iff (\rho - 1)(\Delta^2 + c\Delta)v = -b\chi(u_*)v
\]
or \( \rho = \frac{\nu - 1}{\nu} \). It follows from Lemma 3.6 and (3.7) that

\[
0 < \nu_1(b\chi(u)) \ldots < \nu_n(b\chi(u)) < 1 < \nu_{n+1}(b\chi(u)) \ldots
\]
and thus there are \((-1)^n\) negative eigenvalues \( \rho \). Thus the desired degree is \(+1\) or \(-1\). So the lemma is proved.

**Proof of Theorem 2.1.**

The equation (2.2) can be written in the form

\[
u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s) = 0.
\]
The degree of \( u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s) \) on a large ball of radius \( R > R_0 \) is \(+1\) by Lemma 3.3. From Lemma 3.4 and 3.5, the constant sign solution of (2.2) is only the negative solution \( u_0 \) and the degree on the small neighborhood \( B_d(u_0) \) is \(+1\). Choi and Jung [4] proved that under the same assumptions of Theorem 2.1, there exists another solution of problem (2.2) which changes sign. If \( u_* \) is a solution of (2.2) which changes sign, then from Lemma 3.7, the degree on the ball \( B_{r_*}(u_*)) \) is \(+1\) or \(-1\). Choosing \( R > R_0 \) so that \( B_R \) contains all solutions of (2.2), we can conclude that

\[
d_{LS}(u - (\Delta^2 + c\Delta)^{-1}(bu^+ + s), B_R \setminus \{B_d(u_0) \cup B_{r_*}(u_*)\}, 0) = -1 \quad \text{or} \quad +1.
\]
Thus there exist at least three solutions in \( B_R \).

**References**


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