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ON THE STABILITY OF A CAUCHY-JENSEN FUNCTIONAL EQUATION III

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation

$$2f(x+y, \frac{z+w}{2}) = f(x,z) + f(x,w) + f(y,z) + f(y,w)$$

in the spirit of P.Găvruta.

1. Introduction

In 1940, S.M.Ulam [11] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D.H.Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th.M.Rassias [10] gave a generalization. Recently, P.Găvruta [1] also obtained a further generalization of the Hyers-Ulam result in the following theorem.

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THEOREM 1.1. *Let X be a vector space, let Y a Banach space and let $\varphi : X \times X \to [0, \infty)$ be a function satisfying*

$$\psi_(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in X$. If a function $f : X \to Y$ satisfies the functional inequality $\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y), \quad x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ which satisfies

$$\|f(x) - T(x)\| \le \psi(x, x)$$

for all $x \in X$.

Since then, a further generalization of the Hyers-Ulam theorem has been extensively investigated by a number of mathematicians[5,7,8].

Throughout this paper, let X be a real vector space and Y a Banach space. A mapping $g : X \to Y$ is called a Cauchy mapping (respectively, a Jensen mapping) if g satisfies the functional equation $g(x + y) = g(x) + g(y)$ (respectively, $2g(\frac{x+y}{2}) = g(x) + g(y)$). For a given mapping $f : X \times X \to Y$, we define

$$Cf(x, y, z, w) := 2f(x+y, \frac{z + w}{2}) - f(x, z) - f(x, w) - f(y, z) - f(y, w)$$

for all $x, y, z, w \in X$. A mapping $f : X \times X \to Y$ is called a Cauchy-Jensen mapping if f satisfies the functional equation

$$Cf(x, y, z, w) = 0$$

for all $x, y, z, w \in X$ and the functional equation $Cf = 0$ is called a Cauchy-Jensen functional equation. In 2006, Park and Bae [9] obtained the generalized Hyers-Ulam stability of the Cauchy-Jensen functional equation. The authors[3] obtained the stability of the Cauchy-Jensen functional equation in the spirit of Th.M.Rassias in the following theorem.

THEOREM 1.2. *Let $p, q \neq 1$, $p, q \geq 0$ and $\theta > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Cf(x, y, z, w)\| \leq \theta(\|x\|^p + \|y\|^p)(\|z\|^q + \|w\|^q)$$

for all $x, y, z, w \in X$. Then there exist a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \frac{2\theta}{|2 - 2^p|}\|x\|^p\|y\|^q$$

for all $x, y \in X$.

In this paper, we investigate the generalized Hyers-Ulam stability of a Cauchy-Jensen functional equation. We have better stability results than those of Park and Bae[9].

2. Stability of a Cauchy-Jensen mapping.

THEOREM 2.1. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\sum_{j=0}^{\infty} 2^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}, z, w) < \infty$$

for all $x, y, z, w \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|Cf(x, y, z, w)\| \leq \varphi(x, y, z, w) \tag{1}$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \sum_{j=0}^{\infty} 2^{j-1} \varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, y, y) \tag{2}$$

for all $x, y \in X$. The mapping $F : X \times X \to Y$ is given by

$$F(x, y) := \lim_{n \to \infty} 2^n f(\frac{x}{2^n}, y)$$

for all $x, y \in X$.

Proof. From $\varphi(0,0,0,0) = 0$ and (1), we have $f(0,0) = 0$. Since

$$\|2^j f(\frac{x}{2^j}, y) - 2^{j+1} f(\frac{x}{2^{j+1}}, y)\| \le 2^{j-1}\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, y, y)$$

for all $x, y \in X$ and $n \in \mathbb{N}$, we get

$$\|2^l f(\frac{x}{2^l}, y) - 2^m f(\frac{x}{2^m}, y)\| \le \sum_{j=l}^{m-1} 2^{j-1}\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, y, y) \tag{3}$$

for all $x, y \in X$ and integers l, m $(0 \le l < m)$. Since Y is complete and the sequence $\{2^j f(\frac{x}{2^j}, y)\}$ is a Cauchy sequence for all $x, y \in X$, the sequence $\{2^j f(\frac{x}{2^j}, y)\}$ converges for all $x, y \in X$. Define a map $F : X \times X \to Y$ by

$$F(x, y) = \lim_{n\to\infty} 2^n f(\frac{x}{2^n}, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (3), one can obtain the inequality

$$\|f(x, y) - F(x, y)\| \le \sum_{j=0}^{\infty} 2^{j-1}\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, y, y)$$

for all $x, y \in X$. Since

$$CF(x, y, z, w) = \lim_{n\to\infty} 2^n Cf(\frac{x}{2^n}, \frac{y}{2^n}, y, y) = 0$$

for all $x, y, z, w \in X$, F is a Cauchy-Jensen mapping satisfying (2). Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (2), we have

$$\begin{aligned}\|F(x, y) - F'(x, y)\| &\le \|2^n(F - F')(\frac{x}{2^n}, y)\| \\ &\le 2^n\|(F - f)(\frac{x}{2^n}, y)\| + 2^n\|(f - F')(\frac{x}{2^n}, y)\| \\ &\le \sum_{j=n}^{\infty} 2^{j-1}\varphi(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}, y, y)\end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Taking $n \to \infty$, we have $F(x, y) = F'(x, y)$ as desired. □

As an application of Theorems 2.1, we have the stability result for the case $p > 1$ in the sense of Th.M.Rassias.

Corollary 2.2. *Let X be a normed space and let p, q, θ be non-negative real numbers with $p > 1$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Cf(x, y, z, w)\| \leq \theta(\|x\|^p + \|y\|^p)(\|z\|^q + \|w\|^q)$$

for all $x, y, z, w \in X$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x, y) - F(x, y)\| \leq \frac{2\theta}{2^p - 2}\|x\|^p\|y\|^q$$

for all $x, y \in X$.

3.Stability of a Cauchy-Jensen mapping on the punctured domain.

We need the following lemma(Lemma 3.1 in [3]) to prove Theorem 3.2.

Lemma 3.1. *Let a set $A(\subset X)$ satisfy the following condition : for every $x \neq 0$, there exists a positive integer n_x such that $nx \notin A$ for all $|n| \geq n_x$ and $nx \in A$ for all $|n| < n_x$. If $F : X \times X \to Y$ satisfies the equality*

$$CF(x, y, z, w) = 0$$

for all $x, y, z, w \in X \backslash A$, then the map $F : X \times X \to Y$ is a Cauchy-Jensen mapping.

From Lemma 3.1, we have better stability result of a Cauchy-Jensen mapping in the following theorem.

Theorem 3.2. *Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a function satisfying*

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y, z, w) < \infty$$

for all $x, y, z, w \in X$. Let $f : X \times X \to Y$ be a mapping such that

$$\|Cf(x, y, z, w)\| \leq \varphi(x, y, z, w) \tag{4}$$

for all $x, y, z, w \in X \backslash A$. Then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \leq \tilde{\varphi}(x,y) \tag{5}$$

for all $x, y \in X\backslash A$, where $\tilde{\varphi}$ is the map defined by

$$\tilde{\varphi}(x,y) = \sum_{j=0}^{\infty} \frac{1}{2^{j+2}} \varphi(2^j x, 2^j x, y, y)$$

for all $x, y \in X\backslash A$. Moreover, the mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$.

Proof. Since

$$\begin{aligned}\|\frac{1}{2^j} f(2^j x, y) - \frac{1}{2^{j+1}} f(2^{j+1} x, y)\| &= \frac{1}{2^{j+2}} \|Cf(2^j x, 2^j x, y, y)\| \\ &\leq \frac{1}{2^{j+2}} \varphi(2^j x, 2^j x, y, y)\end{aligned}$$

for all $x, y \in X\backslash A$ and all $j \in N$, we get

$$\|\frac{1}{2^l} f(2^l x, y) - \frac{1}{2^m} f(2^m x, y)\| \leq \sum_{j=l}^{m-1} \frac{1}{2^{j+2}} \varphi(2^j x, 2^j x, y, y) \tag{6}$$

for all $x, y \in X\backslash A$ and integers l, m $(0 \leq l < m)$. Since Y is complete and the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ is a Cauchy sequence for all $x, y \in X\backslash A$, the sequence $\{\frac{1}{2^j} f(2^j x, y)\}$ converges for all $x, y \in X\backslash A$. Define $F_1 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X \backslash A$. Let $A_x = \{n \in \mathbb{N} | nx \in X \backslash A\}$ for each $x \in X \backslash \{0\}$. We easily obtain the equalities

$$\begin{aligned}
\lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y) &= \frac{F_1(2^m x, (2^m+2)y) + F_1(2^m x, -2^m y)}{2^{m+1}} \\
&\quad + \lim_{j\to\infty} \frac{1}{2^{j+2}} Cf(2^j x, 2^j x, (2^m+2)y, -2^m y) \\
&= \frac{F_1(2^m x, (2^m+2)y) + F_1(2^m x, -2^m y)}{2^{m+1}}, \\
\lim_{j\to\infty} \frac{1}{2^j} f(2^j x, 0) &= \frac{F_1(2^m x, 2^m y) + F_1(2^m x, -2^m y)}{2^{m+1}} \\
&\quad + \lim_{j\to\infty} \frac{1}{2^{j+2}} Cf(2^j x, 2^j x, 2^m y, -2^m y) \\
&= \frac{F_1(2^m x, 2^m y) + F_1(2^m x, -2^m y)}{2^{m+1}}, \\
\lim_{j\to\infty} \frac{1}{2^j} f(0, y) &= 0
\end{aligned}$$

hold for all $x, y \in X \backslash \{0\}$, where $2^m \in A_x \cap A_y$. Hence we can define $F : X \times X \to Y$ by

$$F(x, y) := \lim_{j\to\infty} \frac{1}{2^j} f(2^j x, y)$$

for all $x, y \in X$. Putting $l = 0$ and taking $m \to \infty$ in (6), one can obtain the inequality (5). From (4) and the definition of F, we obtain

$$CF(x, y, z, w) = \lim_{j\to\infty} \frac{1}{2^j} Cf(2^j x, 2^j y, z, w) = 0$$

for all $x, y, z, w \in X \backslash A$. By Lemma 3.1, F is a Cauchy-Jensen mapping. Now, let $F' : X \times X \to Y$ be another Cauchy-Jensen mapping satisfying (5). Then we have

$$\begin{aligned}
\|F(x, y) - F'(x, y)\| &= \frac{1}{2^n} \|F(2^n x, y) - F'(2^n x, y)\| \\
&\le \frac{1}{2^n} \|f(2^n x, y) - F(2^n x, y)\| + \frac{1}{2^n} \|f(2^n x, y) - F'(2^n x, y)\| \\
&\le \sum_{j=n}^{\infty} \frac{1}{2^{j+1}} \varphi(2^j x, 2^j x, y, y)
\end{aligned}$$

for all $n \in \mathbb{N}$ and $x, y \in X \backslash A$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X \backslash A$. By Lemma 3.1, $F(x,y) = F'(x,y)$ for all $x, y \in X$ as we desired. □

As an application, we have the stability result in the sense of Th. M. Rassias(See [3]).

COROLLARY 3.3. *Let X be a normed space and $B = \{x \in X | \|x\| \leq 1\}$. If a mapping $f : X \times X \to Y$ satisfies the inequality*

$$\|Cf(x,y,z,w)\| \leq \theta(\|x\|^p + \|y\|^p)(\|z\|^q + \|w\|^q)$$

for all $x, y, z, w \in X \backslash B$ with fixed real numbers $p < 1$ and $\theta > 0$, then there exists a unique Cauchy-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p \|y\|^q$$

for all $x, y \in X \backslash B$.

THEOREM 3.4. *Let A, f be as in Theorem 3.2. Let $\varphi : X \times X \times X \times X \to [0, \infty)$ be a map such that*

$$\lim_{(i,j) \to (\infty,\infty)} \varphi(ix, jy, z, w) = 0$$

for all $x, y, z, w \in X$, where i, j are positive integers. Then f is a Cauchy-Jensen map.

Proof. From (5) and the equality

$$\begin{aligned} &\|f(x,y) - F(x,y)\| \\ &\quad = \frac{1}{2}\|Cf((k+1)x, -kx, (k'+2)y, -k'y) + (f-F)(-kx, -k'y) \\ &\quad + (f-F)((k+1)x, -k'y) + (f-F)((k+1)x, (k'+2)y) \\ &\quad + (f-F)(-kx, (k'+2)y) - CF((k+1)x, -kx, (k'+2)y, -k'y)\| \end{aligned}$$

for all $x, y \neq 0, k, k' \in \mathbb{N}$ with $kx, k'y \notin A$, we get the inequality

$$\begin{aligned} &\|f(x,y) - F(x,y)\| \\ &\quad \leq \frac{1}{2}[\varphi((k+1)x, -kx, (k'+2)y, -k'y) + \tilde{\varphi}((k+1)x, (k'+2)y) \\ &\quad + \tilde{\varphi}((k+1)x, -k'y) + \tilde{\varphi}(-kx, (k'+2)y) + \tilde{\varphi}(-kx, -k'y)] \end{aligned}$$

for all $x, y \neq 0, k, k' \in \mathbb{N}$ with $kx, k'y \notin A$. Similarly we get the inequalities

$$\begin{aligned}
\|f(0,y) - F(0,y)\| &\leq \frac{1}{2}[\varphi(kx, -kx, (k'+2)y, -k'y) + \tilde{\varphi}(kx, (k'+2)y) \\
&\quad + \tilde{\varphi}(kx, -k'y) + \tilde{\varphi}(-kx, (k'+2)y) + \tilde{\varphi}(-kx, -k'y)], \\
\|f(x,0) - F(x,0)\| &\leq \frac{1}{2}[\varphi((k+1)x, -kx, z, -z) + \tilde{\varphi}((k+1)x, z) \\
&\quad + \tilde{\varphi}((k+1)x, -z) + \tilde{\varphi}(-kx, z) + \tilde{\varphi}(-kx, -z)], \\
\|f(0,0) - F(0,0)\| &\leq \frac{1}{2}[\varphi(kx, -kx, z, -z) + \tilde{\varphi}(kx, z) \\
&\quad + \tilde{\varphi}(kx, -z) + \tilde{\varphi}(-kx, z) + \tilde{\varphi}(-kx, -z)]
\end{aligned}$$

for all $x, y \neq 0, z \notin A, k, k' \in \mathbb{N}$ with $kx, k'y \notin A$. Since the limits of the right side of the above inequalities are 0 as $k \to \infty$, we have

$$f(x,y) = F(x,y)$$

for all $x, y \in X$. □

Corollary 3.5. *Let $p < 0$ and let $f : X \times X \to Y$ be as in Corollary 3.3. Then f is a Cauchy-Jensen mapping.*

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