Abstract. Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T : C \to K(E)$ a multivalued nonself-mapping such that $P_T$ is nonexpansive, where $P_T(x) = \{u_x \in Tx : \|x - u_x\| = d(x, Tx)\}$. For $f : C \to C$ a contraction and $t \in (0, 1)$, let $x_t$ be a fixed point of a contraction $S_t : C \to K(E)$, defined by $S_t x := tP_T(x) + (1 - t)f(x)$, $x \in C$. It is proved that if $C$ is a nonexpansive retract of $E$ and $\{x_t\}$ is bounded, then the strong limit $\lim_{t \to 1} x_t$ exists and belongs to the fixed point set of $T$. Moreover, we study the strong convergence of $\{x_t\}$ with the weak inwardness condition on $T$ in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our results provide a partial answer to Jung’s question.

1. Introduction

Let $E$ be a Banach space and $C$ a nonempty closed subset of $E$. We shall denote by $\mathcal{F}(E)$ the family of nonempty closed subsets of $E$, by $\mathcal{CB}(E)$ the family of nonempty closed bounded subsets of $E$, by $K(E)$ the family of nonempty compact subsets of $E$, and by $\mathcal{KC}(E)$ the family of nonempty compact convex subsets of $E$. Let $H(\cdot, \cdot)$ be the Hausdorff distance on $\mathcal{CB}(E)$, that is,

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}$$
for all $A, B \in CB(E)$, where $d(a, B) = \inf\{\|a - b\| : b \in B\}$ is the distance from the point $a$ to the subset $B$. Recall that a mapping $f : C \to C$ is a contraction on $C$ if there exists a constant $k \in (0, 1)$ such that
\[ \|f(x) - f(y)\| \leq k\|x - y\|, \quad x, y \in C. \]
We use $\Sigma_C$ to denote the collection of mappings $f$ verifying the above inequality. That is, $\Sigma_C = \{f : C \to C | f$ is a contraction with constant $k\}$. Note that each $f \in \Sigma_C$ has a unique fixed point in $C$.

A multivalued mapping $T : C \to \mathcal{F}(E)$ is said to be a contraction if there exists a constant $k \in [0, 1)$ such that
\[ H(Tx, Ty) \leq k\|x - y\| \]
for all $x, y \in C$. If (1) is valid when $k = 1$, the $T$ is called nonexpansive. A point $x$ is a fixed point for a multi-valued mapping $T$ if $x \in Tx$. Banach’s Contraction Principle was extended to a multivalued contraction by Nadler [18] in 1969. The set of fixed points is denoted by $F(T)$.

Given a $f \in \Sigma_C$ and a $t \in (0, 1)$, we can define a contraction $G_t : C \to K(C)$ by
\[ G_t x := tTx + (1 - t)f(x), \quad x \in C. \]
Then $G_t$ is a multivalued and hence it has a (non-unique, in general) fixed point $x_t := x_t^f \in C$ (see [18]): that is
\[ x_t \in tTx_t + (1 - t)f(x_t). \]
If $T$ is single valued, we have
\[ x_t = tTx_t + (1 - t)f(x_t). \]

A special case of (4) has been considered by Browder [2] in a Hilbert space as follows. Fix $u \in C$ and define a contraction $G_t$ on $C$ by
\[ G_t x = tTx + (1 - t)u, \quad x \in C. \]
Let $z_t \in C$ be the unique fixed point of $G_t$. Thus
\[ z_t = tTz_t + (1 - t)u. \]
(Such a sequence $\{z_t\}$ is said to be an approximating fixed point of $T$ since it possesses the property that if $\{x_t\}$ is bounded, then $\lim_{t \to 1} \|Tz_t - z_t\| = 0$.) The strong convergence of $\{z_t\}$ as $t \to 1$ for a single-valued nonexpansive self or non-self mapping $T$ was studied in Hilbert space or certain Banach spaces by many authors (see for instance, Browder [2], Halpern [8], Jung and Kim [11], Jung and Kim [12], Kim and Takahashi...
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[13], Reich [26], Singh and Waston [23], Takahashi and Kim [30], Xu [32], and Xu and Yin [36]).

In 1967, Browder [2] proved the following.

**Theorem B.** ([2]). In a Hilbert space, as $t \to 1$, $z_t$ defined by (5) converges strongly to a fixed point of $T$ that is closest to $u$, that is, the nearest point projection of $u$ onto $F(T)$.

However, Pietramala [19] (see also Jung [10]) provided an example showing that Browder’s theorem [2] cannot be extended to the multivalued case without adding an extra assumption even if $E$ is Euclidean. López Acedo and Xu [15] gave the strong convergence of $\{x_t\}$ defined by $x_t \in tTx_t + (1-t)u$, $u \in C$ under the restriction $F(T) = \{z\}$ in Hilbert space. Kim and Jung [14] extended the result of López Acedo and Xu [15] to a Banach space with a weakly sequentially continuous duality mapping. Sahu [20] also studied the multi-valued case in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Recently, Jung [10] gave the strong convergence of $\{x_t\}$ defined by $x_t \in tTx_t + (1-t)u$, $u \in C$ for the multivalued nonexpansive nonself-mapping $T$ in a uniformly convex or reflexive Banach space having a uniformly Gâteaux differentiable norm and mentioned that the condition $F(T) = \{z\}$ should be added in the main results of Sahu [20]. More precisely, he established the following extensions of Browder’s theorem [2].

**Theorem J1.** ([10]). Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T : C \to K(E)$ a nonexpansive nonself-mapping. Suppose that $C$ is a nonexpansive retract of $E$. Suppose that $T(y) = \{y\}$ for any fixed point $y$ of $T$ and that for each $u \in C$ and $t \in (0,1)$, the contraction $G_t$ defined by $G_t x := tTx + (1-t)u$, $x \in C$. has a fixed point $x_t \in C$. Then $T$ has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of $T$.

**Theorem J2.** ([10]). Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T : C \to K\mathcal{C}(E)$ a nonexpansive nonself-mapping satisfying the inwardness condition. Assume that every closed bounded...
convex subset of \( C \) is compact. If the fixed point set \( F(T) \) of \( T \) is nonempty and \( Ty = \{y\} \) for any \( y \in F(T) \), then the sequence \( \{x_t\} \) defined by \( x_t \in tTx_t + (1 - t)u, \ u \in C \) converges strongly as \( t \to 1 \) to a fixed point of \( T \).

Very recently, in order to give a partial answer to Jung’s open question [10]: Can the assumption \( Tz = \{z\} \) in Theorem J1 and J2 be omitted ?, Shahzad and Zegeye [21] considered a class of multivalued mapping under some mild conditions as follows.

Let \( C \) be a closed convex subset of a Banach space \( E \). Let \( T : C \to \mathcal{K}(E) \) be a multivalued nonself-mapping and

\[
P_Tx = \{ux \in Tx : \|x - ux\| = d(x, Tx)\}.
\]

Then \( P_T : C \to \mathcal{K}(E) \) is multivalued and \( P_Tx \) is nonempty and compact for every \( x \in C \). Instead of

\[
G_tx = tTx + (1 - t)u, \ u \in C,
\]

we consider for \( t \in (0, 1) \),

\[
S_tx = tP_Tx + (1 - t)u, \ u \in C,
\]

It is clear that \( S_t \subseteq G_t \) and if \( P_T \) is nonexpansive and \( T \) is weakly inward, then \( S_t \) is weakly inward contraction. Theorem 1 of Lim [16] guarantees that \( S_t \) has a fixed point point \( x_t \), that is,

\[
x_t \in tP_Tx_t + (1 - t)u \subseteq tTx_t + (1 - t)u.
\]

It \( T \) is single-valued, then (8) is reduced to (5).

On the other hand, Xu [35] studied the strong convergence of \( x_t \) defined by (4) as \( t \to 1 \) in either a Hilbert space or a uniformly smooth Banach space and showed that the strong \( \lim_{t \to 1} x_t \) is the unique solution of certain variational inequality. This result of Xu [35] also improved Theorem 2.1 of Moudafi [17] as the continuous version. In 2006, Jung [9] also established the strong convergence of \( x_t \) defined by (4) for finite nonexpansive mappings in a reflexive Banach space Banach space having a uniformly Gâteaux differentiable norm with the condition that every weakly compact convex subset of \( E \) has the fixed point property for nonexpansive mappings.

In this paper, motivated by [10, 21, 35], we establish the strong convergence of \( \{x_t\} \) defined by

\[
x_t \in tP_Tx_t + (1 - t)f(x_t), \ f \in \Sigma_C,
\]
for the multivalued nonself-mapping $T$ in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. We also study the strong convergence of $\{x_t\}$ for the multivalued nonself-mapping $T$ satisfying the inwardness condition in a reflexive Banach space with a uniformly Gâteaux differentiable norm. Our results improve and extend the results in [2, 10, 11, 12, 20, 21, 32, 36] to the viscosity approximation method for multivalued nonself-mapping case. We also point out that our results give a partial answer to Jung’s question [10].

2. Preliminaries

Let $E$ be a real Banach space with norm $\| \cdot \|$ and let $E^*$ be its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$.

A Banach space $E$ is called uniformly convex if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$, where the modulus $\delta(\varepsilon)$ of convexity of $E$ is defined by

$$\delta(\varepsilon) = \inf\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$$

for every $\varepsilon$ with $0 \leq \varepsilon \leq 2$. It is well-known that if $E$ is uniformly convex, then $E$ is reflexive and strictly convex (cf. [5]).

The norm of $E$ is said to be Gâteaux differentiable (and $E$ is said to be smooth) if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y$ in its unit sphere $U = \{x \in E : \|x\| = 1\}$. It is said to be uniformly Gâteaux differentiable if for each $y \in U$, this limit is attained uniformly for $x \in U$. Finally, the norm is said to be uniformly Fréchet differentiable (and $E$ is said to be uniformly smooth if the limit in (9) is attained uniformly for $(x, y) \in U \times U$. A discussion of these and related concepts may be found in [3].

The normalized duality mapping $J$ from $E$ into the family of nonempty (by Hahn-Banach theorem) weak-star compact subsets of its dual $E^*$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

for each $x \in E$. It is single valued if and only if $E$ is smooth.

Let $D$ be a subset of $C$. Then a mapping $Q : C \to D$ is said to be retraction if $Qx = x$ for all $x \in D$. A retraction $Q : C \to D$ is said to
be sunny if each point on the ray \( \{ Qx + t(x - Qx) : t > 0 \} \) is mapped by \( Q \) back onto \( Qx \), in other words, \( Q(Qx + t(x - Qx)) = Qx \) for all \( t \geq 0 \) and \( x \in C \). A subset \( D \) of \( C \) is said to be a sunny nonexpansive retract of \( C \) if there exists a sunny nonexpansive retraction of \( C \) onto \( D \) (cf. [5, 25]). In a smooth Banach space \( E \), it is known (cf. [5, p. 48]) that \( Q \) is a sunny nonexpansive retraction from \( C \) onto \( D \) if and only if the following inequality holds:

\[
\langle x - Qx, J(z - Qx) \rangle \leq 0, \quad x \in C, \quad z \in D.
\]

A mapping \( T : C \to CB(E) \) is \(*\)-nonexpansive ([7]) if for all \( x, y \in C \) and \( u_x \in Tx \) with \( \|x - u_x\| = \inf \{\|x - z\| : z \in Tx\} \), there exists \( u_y \in Ty \) with \( \|y - u_y\| = \inf \{\|y - w\| : w \in Ty\} \) such that

\[
\|u_x - u_y\| \leq \|x - y\|.
\]

It is known that \(*\)-nonexpansiveness is different from nonexpansiveness for multivalued mappings. There are some \(*\)-nonexpansiveness multivalued mappings which are not nonexpansive and some nonexpansive multivalued mappings which are not \(*\)-nonexpansive [31].

Let \( \mu \) be a linear continuous functional on \( \ell^\infty \) and let \( a = (a_1, a_2, ...) \in \ell^\infty \). We will sometimes write \( \mu_n(a_n) \) in place of the value \( \mu(a) \). A linear continuous functional \( \mu \) such that \( \|\mu\| = 1 = \mu(1) \) and \( \mu_n(a_n) = \mu_n(a_{n+1}) \) for every \( a = (a_1, a_2, ...) \in \ell^\infty \) is called a Banach limit. We know that if \( \mu \) is a Banach limit, then

\[
\liminf_{n \to \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \to \infty} a_n
\]

for every \( a = (a_1, a_2, ...) \in \ell^\infty \). Let \( \{x_n\} \) be a bounded sequence in \( E \). Then we can define the real valued continuous convex function \( \phi \) on \( E \) by

\[
\phi(z) = \mu_n\|x_n - z\|^2
\]

for each \( z \in E \).

The following lemma which was given in [6, 28] is, in fact, a variant of Lemma 1.3 in [25].

**Lemma 1.** Let \( C \) be a nonempty closed convex subset of a Banach space \( E \) with a uniformly Gâteaux differentiable norm and let \( \{x_n\} \) be a bounded sequence in \( E \). Let \( \mu \) be a Banach limit and \( u \in C \). Then

\[
\mu_n\|x_n - u\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2
\]
if and only if
\[ \mu_n \langle x - u, J(x_n - u) \rangle \leq 0 \]
for all \( x \in C \).

We also need the following result, which was essentially given by Reich [27, pp. 314-315] and was also proved by Takahashi and Jeong [29].

**Lemma 2.** Let \( E \) be a uniformly convex Banach space, \( C \) a nonempty closed convex subset of \( E \), and \( \{x_n\} \) a bounded sequence of \( E \). Then the set
\[ M = \{ u \in C : \mu_n \|x_n - u\|^2 = \min_{z \in C} \mu_n \|x_n - z\|^2 \} \]
consists of one point.

We introduce some terminology for boundary conditions for non-self mappings. The **inward set** of \( C \) at \( x \) is defined by
\[ I_C(x) = \{ z \in E : z = x + \lambda(y-x) : y \in C, \lambda \geq 0 \}. \]
Let \( \bar{I}_C(x) = x + T_C(x) \) with
\[ T_C(x) = \left\{ y \in E : \liminf_{\lambda \to 0^+} \frac{d(x + \lambda y, C)}{\lambda} = 0 \right\} \]
for any \( x \in C \). Note that for a convex set \( C \), we have \( \bar{I}_C(x) = \overline{I_C(x)} \), the closure of \( I_C(x) \). A multivalued mapping \( T : C \to F(E) \) is said to satisfy the **inwardness condition** if \( Tx \subset I_C(x) \) for all \( x \in C \) and respectively, to satisfy the **weak inwardness condition** if \( Tx \subset \overline{I_C(x)} \) for all \( x \in C \). We notice that a fixed point theorem for nonexpansive mappings satisfying the inwardness condition is given in Corollary 3.5 of Reich [24]. A fixed point theorem for multi-valued strict contractions was given in Theorem 3.4 of Reich [24], too. It is also well-known that if \( C \) is a nonempty closed subset of a Banach space \( E \), \( T : C \to F(E) \) is a contraction satisfying the weak inwardness condition, and \( x \in E \) has a nearest point in \( Tx \), then \( T \) has a fixed point ([Theorem 11.4 of Deimling [4]]).

Finally, the following lemmas were given by Xu [34] (also see Lemma 2.3.2 of Xu [33] for Lemma 4).

**Lemma 3.** If \( C \) is a closed bounded convex subset of a uniformly convex Banach space \( E \) and \( T : C \to K(E) \) is a nonexpansive mapping satisfying the weak inwardness condition, then \( T \) has a fixed point.
Lemma 4. If $C$ is a compact convex subset of a Banach space $E$ and $T : C \to \mathcal{K}(E)$ is a nonexpansive mapping satisfying the boundary condition:
$$Tx \cap \overline{I}_C(x) \neq \emptyset, \quad x \in C,$$
then $T$ has a fixed point.

3. Main results

Now, we first prove a strong convergence theorem.

Theorem 1. Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T : C \to \mathcal{K}(E)$ a multivalued nonself-mapping such that $P_T$ is nonexpansive. Suppose that $C$ is a nonexpansive retract of $E$. Suppose that for $f \in \Sigma_C$ and $t \in (0, 1)$, the contraction $S_t$ defined by $S_t x = tP_T x + (1 - t)f(x)$ has a fixed point $x_t \in C$. Then $T$ has a fixed point if and only if \{x_t\} remains bounded as $t \to 1$ and in this case, \{x_t\} converges strongly as $t \to 1$ to a fixed point of $T$.

If we define $Q : \Sigma_C \to F(T)$ by $Q(f) := \lim_{t \to 1} x_t$ for $f \in \Sigma_C$, then $Q(f)$ solves the variational inequality
$$((I - f)(Q(f)), J(Q(f) - z)) \leq 0, \quad f \in \Sigma_C, \quad z \in F(T).$$

Proof. For given any $x_t \in C$, we can find some $y_t \in P_T x_t$ such that
$$x_t = t y_t + (1 - t)f(x_t).$$
Let $z \in F(T)$. Then \{x_t\} is uniformly bounded. In fact, noting that $P_T y = \{y\}$ whenever $y$ is a fixed point of $T$, we have $z \in P_T z$ and
$$\|y_t - z\| = d(y_t, P_T z) \leq H(P_T x_t, P_T z) \leq \|x_t - z\| \quad \text{for all} \ t \in (0, 1).$$
Thus we have
$$\|x_t - z\| \leq t\|y_t - z\| + (1 - t)\|f(x_t) - z\| \leq t\|x_t - z\| + (1 - t)(\|f(x_t) - f(z)\| + \|f(z) - z\|) \leq t\|x_t - z\| + (1 - t)(k\|x_t - z\| + \|f(z) - z\|).$$
This implies that
$$\|x_t - z\| \leq \frac{1}{1 - k}\|f(z) - z\|$$
and so \{x_t\} is uniformly bounded. Also \{f(x_t)\} is bounded.
Suppose conversely that \( \{x_t\} \) remains bounded as \( t \to 1 \). We now show that \( T \) has a fixed point \( z \) and that \( \{x_t\} \) converges strongly as \( t \to 1 \) to a fixed point of \( T \). To this end, let \( t_n \to 1 \) and \( x_n = x_{t_n} \). Define \( \phi : E \to [0, \infty) \) by \( \phi(z) = \mu_n \|x_n - z\|^2 \). Since \( \phi \) is continuous and convex, \( \phi(z) \to \infty \) as \( \|z\| \to \infty \), and \( E \) is reflexive, \( \phi \) attains its infimum over \( C \) (cf. [1, p. 79]). Let \( z \in C \) be such that
\[
\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2
\]
and let
\[
M = \{x \in C : \mu_n \|x - x\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2\}.
\]
Then \( M \) is a nonempty bounded closed convex subset of \( C \). Since \( C \) is a nonexpansive retract of \( E \), the point \( z \) is the unique global minimum (over all of \( E \)). In fact, let \( Q \) be a nonexpansive retraction of \( E \) onto \( C \). Then for any \( y \in E \), we have
\[
\mu_n \|x_n - z\|^2 \leq \mu_n \|x_n - Qy\|^2 = \mu_n \|Qx_n - Qy\|^2 \leq \mu_n \|x_n - y\|^2
\]
and hence
\[
\mu_n \|x_n - z\|^2 = \min_{y \in E} \mu_n \|x_n - y\|^2.
\]
This global minimum point \( z \) is also unique by Lemma 2.

On the other hand, since \( x_t = ty_t + (1 - t)f(x_t) \) for some \( y_t \in P_T x_t \), it follows that
\[
\|x_t - y_t\| = (1 - t)\|f(x_t) - y_t\| \to 0
\]
as \( t \to 1 \). Since \( P_T \) is compact valued, we have for each \( n \geq 1 \), some \( w_n \in P_T z \) for \( z \in M \) such that
\[
\|y_n - w_n\| = d(y_n, P_T z) \leq H(P_T x_n, P_T z) \leq \|x_n - z\|.
\]
Let \( w = \lim_{n \to \infty} w_n \in P_T z \). It follows from (14) and (15) that
\[
\mu_n \|x_n - w\|^2 \leq \mu_n \|y_n - w_n\|^2 \leq \mu_n \|x_n - z\|^2.
\]
Since \( z \) is the unique global minimum, we have \( w = z \in P_T z \subset T z \) and hence \( F(T) \neq \emptyset \). We have also that \( P_T z = \{z\} \).

On the another hand, for \( P_T z = \{z\} \subset C \), we have from (13)
\[
\langle x_n - y_n, J(x_n - z) \rangle = \langle (x_n - z) + (z - y_n), J(x_n - z) \rangle \\
\geq \|x_n - z\|^2 - \|y_n - z\|\|x_n - z\| \\
\geq \|x_n - z\|^2 - \|x_n - z\|^2 = 0,
\]
and it follows that
\[ 0 \leq \langle x_n - y_n, J(x_n - z) \rangle = (1 - t_n)\langle f(x_n) - y_n, J(x_n - z) \rangle. \]

Hence from (14) and (16), we obtain
\[ \mu_n \langle x_n - f(x_n), J(x_n - z) \rangle \leq 0 \]
for \( P_Tz = \{ z \} = M \). But, from (11) in Lemma 1, we have
\[ \mu_n \langle x - z, J(x_n - z) \rangle \leq 0 \]
for all \( x \in C \). In particular, we have
\[ \mu_n \langle f(z) - z, J(x_n - z) \rangle \leq 0. \]

Combining (17) and (18), we get
\[ \mu_n \|x_n - z\|^2 = \mu_n \langle x_n - z, J(x_n - z) \rangle \]
\[ \leq \mu_n \langle f(x_n) - f(z), J(x_n - z) \rangle + \mu_n \langle f(z) - z, J(x_n - z) \rangle \]
\[ \leq k\mu_n \|x_n - z\|^2 \]
and hence \( \mu_n \|x_n - z\|^2 \leq 0 \). Therefore, there is a subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) which converges strongly to \( z \). To complete the proof, suppose that there is another subsequence \( \{ x_{n_j} \} \) of \( \{ x_n \} \) which converges strongly to \( y \) (say). Since
\[ d(x_{n_k}, P_Tx_{n_k}) \leq \|x_{n_k} - y_{n_k}\| = (1 - t_{n_k})\|f(x_{n_k}) - y_{n_k}\| \to 0 \]
as \( k \to \infty \), we have \( d(y, Ty) = 0 \) and hence \( y \in P_Ty \subset Ty \). Noting that \( P_Ty = \{ y \} \), from (17) we have
\[ \langle z - f(z), J(z - y) \rangle \leq 0 \text{ and } \langle y - f(y), J(y - z) \rangle \leq 0. \]

Adding these two inequalities yields
\[ \|z - y\|^2 \leq \langle f(z) - f(y), J(z - y) \rangle = k\|z - y\|^2 \]
and thus \( z = y \). This proves the strong convergence of \( \{ x_t \} \) to \( z \).

Define \( Q : \Sigma_C \to F(T) \) by \( Q(f) := \lim_{t \to 1} x_t \). Since \( x_t = ty_t + (1 - t) f(x_t) \) for some \( y_t \in P_Tx_t \),
\[ (I - f)(x_t) = -\frac{t}{1 - t}(x_t - y_t). \]
From (13), we have for $z \in F(T)$
\[
\langle (I - f)(x_t), J(x_t - z) \rangle = - \frac{t}{1 - t} \langle (x_t - z) + (z - y_t), J(x_t - z) \rangle \\
\leq - \frac{t}{1 - t} (\|x_t - z\|^2 - \|y_t - z\| \|x_t - z\|) \\
\leq - \frac{t}{1 - t} (\|x_t - z\|^2 - \|x_t - z\|^2) = 0.
\]

Letting $t \to 1$ yields
\[
\langle (I - f)(Q(f)), J(Q(f) - z) \rangle \leq 0, \quad f \in \Sigma_C, \quad z \in F(T).
\]

**Remark 1.** In Theorem 1, if $f(x) = u, \ x \in C,$ is a constant mapping, then it follows from (12) that
\[
\langle Q(u) - u, J(Q(u) - z) \rangle \leq 0, \quad u \in C, \quad z \in F(T).
\]
Hence by (10), $Q$ reduces to the sunny nonexpansive retraction from $C$ onto $F(T)$.

By definition of the Hausdorff metric, we obtain that if $T$ is $*$-nonexpansive, then $P_T$ is nonexpansive. Hence, as a direct consequence of Theorem 1, we have the following result.

**Corollary 1.** Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T : C \to K(E)$ a multivalued $*$-nonexpansive nonself-mapping. Suppose that $C$ is a nonexpansive retract of $E$. Suppose that for $f \in \Sigma_C$ and $t \in (0, 1)$, the contraction $S_t$ defined by $S_t x = tP_T x + (1 - t)f(x)$ has a fixed point $x_t \in C$. Then $T$ has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of $T$.

It is well-known that every nonempty closed convex subset $C$ of a strictly convex reflexive Banach space $E$ is Chebyshev, that is, for any $x \in E$, there is a unique element $u \in C$ such that $\|x - u\| = \inf\{\|x - v\| : v \in C\}$. Thus, we have the following corollary.
Corollary 2. Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T : C \to K\mathcal{C}(E)$ a multivalued nonself-mapping such that $P_T$ is nonexpansive. Suppose that $C$ is a nonexpansive retract of $E$. Suppose that for $f \in \Sigma_C$ and $t \in (0, 1)$, the contraction $S_t$ defined by $S_t x = tP_T x + (1 - t)f(x)$ has a fixed point $x_t \in C$. Then $T$ has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of $T$.

Proof. In this case, $Tx$ is Chebyshev for each $x \in C$. So $P_T$ is a selector of $T$ and $P_T$ is single valued. Thus the result follows from Theorem 1.

Corollary 3. Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T : C \to K\mathcal{C}(E)$ a multivalued nonexpansive nonself-mapping. Suppose that $C$ is a nonexpansive retract of $E$. Suppose that for $f \in \Sigma_C$ and $t \in (0, 1)$, the contraction $S_t$ defined by $S_t x = tP_T x + (1 - t)f(x)$ has a fixed point $x_t \in C$. Then $T$ has a fixed point if and only if $\{x_t\}$ remains bounded as $t \to 1$ and in this case, $\{x_t\}$ converges strongly as $t \to 1$ to a fixed point of $T$.

Corollary 4. Let $E$ be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed bounded convex subset of $E$, and $T : C \to K\mathcal{C}(E)$ a multivalued nonself-mapping satisfying the weak inwardness condition such that $P_T$ is nonexpansive. Suppose that $C$ is a nonexpansive retract of $E$. Let $f \in \Sigma_C$ and $t \in (0, 1)$. Then $\{x_t\}$ defined by $x_t \in tP_T x_t + (1 - t)f(x_t)$ converges strongly as $t \to 1$ to a fixed point of $T$.

Proof. Fix $f \in \Sigma_C$ and define for each $t \in (0, 1)$, the contraction $S_t : C \to K\mathcal{C}(E)$ by

$$S_t x := tP_T x + (1 - t)f(x), \quad x \in C.$$ 

As it is easily seen that $S_t$ also satisfies the weak inwardness condition: $S_t x \subset \overline{I}_C(x)$ for all $x \in C$, it follows from Lemma 3 that $S_t$ has a fixed point denoted by $x_t$. Thus the result follows from Theorem 1. \qed
Remark 2. (1) As in [31], Shahzad and Zegeye [21] gave the following example of a multivalued $T$ such that $P_T$ is nonexpansive: Let $C = [0, \infty)$ and $T$ be defined by $Tx = [x, 2x]$ for $x \in C$. Then $P_Tx = \{x\}$ for $x \in C$. Also $T$ is $*$-nonexpansive but not nonexpansive (see [31]).

(2) Theorem 1 (and Corollaries 1-4) generalizes Theorem 3.1 (and Corollaries 3.3-3.5) of Shahzad and Zegeye [21] to the viscosity approximation method.

(3) Theorem 1 also improves and complements the corresponding results of Jung [10], Kim and Jung [14] and Sahu [20]. Theorem 1 extends the corresponding results of Jung and Kim [11], Jung and Kim of [12] and Xu and Yin [36], to the multivalued mapping case, too.

(4) Our results apply to all $L^p$ spaces or $\ell^p$ spaces for $1 < p < \infty$.

Theorem 2. Let $E$ be a reflexive Banach space with a uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $E$, and $T : C \to KC(E)$ a multivalued nonself-mapping satisfying the inwardness condition such that $P_T$ is nonexpansive. Let $f \in \Sigma_C$ and $t \in (0, 1)$. Assume that every closed bounded convex subset of $C$ is compact. If $P_T$ has a fixed point, then the sequence $\{x_t\}$ defined by

\[ x_t \in tP_Tx_t + (1 - t)f(x_t), \]

converges strongly as $t \to 1$ to a fixed point of $T$.

Proof. Let $z \in P_Tz$. As in proof of Theorem 1, we have $\|x_t - z\| \leq \frac{1}{1-t}\|f(z) - z\|$ for all $t \in (0, 1)$ and hence $\{x_t\}$ is uniformly bounded.

We now show that $\{x_t\}$ converges strongly as $t \to 1^-$ to a fixed point of $T$. To this end, let $t_n \to 1$ and $x_n = x_{t_n}$. As in the proof of Theorem 1, we define the same function $\phi : E \to [0, \infty)$ by $\phi(z) = \mu_n\|x_n - z\|^2$ and let

\[ M = \{x \in C : \mu_n\|x_n - x\|^2 = \min_{y \in C} \mu_n\|x_n - y\|^2\}. \]

Then $M$ is a nonempty closed bounded convex subset of $C$ and by assumption, $M$ is compact convex. Clearly, $P_T$ satisfies the inwardness condition. By using the same argument as in Theorem 2 of Jung [10], we can prove that the inwardness condition of $P_T$ on $C$ implies a weaker inwardness of $P_T$ on $M$, that is,

\[ P_Tz \cap I_M(z) \neq \emptyset, \quad z \in M. \]
So, by Lemma 4, there exists \( z \in M \) such that \( z \in P_Tz \subseteq Tz \) and so \( P_Tz = \{z\} \). The strong convergence of \( \{x_t\} \) to \( z \) is the same as given in the proof of Theorem 1.

**Corollary 5.** Let \( E \) be a reflexive Banach space with a uniformly Gâteaux differentiable norm, \( C \) a nonempty closed convex subset of \( E \), and \( T : C \to \mathcal{K}C(E) \) a multivalued \(*\)-nonexpansive nonself-mapping satisfying the inwardness condition such that \( P_T \) is nonexpansive. Let \( f \in \Sigma_C \) and \( t \in (0, 1) \). Assume that every closed bounded convex subset of \( C \) is compact. If \( P_T \) has a fixed point, then the sequence \( \{x_t\} \) defined by (19) converges strongly as \( t \to 1 \) to a fixed point of \( T \).

**Corollary 6.** Let \( E \) be a uniformly smooth Banach space, \( C \) a nonempty closed convex subset of \( E \), and \( T : C \to \mathcal{K}C(E) \) a multivalued nonself-mapping satisfying the inwardness condition such that \( P_T \) is nonexpansive. Let \( f \in \Sigma_C \) and \( t \in (0, 1) \). Assume that every closed bounded convex subset of \( C \) is compact. If \( P_T \) has a fixed point, then the sequence \( \{x_t\} \) defined by (19) converges strongly as \( t \to 1 \) to a fixed point of \( T \).

**Corollary 7.** Let \( E \) be a reflexive Banach space with a uniformly Gâteaux differentiable norm, \( C \) a nonempty compact convex subset of \( E \), and \( T : C \to \mathcal{K}C(E) \) a multivalued nonself-mapping satisfying the inwardness condition such that \( P_T \) is nonexpansive. Let \( f \in \Sigma_C \) and \( t \in (0, 1) \). If \( P_T \) has a fixed point, then the sequence \( \{x_t\} \) defined by (19) converges strongly as \( t \to 1 \) to a fixed point of \( T \).

**Corollary 8.** Let \( E \) be a uniformly smooth Banach space, \( C \) a nonempty compact convex subset of \( E \), and \( T : C \to \mathcal{K}C(E) \) a multivalued nonself-mapping satisfying the inwardness condition such that \( P_T \) is nonexpansive. Let \( f \in \Sigma_C \) and \( t \in (0, 1) \). If \( P_T \) has a fixed point, then the sequence \( \{x_t\} \) defined by (19) converges strongly as \( t \to 1 \) to a fixed point of \( T \).

**Remark 3.** (1) Theorem 2 (and Corollaries 5-8) also improves Theorem 3.9 (and Corollaries 3.10-3.12) of Shahzad and Zegeye [21] to the
viscosity approximation method. Theorem 2 (and Corollaries 6-7) complements Theorem 2 (and Corollaries 4-5) of Jung [10], too.

2) Theorem 2 is also a multivalued version of Theorem 1 and Corollary 1 of Jung and Kim [12] and Theorem 1 of Xu [32].

3) A fixed point theorem for \( T : C \to KC(E) \) a \( \ast \)-nonexpansive, 1-\( \chi \)-contractive multivalued mapping satisfying the inwardness condition in a special Banach space was recently given by Shahzad and Lone [22]. In this case, one can relax the assumption that \( F(T) \neq \emptyset \).

References


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