SIGN CHANGING PERIODIC SOLUTIONS OF A NONLINEAR WAVE EQUATION

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Abstract. We seek the sign changing periodic solutions of the nonlinear wave equation \( u_{tt} - u_{xx} = a(x, t)g(u) \) under Dirichlet boundary and periodic conditions. We show that the problem has at least one solution or two solutions whether \( \frac{1}{2}g(u)u - G(u) \) is bounded or not.

1. Introduction

In this paper we seek the sign changing solutions of the following nonlinear wave equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = a(x, t)g(u),
\]

under Dirichlet boundary condition and periodic condition:

\[
\begin{align*}
    u(0, t) &= u(\pi, t) = 0, \\
    u(x, t + T) &= u(x, t),
\end{align*}
\]

where \( a : [0, \pi] \times \mathbb{R} \to \mathbb{R} \) is a continuous function which changes sign such that \( a(x, t) = -a(x, t + \frac{T}{2}) \), and the open sets

\[
\{(x, t) \mid a(x, t) > 0\}, \quad \{(x, t) \mid a(x, t) < 0\}
\]

are both nonempty. We shall write \( a = a^+ - a^- \), where \( a^+ = a \cdot \chi_{\Omega^+} \) and \( a^- = -a \cdot \chi_{\Omega^-} \). In what follows we assume systematically that \( T \) is a rational multiple of \( \pi \). We assume that \( g \) satisfies the following conditions:

\( g \) 1) \( g \in C(\mathbb{R}, \mathbb{R}), \)

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(g2) $g(u) = o(u)$,
(g3) there exists a constant $\mu > 2$ such that
$$g(u)u \geq \mu \int_0^u g(s)ds > 0,$$
(g4) there exist constants $a_1, a_2 > 0$ and $p > 1$ such that
$$|g(u)| \leq a_1|u|^p + a_2 \quad \text{for all } u,$$
where $G(u) = \int_0^x g(t)dt$.

Integrating condition (g3) shows that there exist constants $a_3, a_4 > 0$ such that
$$G(u) \geq a_3|u|^\mu - a_4. \quad (1.2)$$

The purpose of this paper is to show the existence of solutions of the problem (1.1) when $\frac{1}{2}g(u)u - G(u)$ is bounded or $\frac{1}{2}g(u)u - G(u)$ is not bounded. Our main results are as follows:

**Theorem 1.1.** Assume that $g$ satisfies (g1) – (g4) and $\frac{1}{2}g(u)u - G(u)$ is bounded. Then the problem (1.1) has at least one bounded solution provided that $p$ in (g4) is further restricted by $p + 1 < \mu$.

**Theorem 1.2.** Assume that $g$ satisfies (g1) – (g4), $\frac{1}{2}g(u)u - G(u)$ is not bounded. We also assume that there exists a small $\epsilon > 0$ such that $\int_\Omega a^-(x, t) < \epsilon$. Then for each $T$ the problem (1.1) has at least two solutions, (1) one of which is bounded and (2) the other is a large norm solution such that for each real number $M$,
$$\max_{x \in [0, \pi]} \max_{t \in [0, T]} |u(x, t)| > M$$
provided that $p$ in (g4) is further restricted by $p + 1 < \mu$.

Theorem 1.1 and Theorem 1.2 will be proved in Section 3 and 4 via variational methods.

An outline of this paper is as follows: in Section 2 we introduce a subspace $H$ of functions satisfying some symmetry properties, stable by $A(Au = u_{tt} - u_{xx})$, $g$ such that the intersection of $H$ with the kernel of $A$ is reduced to 0. The search of a solution of the problem (1.1) in the space $H$ reduces the problem to a situation where $A^{-1}$ is a compact operator. In Section 3 we prove Theorem 1.1 and 1.2(1). We introduce a functional $I$ whose critical points and weak solutions of (1.1) possess one-to-one correspondence. Next we prove that $I \in C^1(E, \mathbb{R})$ and satisfies the Palais-Smale condition. Then, we show that there exist $\rho > 0, \delta > 0,
and \( u_0 \in E \) satisfying \( \| u_0 \| > \rho \) such that if \( \| u \| = \rho \), then \( I(u) \geq \delta \), and \( I(u_0) \leq 0 \). By critical point theorem for indefinite functionals (cf. [3]) there exists at least one solution of (1.1) which is bounded. In Section 4, we prove Theorem 1.2(2) by the method of Rabinowitz (cf. [13]). We introduce a functional \( J \) such that large critical values of \( J \) induce large critical values of \( I \).

2. Invariant spaces

Let \( \Omega = (0, \pi) \times (0, T) \); \( T \) is a rational multiple of \( \pi \), that is, \( T = \frac{2\pi b}{a} \), where \( a \) and \( b \) are coprime integers. Let \( \mathcal{A} \) be the operator defined by
\[
\mathcal{A}u = u_{tt} - u_{xx}
\]
and \( D(\mathcal{A}) \) be a collection of functions which belongs to the domain of an operator \( \mathcal{A} \) and which satisfies some boundary conditions. Let \( A \) be the adjoint of \( \mathcal{A} \) in \( L^2(\Omega) \). We investigate solutions of
\[
Au = a(x, t)g(u).
\]
We note that the eigenvalues of \( A \) are \( j^2 - \left( \frac{2\pi k}{T} \right)^2 \), \( j = 1, 2, \ldots \) and \( k = 0, 1, 2, \ldots \) and the corresponding eigenfunctions are
\[
\sin jx \sin \frac{2\pi kt}{T} \quad \text{and} \quad \sin jx \cos \frac{2\pi kt}{T}.
\]
We also note that the set of functions \( \sin jx \sin \frac{2\pi kt}{T} \), \( \sin jx \cos \frac{2\pi kt}{T} \) is an orthogonal base for \( L^2(\Omega) \). Let \( u \) be a function of \( L^2(\Omega) \). Then there exists one and only one function of \( L^2([0, \pi] \times \mathbb{R}) \) which is \( T \) periodic in \( t \) and equals \( u \) on \( \Omega \). We shall again denote this function by \( u \). Let us denote an element \( u \), in \( L^2(\Omega) \), as
\[
u = \sum_{j>0} \sum_{k>0} u_{j,k} \sin jx \exp ik \frac{a}{b} t \]
with \( u_{j,k} = \pi_{j, -k} \). We assume that \( b \) is even and \( a \) is odd. Let \( H \) be the closed subspace of \( L^2(\Omega) \) defined by
\[
H = \{ u \in L^2(\Omega) | u(x, t) = -u \left( x, t + \frac{T}{2} \right) \quad \text{a.e.} \ x \in (0, \pi), t \in \mathbb{R} \}.
\]
Then \( H \) is invariant under shifts: Let \( u \in H \) and \( \tau \) be a real number. If \( v(x, t) = u(x, t + \tau) \), then \( v \in H \). \( H \) is invariant by \( g \): Let \( u \in H \) such
that \( g(u) \in L^2(\Omega) \). Then \( g(u) \in H \).

Let \( \tilde{u}(x,t) = u(x,t + \frac{T}{2}) \). Then
\[
\tilde{u} = \sum_{j > 0} u_{j,k} (-1)^k \sin jx \exp ik \frac{a}{b} t.
\]

Therefore
\[
u \in H \iff u_{j,k} = 0 \quad \text{for any even } k. \tag{2.1}
\]

Let \( A_1 \) be the linear operator of \( H \) defined by
\[
D(A_1) = D(A) \cap H
\]
\[
A_1 u = Au \quad \text{for every } u \in H.
\]

Then it follows from (2.1) that \( A_1 \) is self adjoint in \( H \).

We claim that \( H \cap N(A) = \{0\} \),
where \( N(A) \) is the kernel of \( A \). In fact, let \( u \in H \cap N(A) \). Then
\[
u = \sum u_{j,k} \sin jx \exp ik \frac{a}{b} t,
\]
\[
j^2 - \frac{k^2 a^2}{b^2} \neq 0 \implies u_{j,k} = 0.
\]

Let \( j \) and \( k \) be such that
\[
j^2 - \frac{k^2 a^2}{b^2} = 0.
\]

Since \( b \) is even and \( a \) is odd, \( k \) is even. Using (2.1) we have \( u_{j,k} = 0 \) and therefore \( H \cap N(A) = \{0\} \).

3. Proof of Theorem 1.1 and Theorem 1.2(1)

To prove Theorem 1.1 we shall show that the corresponding functional \( I(u) \) of the problem (1.1) satisfies the geometric assumptions of the critical point theorem for indefinite functionals (cf. [3]). Then, by critical point theorem we shall seek solutions of (1.1). Now, we are going to seek a function \( u \) in \( H \) such that
\[
A_1 u = a(x,t) g(u). \tag{3.1}
\]
The eigenvalues of $A_1$ are $j^2 - \left(\frac{2\pi k}{T}\right)^2$, where $j$ is odd and $k$ is even. Given $u \in H$, we write

$$u = \sum_{j>0 \atop j \text{ odd}} \sum_{k \text{ even}} u_{j,k} \sin jx \exp i \frac{2\pi k t}{T}$$

with $u_{j,k} = \overline{u}_{j,-k}$. Let

$$E = \{ u \in H \mid \sum_{j,k} \left| j^2 - \frac{a^2 k^2}{b^2} \right| \cdot |u_{j,k}|^2 < +\infty \},$$

$$(u,v) = \sum_{j,k} |j^2 - \frac{a^2 k^2}{b^2}| u_{j,k} \cdot \overline{v}_{j,k} \text{ for } u, v \in E,$$

where $(\ , \ )$ is a scalar product on $E$. With this scalar product $E$ is a Hilbert space with a norm

$$\|u\| = (u,u)^{\frac{1}{2}}, \quad u \in E.$$

Let

$$\|u\|_r = \left( \int_{\Omega} |u|^r \right)^{\frac{1}{r}}, \quad r \geq 1.$$

By the classical theorem of Riesz (cf. [9, p525]), we have

$$\|u\|_r \leq \left( \frac{\pi T}{2} \right)^{\frac{1}{2}} \left( \sum_{j,k} |u_{j,k}|^{r'} \right)^{\frac{1}{r'}}, \quad r \geq 2, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Since for every $\epsilon > 0$

$$\sum_{j \text{ odd} \atop k \text{ even}} \frac{1}{|j^2 - \frac{a^2 k^2}{b^2}|^{1+\epsilon}} < \infty,$$

it follows that for every $r \in [2, +\infty)$ there is $c_r \in \mathbb{R}$ such that

$$\|u\|_r \leq c_r \|u\|.$$ (3.2)

Let

$$E_+ = \{ u \mid u \in E, \ u_{j,k} = 0 \text{ if } j^2 - \frac{a^2 k^2}{b^2} < 0 \},$$

$$E_- = \{ u \mid u \in E, \ u_{j,k} = 0 \text{ if } j^2 - \frac{a^2 k^2}{b^2} > 0 \}.$$

Then $E = E_+ \oplus E_-$, for $u \in E$, $u = u^+ + u^- \in E_+ \oplus E_-$. Let $P_+$ be the orthogonal projection on $E_+$ and $P_-$ be the orthogonal projection
on $E_\pm$. We can write $P_+u = u^+, P_-u = u^-$, for $u \in E$. We consider the following functional associated with (1.1),

$$I(u) = \frac{1}{2}\int_\Omega [\vert u_t \vert^2 + \vert u_x \vert^2] \, dx \, dt - \int_\Omega a(x, t)G(u) \, dx \, dt.$$  \hfill (3.3)

$$= \frac{1}{2} (\Vert P_+u \Vert^2 - \Vert P_-u \Vert^2) - \int_\Omega a(x, t)G(u) \, dx \, dt,$$

where

$$G(u) = \int_0^u g(s) \, ds.$$

From (g4) and (3.2), $I$ is well defined. The solutions of (1.1) coincide with the nonzero critical points of $I(u)$. The following proposition shows that $I(u) \in C^1(E, R)$ (For the proof, refer to [3]).

**Proposition 1.** Assume that $g$ satisfies (g1) – (g4). Then $I(u)$ is continuous and Fréchet differentiable in $E$ with Fréchet derivative

$$I'(u)h = \int_\Omega [-u_t \cdot h_t + u_x \cdot h_x - a(x, t)g(u)h] \, dx \, dt$$  \hfill (3.4)

$$= (P_+u, P_+h) - (P_-u, P_-h) - \int_\Omega a(x, t)g(u)h \, dx \, dt$$

for all $h \in E$. Moreover if we set

$$F(u) = \int_\Omega \int_\Omega a(x, t)G(u) \, dx \, dt,$$

then $F'(u)$ is continuous with respect to weak convergence, $F'(u)$ is compact, and

$$F'(u)h = \int_\Omega a(x, t)g(u)h \, dx \, dt \quad \text{for all } h \in E.$$

This implies that $I \in C^1(E, R)$ and $F(u)$ is weakly continuous.

The following proposition shows that $I(u)$ satisfies (PS) condition when $\frac{1}{2}g(u)u - G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_\Omega a^- (x, t) < \epsilon$.

**Proposition 2.** Assume that $g$ satisfies (g1) – (g4). We also assume that $\frac{1}{2}g(u)u - G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_\Omega a^- (x, t) \, dx \, dt < \epsilon$. Then $I(u)$ satisfies the Palais-Smale condition provided that $p$ in (g4) is restricted by $p + 1 < \mu$ : If for a sequence $(u_m), I(u_m)$ is bounded from above and $I'(u_m) \to 0$ as $m \to \infty$, then $(u_m)$ is bounded.
Proof. Suppose that \((u_m)\) is a sequence with \(I(u_m) \leq M\) and \(I'(u_m) \to 0\) as \(m \to \infty\). Then, by (g3), (g4), (3.2), (1.2) and the H"{o}lder inequality, we have: for large \(m\) with \(u = u_m\),

\[
M + \frac{1}{2}\|u\|_2 \geq I(u) - \frac{1}{2}I'(u)u = \int_{\Omega} \frac{1}{2}a(x,t)g(u)u - a(x,t)G(u)
\]

\[
= \int_{\Omega} a^+(x,t)\left[\frac{1}{2}g(u)u - G(u)\right] - \int_{\Omega} a^-(x,t)\left[\frac{1}{2}g(u)u - G(u)\right]
\]

\[
\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu \int_{\Omega} a^+(x,t) \cdot G(u)
\]

\[
- \max_{\Omega^-} \left|\frac{1}{2}g(u)u - G(u)\right| \int_{\Omega^-} a^- (x,t) dx \, dt
\]

\[
\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu \int_{\Omega} a^+(x,t) \cdot (a_3|u|^{\mu} - a_4)
\]

\[
- \max_{\Omega^-} \left|\frac{1}{2}g(u)u - G(u)\right| \int_{\Omega^-} a^- (x,t) dx \, dt
\]

Thus if \(\frac{1}{2}g(u)u - G(u)\) is bounded or there exists an \(\epsilon > 0\) such that \(\int_{\Omega^-} a^- (x,t) < \epsilon\), then we have

\[
1 + \|u\| \geq M_1 \int_{\Omega} |u|^{\mu} \geq M_2 \left(\int_{\Omega} |u|^2 dx \, dt\right)^{\frac{1}{2}} |u|^{\mu}.
\] (3.5)

Moreover since

\[
|I'(u_m)\varphi| \leq \|\varphi\|
\] (3.6)
for large \( m \) and all \( \varphi \in E \), choosing \( \varphi = u_m^+ \in E_+ \) gives
\[
\|u_m^+\|^2 = \int_\Omega ((u_m)_t - (u_m)_{xx}) \cdot u_m^+ \\
\leq \int_\Omega a(x, t)g(u_m)u_m^+ + \|u_m^+\| \\
\leq \int_\Omega |a(x, t)||g(u_m)||u_m| + \|u_m\| \\
\leq \|a\|_\infty \int_\Omega (a_1 |u_m|^{p+1} + a_2 |u_m|) + \|u_m\| \\
\leq C_1 \int_\Omega |u_m|^{p+1} + C_2 \|u_m\|_{L^2(\Omega)} + \|u_m\| \\
\leq C_1 \int_\Omega |u_m|^{p+1} + C'_2 \|u_m\|.
\]
Taking \( \varphi = -u_m^- \) in (3.6) yields
\[
\|u_m^-\|^2 = \int_\Omega ((u_m)_t - (u_m)_{xx}) \cdot (-u_m^-) \\
\leq \int_\Omega a(x, t)g(u_m) \cdot (-u_m^-) + \| - u_m^-\| \\
\leq \int_\Omega |a(x, t)||g(u_m)||u_m| + \|u_m\| \\
\leq \|a\|_\infty \int_\Omega (a_1 |u_m|^{p+1} + a_2 |u_m|) + \|u_m\| \\
\leq C_3 \int_\Omega |u_m|^{p+1} + C_4 \|u_m\|_{L^2(\Omega)} + \|u_m\| \\
\leq C_3 \int_\Omega |u_m|^{p+1} + C'_4 \|u_m\| + \|u_m\|.
\]
Thus, by (3.5), if \( p + 1 \leq \mu \), we have
\[
\|u_m\|^2 = \|u_m^+\|^2 + \|u_m^-\|^2 \leq M_3 \int_\Omega |u_m|^{p+1} + M_4 \|u_m\| \\
\leq M_3 \int_\Omega |u_m|^{p} + M_4 \|u_m\| \\
\leq M_5 1 + \|u_m\| + M_4 \|u_m\| \leq M_6 (1 + \|u_m\|).
\]
from which the boundedness of \((u_m)\) follows. Thus \((u_m)\) converges weakly in \(E\). Since \(P_\pm u_m = \pm P_\pm u_m + P_\pm \overline{P}(u_m)\) with \(\overline{P}\) compact and the weak convergence of \(P_\pm u_m\) imply the strong convergence of \(P_\pm u_m\) and hence \((P.S.)\) condition holds.

Next, we will prove that \(I(u)\) satisfies one of geometrical assumptions of the critical point theorem of indefinite functional \(I(u)\).

**Proposition 3.** Assume that \(g\) satisfies \((g1)-(g4)\). Then there exist a small real number \(\rho > 0\), \(\delta > 0\), \(u_0 \in E\) satisfying \(\|u_0\| > \rho\) such that

1. if \(\|u\| = \rho\), then \(I(u) \geq \delta\) and
2. \(I(u_0) \leq 0\).

**Proof.** (1) By \((g4)\), (1.2), (3.2) and the Hölder inequality, we have

\[
I(u) = \frac{1}{2}\|P_+ u\|^2 - \frac{1}{2}\|P_- u\|^2 - \int_\Omega a(x,t)G(u)
\]

\[
\geq \frac{1}{2}\|P_+ u\|^2 - \frac{1}{2}\|P_- u\|^2 - \|a\|\int_\Omega C_1|u|^{p+1}
\]

\[
\geq \frac{1}{2}\|P_+ u\|^2 - \frac{1}{2}\|P_- u\|^2 - \|a\|\int_\Omega C_1'\|u\|^{p+1}
\]

for \(C_1, C_1' > 0\). Since \(p + 1 > 2\), there exist \(\rho > 0\) and \(\delta > 0\) such that if \(\|u\| = \rho\), then \(I(u) \geq \delta\).

(2) If we choose \(\psi \in E\) such that \(\|\psi\| = 1\), \(\psi \geq 0\) in \(\Omega\) and \(\text{support}(\psi) \subset \Omega^+\), then we have

\[
I(t\psi) \leq \frac{1}{2}\|P_+ (t\psi)\|^2 - \frac{1}{2}\|P_- (t\psi)\|^2 - \int_{\Omega^+} a(x,t) (a_3 t^\mu \psi^\mu - a_4)
\]

\[
\leq \frac{1}{2}\|t\psi\|^2 - \int_{\Omega^+} (a_3 t^\mu \psi^\mu - a_4)
\]

\[
= \frac{1}{2} t^2 - \int_{\Omega^+} a(x,t) (a_3 t^\mu \psi^\mu - a_4)
\]

for all \(t > 0\). Since \(\mu > 2\), for \(t_0\) great enough, \(u_0 = t_0 \psi\) is such that \(\|u_0\| > \rho\) and \(I(u_0) \leq 0\). \(\square\)

**Proof of Theorem 1.1 and Theorem 1.2(1)**
By Proposition 3.1 and 3.2 \( I(u) \in C^1(E, \mathbb{R}) \) and satisfies the Palais-Smale condition. By Proposition 3.3 there exist \( \rho > 0, \delta > 0, \) \( u_0 \in E \) satisfying \( \|u_0\| > \rho \) such that if \( \|u\| = \rho \), then \( I(u) \geq \delta \), and \( I(u_0) \leq 0 \). By the critical point theorem for indefinite functional, \( I(u) \) has a critical value \( b \geq \delta \) given by

\[
b = \inf_{\gamma \in \Gamma} \max_{[0,1]} I(\gamma(t)),
\]

where \( \Gamma = \{ \gamma \in C([0,1], E) \mid \gamma(0) = 0 \text{ and } \gamma(1) = u_0 \} \).

We denote by \( \tilde{u} \) a critical point of \( I \) such that \( I(\tilde{u}) = b \). We claim that there exists a constant \( C > 0 \) such that

\[
\|a^+(x, t)^{\frac{1}{p}} \tilde{u}\|_{L^2(\Omega)} \leq C \left( 1 + L \int_{\Omega} a^- (x, t) dx \right)^{\frac{1}{p}},
\]

where \( L = \max_{\Omega} |\frac{1}{2} g(\tilde{u}) \tilde{u} - G(\tilde{u})| \).

In fact, we have

\[
b \leq \max_{0 \leq t \leq 1} I(tu_0),
\]

and

\[
I(tu_0) = t^2 \left( \frac{1}{2} \|P_+ u_0\|^2 - \frac{1}{2} \|P_- u_0\|^2 \right) - \int_{\Omega} a(x, t) G(tu_0) dx dt
\]

\[
\leq t^2 \|u_0\|^2 - \int_{\Omega} a^+(x, t) G(tu_0) dx dt + \int_{\Omega} a^-(x, t) G(tu_0) dx dt
\]

\[
\leq t^2 \|u_0\|^2 - a_3 t^\mu \int_{\Omega} a^+(x, t) u_0^\mu + a_4 \int_{\Omega} a^+(x, t) + a_5 t^{p+1} \int_{\Omega} a^-(x, t) u_0^{p+1}
\]

\[
= Ct^2 - Ct^\mu + C + C't^{p+1}.
\]
Since $0 \leq t \leq 1$, $b$ is bounded: $b < \tilde{C}$.

We can write
\[
b = I(\tilde{u}) - \frac{1}{2} I'(\tilde{u}) \tilde{u}
\]
\[
= \int_{\Omega} a(x,t) \left( \frac{1}{2} g(\tilde{u}) \tilde{u} - G(\tilde{u}) \right) \, dx \, dt
\]
\[
= \int_{\Omega} a^+(x,t) \left( \frac{1}{2} g(\tilde{u}) \tilde{u} - G(\tilde{u}) \right) \, dx \, dt
\]
\[
- \int_{\Omega} a^-(x,t) \left( \frac{1}{2} g(\tilde{u}) \tilde{u} - G(\tilde{u}) \right) \, dx \, dt
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \int_{\Omega} a^+(x,t) g(\tilde{u}) \tilde{u} - \max_{\Omega} \left| \frac{1}{2} g(\tilde{u}) \tilde{u} - G(\tilde{u}) \right| \int_{\Omega} a^-(x,t) \, dx \, dt
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \mu \int_{\Omega} a^+(x,t) (a_3 |\tilde{u}|^\mu - a_4) - L \int_{\Omega^1} a^-(x,t) \, dx \, dt,
\]

where $L = \max_{\Omega} |\frac{1}{2} g(\tilde{u}) \tilde{u} - G(\tilde{u})|$. Thus we have
\[
C \left( 1 + L \int_{\Omega^1} a^-(x,t) \, dx \, dt \right) \geq \int_{\Omega} a^+(x,t) |\tilde{u}|^\mu
\]
\[
\geq \left[ \int_{\Omega} \left( a^+(x,t) \frac{1}{\mu} |\tilde{u}| \right)^{\frac{\mu}{2}} \right]^{\frac{2}{\mu}}, \quad (3.8)
\]

from which we can conclude that $\tilde{u}$ is bounded. In fact, suppose that $\tilde{u}$ is not bounded. Then for any $R > 0$, $|\tilde{u}| \geq R$. Thus we have
\[
\int_{\Omega} a^+(x,t) |\tilde{u}|^\mu \geq R^\mu \int_{\Omega} a^+(x,t) \, dx \, dt
\]

for any $R$, which contradicts the fact (3.8) and the proof of Theorem 1.1 is complete. On the other hand, by Proposition 3.2, if $\frac{1}{2} g(u) u - G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^1} a^-(x,t) \, dx \, dt < \epsilon$, then $I(u)$ satisfies the Palais-Smale condition. Proposition 3.3 and the critical point theorem for indefinite functional show that $I(u)$ has a critical value $b$ with critical point $\tilde{u}$ such that $I(\tilde{u}) = b$. If $\int_{\Omega^1} a^-(x,t) \, dx \, dt$ is sufficiently small, by (3.8), we have
\[
\int_{\Omega} a^+(x,t) |\tilde{u}|^\mu \leq C
\]

for $C > 0$, from which we can conclude that $\tilde{u}$ is bounded and the proof of Theorem 1.2(1) is complete.
4. Proof of Theorem 1.2(2)

In this section we assume that \( \frac{1}{2}g(u)u - G(u) \) is not bounded and there exists an \( \epsilon > 0 \) such that \( \int_{\Omega} a^{-}(x,t) < \epsilon \). Then \( I \in C^{1}(E,R) \) and satisfies the Palais-Smale condition (cf. Proposition 3.1 and 3.2). Now, we define a functional

\[
J(u) = \frac{1}{2} (\| P_{+}u \|^2 - \| P_{-}u \|^2) - \| a \|_{\infty} \int_{\Omega} \frac{a_{1}}{p + 1}|u|^{p + 1}.
\]

Then \( J \in C^{1}(E,R) \) and satisfies the Palais-Smale condition, and

\[
J(u) - \| a \|_{\infty}a_{2}\pi T \leq I(u).
\]

Let \((E_{i})_{i \geq 0}\) be a sequence of subspaces of \( E \) such that there exist an odd integer \( j_{i} \) and an even integer \( k_{i} \) such that

1. \( E_{i} \) is spanned by \( \sin j_{i}x \sin k_{i}a_{b}t, \sin j_{i}x \cos k_{i}a_{b}t \).
2. \( i \leq i' \Rightarrow (k_{i}a_{b})^{2} - (j_{i})^{2} \leq (k_{i'}a_{b})^{2} - (j_{i'})^{2} \),
3. \( E = \bigoplus_{i \in N} E_{i} \).

Let \( V_{m} = \bigoplus_{i \leq m} E_{i} \oplus E_{-} \).

From Proposition 3.3, there exists an \( R_{m} > 0 \) such that

\[
J(u) - \| a \|_{\infty}a_{2}\pi T \leq I(u) \leq 0 \quad \text{for } u \in \left( V_{m} \cap E^{+} \right) \setminus B_{R_{m}}.
\]

For \( u \in E, \theta \in [0, T] \) set:

\[
s_{\theta}u(x,t) = u(x, t + \theta).
\]

If \( u \in E, s_{\theta}u \in E \) and \( I(u) = I(s_{\theta}u), J(u) = J(s_{\theta}u) \).

Let

\[
F = \{ u \in E \mid u \text{ is independent of } t \}.
\]

We have

\[
F = \{ u \in E \mid s_{\theta}u = u \quad \forall \theta \in [0, T] \}.
\]

We remark that

\[
F \subset E^{+}.
\]

We call a subset \( B \) of \( E \) an invariant set if for all \( u \in B, s_{\theta}u \in B \) for all \( \theta \in [0, T] \). Let \( C(B,E) \) be the set of continuous functions from \( B \) into \( E \). If \( B \) is an invariant set we say \( h \in C(B,E) \) is an equivariant map if \( h(s_{\theta}u) = s_{\theta}h(u) \) for all \( \theta \in [0, T] \) and \( u \in B \). Let

\[
\varepsilon = \{ B \mid B \subset E \setminus \{0\}, B \text{ is closed and invariant } \}.
\]

In [10] it is proved that there is an index theory i.e., a mapping \( i : \varepsilon \to N \cup \{\infty\} \) such that if \( B, B_{1} \in \varepsilon \),
(1) \( i(B) \leq i(B_1) \) if there is \( \varphi \in C(B, B_1) \) with \( \varphi \) equivariant.

(2) \( i(B \cup B_1) \leq i(B) + i(B_1) \).

(3) If \( B \subset E \setminus F \) and \( B \) is compact, \( i(B) < +\infty \) and there is a \( \delta > 0 \) such that \( i(N_{\delta}(B)) = i(B) \) where \( N_{\delta}(B) = \{ x \mid |x - B| \leq \delta \} \).

(4) If \( S \subset E \setminus F \) is a \( 2n \) dimensional invariant sphere,

\[
i(S) = n.
\]

Let \( G_m \) denote the class of mapping \( h \in C(D_m, E) \) which satisfy the following properties

1. \( h \) is equivariant
2. \( h(u) = u \) for all \( u \in (\partial B_{R_m} \cap V_m) \cup F \).
3. \( Ph(u) = \alpha(u)Pu + \Psi(u) \) where \( \Psi \) is compact and \( \alpha \in C(D_m, [1, \overline{\alpha}]) \), \( \overline{\alpha} \) depending on \( h \).

Let

\[
\Gamma_j = \{ h(D_m \setminus Y) \mid m \geq j, h \in G_m, Y \in \varepsilon \text{and} i(Y) \leq m - j \};
\]

\[
c_j = \inf_{B \in \Gamma_j} \sup_{u \in B} I(u),
\]

\[
b_j = \inf_{B \in \Gamma_j} \sup_{u \in B} J(u).
\]

As in [13] we have the following lemma.

**LEMMA 4.1.** \( b_j \) is a critical value of \( J \),

\[
b_j - a_2\|a\|_\infty \pi T \leq c_j,
\]

(4.4)

if \( c_j \geq \delta \), then \( c_j \) is a critical value of \( I \),

(4.5)

where \( \delta \) is defined as in [13], i.e.,

\[
\delta = \sup_{E_0} \left( a_4 \int_\Omega a^-(x, t) + \frac{c}{p + 1} \int_\Omega a^-(x, t)|u|^{p+1} \right),
\]

where \( c = \max\{a_1, a_2\} > 0 \) and \( E_0 \) is the null space of \( A \).

**Proof of Theorem 1.2(2)**

We note that

\[
b_j \geq \sup_{\rho} \left( \inf_{u \in V_{j-1}^\perp} J(\rho u) \right).
\]

(4.6)

If \( u \in V_{j-1}^\perp \), by (3.2), there exists \( \epsilon_j \) with

\[
\lim_{j \to \infty} \epsilon_j = 0.
\]
such that \( \|u\|_{p+1} \leq \epsilon_j \|u\| \).
If \( u \in V_{j-i}^\perp \) and \( \|u\| = 1 \), by (g3), (g4),
\[
J(\rho u) \geq \frac{\rho^2}{2} - \epsilon_j \frac{a_1}{p+1} \rho^{p+1} \|a\|_\infty.
\] (4.7)
Thus if \( j \to \infty \), then \( J(\rho u) \geq \frac{a^2}{2} \). Using (4.6) we have
\[
\lim_{j \to \infty} b_j = \infty.
\] (4.8)
Using (4.8), (4.4), and (4.5) we see that for \( j \) large enough \( c_j \) is a critical value of \( I \) and
\[
\lim_{j \to \infty} c_j = +\infty.
\] (4.9)
Note that \( A_1 u = a(x, t)g(u) \) and \( \max_{x \in [0, \pi], t \in [0, T]} |u(x, t)| \leq K \) imply
\[
I(u) \leq \left( \max_{|s| < K} \frac{1}{2} sg(s) - \min_{|s| < K} G(s) \right) \int_{\Omega} a^+(x, t) dx dt.
\]
We conclude the proof using (4.9).

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