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SOME RESULTS OF R-GROUP STRUCTURES

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Abstract. In this paper, we initiate a study of faithful R-group G and some substructures of R and G. Next, we investigate a faithful representation of near-ring R and some properties of monogenic R-groups.

1. Introduction

A (left) near-ring R is an algebraic system $(R, +, \cdot)$ with two binary operations, say $+$ and $\cdot$ such that $(R, +)$ is a group (not necessarily abelian) with neutral element 0, $(R, \cdot)$ is a semigroup and $a(b + c) = ab + ac$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*. An element d in R is called *distributive* if $(a + b)d = ad + bd$ for all a and b in R.

An *ideal* of R is a subset I of R such that (i) $(I, +)$ is a normal subgroup of $(R, +)$, (ii) $a(I + b) - ab \subset I$ for all $a, b \in R$, (iii) $(I + a)b - ab \subset I$ for all $a, b \in R$. If I satisfies (i) and (ii) then it is called a *left ideal* of R. If I satisfies (i) and (iii) then it is called a *right ideal* of R.

On the other hand, an *R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R. In case, $(H, +)$ is normal in above, we say that *normal R-subgroup*, *normal left R-subgroup* and *normal right R-subgroup* instead of R-subgroup, left R-subgroup and right R-subgroup, respectively.

We consider the following notations: Given a near-ring R, $R_0 = \{a \in R \mid 0a = 0\}$ is called the *zero symmetric part* of R, $R_c = \{a \in R \mid 0a =$

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$a\}$ is called the *constant part* of R, and $R_d = \{a \in R \mid a \textit{ is distributive}\}$ is called the *distributive part* of R.

We note that R_0 and R_c are subnear-rings of R, but R_d is not a subnear-ring of R. A near-ring R with the extra axiom $0a = 0$ for all $a \in R$, that is, $R = R_0$ is said to be *zero symmetric,* also, in case $R = R_c$, R is called a *constant* near-ring, and in case $R = R_d$, R is called a *distributive* near-ring.

Let $(G, +)$ be a group (not necessarily abelian). In the set

$$M(G) = \{f \mid f : G \longrightarrow G\}$$

of all the self maps of G, if we define the sum $f+g$ of any two mappings f, g in $M(G)$ by the rule $x(f+g) = xf + xg$ for all $x \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$, then $(M(G), +, \cdot)$ becomes a near-ring. It is called the *self map near-ring* of the group G. Also, if we define the set

$$M_0(G) = \{f \in M(G) \mid 0f = 0\},$$

then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring.

Let R and S be two near-rings. Then a mapping θ from R to S is called a *near-ring homomorphism* if (i) $(a+b)\theta = a\theta + b\theta$, (ii) $(ab)\theta = a\theta b\theta$. We can replace homomorphism by momomorphism, epimorphism, isomorphism, endomorphism and automorphism, if these terms have their usual meanings as for ring theory([1]).

Let R be any near-ring and G an additive group. Then G is called an *R-group* if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a *representation* of R on G, we write that xr (right scalar multiplication in R) for $x(r\theta)$ for all $x \in G$ and $r \in R$. If R is unitary, then R-group G is called *unitary*. Thus an R-group is an additive group G satisfying (i) $x(a+b) = xa + xb$, (ii) $x(ab) = (xa)b$ and (iii) $x1 = x$ (If R has a unity 1), for all $x \in G$ and $a,\ b \in R$. Evidently, every near-ring R can be given the structure of an R-group (unitary if R is unitary) by right multiplication in R. Moreover, every group G has a natural $M(G)$-group structure, from the representation of $M(G)$ on G given by applying the $f \in M(G)$ to the $x \in G$ as a scalar multiplication xf.

A representation θ of R on G is called $faithful$ if $Ker\theta = \{0\}$. In this case, we say that G is called a *faithful R-group*.

For an R-group G, a subgroup T of G such that $TR \subset T$ is called an *R-subgroup* of G, a normal subgroup N of G such that $NR \subset N$ is called a *normal R-subgroup* of G and an *R-ideal* of G is a normal subgroup N of G such that $(N + x)a - xa \subset N$ for all $x \in G,\ a \in R$. Also, note that normal R-subgroups of G are not equivalent to an R-ideals of R.

Let R be a near-ring and let G be an R-group. If there exists x in G such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then G is called a *monogenic R-group* and the element x is called a *generator* of G, more specially, if G is monogenic and for each $x \in G$, $xR = 0$ or $xR = G$, then G is called a *strongly monogenic R-group.* It is clearly proved that $G \neq 0$ if and only if $GR \neq 0$ for any monogenic or strongly monogenic R-group G. For the remainder concepts and results on near-rings, we refer to [6].

2. Some Properties of monogenic R-Groups

A near-ring R is called *distributively generated* (briefly, *d.g.*) if it contains a subsemigroup S of $(R_d, \cdot)$ which generates the additive group $(R, +)$, we denote it by (R, S).

On the other hand, the set of all distributive elements of $M(G)$ are obviously the semigroup $End(G)$ of all endomorphisms of the group G under composition. Here we denote that $E(G)$ is the d.g. near-ring generated by $End(G)$, that is, $E(G)$ is d.g. subnear-ring of $(M_0(G), +, \cdot)$ generated by $End(G)$. It is said to be that $E(G)$ is the *endomorphism near-ring* of the group G.

Let (R, S) and (T, U) be d.g. near-rings. Then a near-ring homomorphism

$$\theta : (R, S) \longrightarrow (T, U)$$

is called a *d.g. near-ring homomorphism* if $S\theta \subseteq U$. Note that a semigroup homomorphism $\theta : S \longrightarrow U$ is a d.g. near-ring homomorphism if it is a group homomorphism from $(R, +)$ to $(T, +)$ (C.G. Lyons and J.D.P. Meldrum [3], [4]).

Example 2.1. *If R is a distributive near-ring with unity 1, then R is a ring. Furthermore, if R is a distributive near-ring with unity 1, then every (R,R)-group is a unitary R-module.*

Proof. Let G be an (R,R)-group. Since G is unitary, $x(1+1) = x+x$, for all $x \in G$. Thus we have that

$$x + y + x + y = (x + y)(1 + 1) = x(1 + 1) + y(1 + 1) = x + x + y + y,$$

for all $x, y \in G$. This implies that $(G,+)$ is abelian. Since $R = S$, the set of all distributive elements, $(x + y)r = xr + yr$, for all $x, y \in G$ and all $r \in R$. Hence G becomes a unitary R-module. □

Lemma 2.2 ([5]). *Let (R,S) be a d.g. near-ring. Then all R-subgroups and all R-homomorphic images of a (R,S)-group are also (R,S)-groups.*

Now, we consider that the substructures of R and G, also quotients of substructure relations between them.

Let G be an R-group and K, K_1 and K_2 be subsets of G. Define

$$(K_1 : K_2) := \{a \in R; K_2a \subset K_1\}.$$

We abbreviate that for $x \in G$

$$(\{x\} : K_2) =: (x : K_2).$$

Similarly for $(K_1 : x)$. $(0 : K)$ is called the *annihilator* of K, denoted it by $A(K)$. We say that G is a *faithful R-group* or that R *acts faithfully* on G if $A(G) = \{0\}$, that is, $(0 : G) = \{0\}$.

A subgroup H of G such that $xa \in H$ for all $x \in H, a \in R$, is an *R-subgroup* of G, and an *R-ideal* of G is a normal subgroup N of G such that

$$(x + g)a - ga \in N$$

for all $g \in G, x \in N$ *and* $a \in R$ (J.D.P. Meldrum [6]).

Lemma 2.3. *Let G be an R-group and K_1 and K_2 subsets of G. Then*

(1) *If K_1 is a normal R-subgroup of G, then $(K_1 : K_2)$ is a normal right R- subgroup of the near-ring R.*

(2) *If* K_1 *is an* R*-subgroup of* G*, then* $(K_1 : K_2)$ *is a right* R*-subgroup.*
(3) *If* K_1 *is an* R*-ideal of* G *and* K_2 *is an* R*-subgroup of* G*, then* $(K_1 : K_2)$ *is a two-sided ideal of* R*.*

Proof. (1) and (2) are proved by J.D.P. Meldrum [6]. Now, we will prove only (3) : Using the condition (1), $(K_1 : K_2)$ is a normal subgroup of R. Let $a \in (K_1 : K_2)$ and $r \in R$. Then

$$K_2(ra) = (K_2r)a \subset K_2a \subset K_1,$$

so that $ra \in (K_1 : K_2)$. Whence $(K_1 : K_2)$ is a left ideal of R.

Next, let $r_1, r_2 \in R$ and $a \in (K_1 : K_2)$. Then

$$k\{(a + r_1)r_2 - r_1r_2\} = (ka + kr_1)r_2 - kr_1r_2 \in K_1$$

for all $k \in K_2$, since $K_2a \subset K_1$ and K_1 is an ideal of G. Thus $(K_1 : K_2)$ is a right ideal of R. Therefore $(K_1 : K_2)$ is a two-sided ideal of R. □

Corollary 2.4 ([6]). *Let* R *be a near-ring and* G *an* R*-group. Then*

(1) *For any* $x \in G$*,* $(0 : x)$ *is a right ideal of* R*.*
(2) *For any* R*-subgroup* K *of* G*,* $(0 : K)$ *is a two-sided ideal of* R*.*
(3) *For any subset* K *of* G*,* $(0 : K) = \bigcap_{x \in K}(0 : x)$*.*

Proposition 2.5. *Let* R *be a near-ring and* G *an* R*-group. Then*

(1) $A(G)$ *is a two-sided ideal of* R*. Moreover* G *is a faithful* $R/A(G)$*-group.*
(2) *For any* $x \in G$*, we get* $xR \cong R/(0 : x)$ *as* R*-groups.*

Proof. (1) By Corollary 2.4 and Lemma 2.3, $A(G)$ is a two-sided ideal of R. We now make G an $R/A(G)$-group by defining, for $r \in R, r + A(G) \in R/A(G)$, the action $x(r + A(G)) = xr$. If $r + A(G) = r' + A(G)$, then $-r' + r \in A(G)$ hence $x(-r' + r) = 0$ for all x in G, that is to say, $xr = xr'$. This tells us that

$$x(r + A(G)) = xr = xr' = x(r' + A(G));$$

thus the action of $R/A(G)$ on G has been shown to be well defined. The verification of the structure of an $R/A(G)$-group is a routine triviality. Finally, to see that G is a faithful $R/A(G)$-group, we note that if $x(r+A(G)) = 0$ for all $x \in G$, then by the definition of $R/A(G)$-group structure, we have $xr = 0$. Hence $r \in A(G)$. This says that only the zero element of $R/A(G)$ annihilates all of G. Thus G is a faithful $R/A(G)$-group.
(2) For any $x \in G$, clearly xR is an R-subgroup of G. The map $\phi : R \longrightarrow xR$ defined by $\phi(r) = xr$ is an R-ephimorphism, so that from the isomorphism theorem, since the kernel of ϕ is $(0 : x)$, we deduce that

$$xR \cong R/(0 : x)$$

as R-groups. □

Corollary 2.6. *Let G be a monogenic R-group with x as a generator. Then we have the following isomorphic relation.*

$$G \cong R/(0 : x).$$

Proposition 2.7. *If R is a near-ring and G an R-group, then $R/A(G)$ is isomorphic to a subnear-ring of $M(G)$.*

Proof. Let $a \in R$. We define $\tau_a : G \longrightarrow G$ by $x\tau_a = xa$ for each $x \in G$. Then τ_a is in $M(G)$. Consider the mapping $\phi : R \longrightarrow M(G)$ defined by $\phi(a) = \tau_a$. Then obviously, we see that

$$\phi(a+b) = \phi(a) + \phi(b) \text{ and } \phi(ab) = \phi(a)\phi(b),$$

that is, ϕ is a near-ring homomorphism from R to $M(G)$.

Next, we must show that $Ker\phi = A(G)$. Indeed, if $a \in Ker\phi$, then $\tau_a = 0$, which implies that $Ga = G\tau_a = 0$, that is, $a \in A(G)$. On the other hand, if $a \in A(G)$, then by the definition of $A(G)$, $Ga = 0$ hence $0 = \tau_a = \phi(a)$, this implies that $a \in Ker\phi$. Therefore from the first isomorphism theorem on $R-$ groups, the image of R is a near-ring isomorphic to $R/A(G)$. Consequently, $R/A(G)$ is isomorphic to a subnear-ring of $M(G)$. □

Thus we can obtain the following important statement as in ring theory.

Corollary 2.8. *If G is a faithful R-group, then R is embedded in $M(G)$.*

Corollary 2.9. *If (R, S) is a d.g. near-ring, then every monogenic R-group is an (R, S)-group.*

Proof. Let G be a monogenic R-group with x as a generator. Then the map $\phi : r \mapsto xr$ is an R-epimorphism from R to G as R-groups. We see that by the Corollary 2.6,

$$G \cong R/A(x),$$

where $A(x) = (0 : x) = Ker\phi$. From the Lemma 2.2, we see that G is an (R, S)-group. □

Now, we get the following useful results of monogenic R-groups to make primitive near-rings.

Proposition 2.10. *Let G be a monogenic R-group with generator x. Then*

(1) *For any right ideal I of R, xI is an R-ideal of G.*
(2) *If I is a left R-subgroup of R and xI is an R-ideal of G, then I is an ideal of R.*
(3) *If e is a right identity of R and if G is a faithful R-group, then e is a two-sided identity of R.*

Proof. (1) Let $a \in G$. Then there exists $t \in R$ such that $a = xt$. Thus for each $xy \in xI, r \in R$, and $a \in G$,

$$(a + xy)r - ar = (xt + xy)r - (xt)r = x(t + y)r - x(tr)$$

$$= x\{(t + y)r - tr\} \in xI$$

By using similar method, it can be easily shown that xI is an additive normal subgroup of G. Therefore xI is an R-ideal of G.
(2) For any $y \in I$ and $a,\ b \in R$, we obtain the following equality:

$$x\{(y + a)b - ab\} = x(y + a)b - x(ab) = (xy + xa)b - (xa)b) \in xI$$

Hence $(y+a)b-ab \in xI$. Similarly, we can show that I is an additive normal subgroup of R. Consequently, I is an ideal of R.
(3) First, let e be a right identity of R and $g = xt$ be any element in G. Then we have the relation that

$$ge = (xt)e = x(te) = xt = g$$

Next, let r be any element of R and g be an arbitrary element in G. Then one gets the following equality that

$$g(er - r) = g(er) + g(-r) = (ge)r - gr = gr - gr = 0$$

Thus $(er - r) \in (0 : G) = A(G)$.
Since G is faithful, it implies that $er-r = 0$, that is, $er = r$. Hence e is a two-sided identity of R. □

We note that, in the above Proposition 2.10 (2), if R satisfies DCCN and G satisfies DCCI, then R satisfies DCCI.

Lemma 2.11 (Wielandt and Betsch [2]). *If R is a zero symmetric near-ring and A, B, K are R-ideals of an R-group G, then*

(1) *We get an additive abelian group:*

$$G' = [(A + K) \cap (B + K)]/[(A \cap B) + K]$$

and for any x, $y \in G'$, and $r \in R$, we have $(x + y)r = xr + yr$.

(2) *We obtain a quotient ring $R/(0 : G')$.*

Proposition 2.12. *Let G be a faithful monogenic R-group with generator x, where R is a zero symmetric near-ring. If I and J are right ideals of R and $I \cap J \subseteq (0 : x)$, then R is a ring.*

Proof. From the Proposition 2.5 (2), we have that

$$G = xR \cong R/(0 : x) = [(I + (0 : x) \cap J + (0 : x)]/[(I \cap J) + (0 : x)] = G'$$

On the other hand, since G is faithful, by the definition, we see that

$$(0 : G') \cong (0 : G) = A(G) = 0$$

Consequently, Lemma 2.11 implies that R is a ring. □

For an R-group G, we have the following:
F or any x in G, xR is an R-subgroup of G.
F or any R-subgroup A of G, we have that $0R = 0R_c \subseteq A$, where 0 is the additive identity of G.

$0R$ is the smallest R-subgroup of G under all R-subgroups of G, So throughout this paper, we will write that

$$0R = 0R_c =: \Omega.$$

We note that if R is zero symmetric, then $\Omega = 0$, and $\Omega = xR_c$ for all $x \in G$.

Also, we can define the following concepts: An R-group G is called *simple* if G has no non-trivial ideal, that is, G has no ideals except o and G. Similarly, we can define simple near-ring as in the case of ring. Also, R-group G is called *R-simple* if G has no R-subgroups except Ω and G.

LEMMA 2.13. *For an R-group G and a subgroup A of G, we have the following:*

(1) *A is an R-ideal of G if and only if A is an R_0-ideal of G.*
(2) *A is an R-subgroup of G if and only if A is an R_0-subgroup of G and $\Omega \subseteq A$.*

Proof. (1) Obviously, an R-ideal of G is an R_0-ideal of G. Conversely, suppose A is an R_0-ideal of G. Let $a \in A,\ x \in G$ and $r \in R$. Then since $R = R_0 \oplus R_c$, we rewrite that $r = s + t$, where $s \in R_0$ and $t \in R_c$. Thus we have

$$(a+x)r - xr = (a+x)(s+t) - x(s+t) = (a+x)s + (a+x)t - xt - xs$$

Here, since $t \in R_c$, (a+x)t-xt=t-t=0 so that $(a + x)r - xr = (a + x)s - xs$. Also since $s \in R_0$ and A is an R_0-ideal of G, $(a+x)s-xs \in A$, that is $(a + x)r - xr \in A$. Consequently, A is an R-ideal of G.
(2) The statement (2) can be proved by using similar method of the proof of case (1). □

THEOREM 2.14. *Let G be a monogenic R-group with generator x. Then we have the following:*

(1) *If I is a left R-subgroup of R and xI is an R-ideal of G, then $(xI : x)$ is an ideal of R.*
(2) *If G is R_0-simple, then either $GR = 0$ or G is strongly monogenic.*

Proof. (1) For any $y \in I$ and $a,\ b \in R$, we obtain the following equality:

$$x\{(y+a)b - ab\} = x(y+a)b - x(ab) = (xy + xa)b - (xa)b) \in xI$$

Hence $(y+a)b - ab \in (xI : x)$. In this way, we can show that $(xI : x)$ is an additive normal subgroup of R. Consequently, $(xI : x)$ is an ideal of R.
(2) Suppose that G is R_0-simple and $G = GR \neq 0$ (See the note below the definition of monogenic R-group). Then G has no R-subgroups except $\Omega = 0$ and G. Let $x \in G$ and $xR \neq 0$. Then since xR is an R-subgroup, moreover an R_0-subgroup by Lemma 2.13 (2) of G, $G = xR$. Hence G is strongly monogenic. □

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