INJECTIVE AND PROJECTIVE PROPERTIES OF REPRESENTATIONS OF QUIVERS WITH \( n \) EDGES

Sangwon Park

Abstract. We define injective and projective representations of quivers with two vertices with \( n \) arrows. In the representation of quivers we denote \( n \) edges between two vertices as \( \Rightarrow \) and \( n \) maps as \( f_1 \sim f_n \), and \( E \oplus E \oplus \cdots \oplus E \) (\( n \) times) as \( \oplus_n E \). We show that if \( E \) is an injective left \( R \)-module, then
\[
\oplus_n E \xrightarrow{p_1 \sim p_n} E
\]
is an injective representation of \( Q = \bullet \Rightarrow \bullet \) where \( p_i(a_1, a_2, \cdots , a_n) = a_i, \; i \in \{1, 2, \cdots , n\} \). Dually we show that if \( M_1 \xrightarrow{f_1 \sim f_n} M_2 \) is an injective representation of a quiver \( Q = \bullet \Rightarrow \bullet \) then \( M_1 \) and \( M_2 \) are injective left \( R \)-modules. We also show that if \( P \) is a projective left \( R \)-module, then
\[
P \xrightarrow{i_1 \sim i_n} \oplus_n P
\]
is a projective representation of \( Q = \bullet \Rightarrow \bullet \) where \( i_k \) is the \( k \)th injection. And if \( M_1 \xrightarrow{f_1 \sim f_n} M_2 \) is an projective representation of a quiver \( Q = \bullet \Rightarrow \bullet \) then \( M_1 \) and \( M_2 \) are projective left \( R \)-modules.

1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) ([1]). We may consider many different types of quivers. We allow multiple edges and multiple arrows, and edges going from a vertex back to the same vertex. Originally a representation of quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex
to the one assigned to the terminal vertex. For example, a representation of the quiver $Q = \bullet \rightarrow \bullet$ is $V_1 \xrightarrow{f} V_2$, $V_1$ and $V_2$ are vector spaces and $f$ is a linear map (morphism). Then we can define a morphism of two representations of the same quiver i.e., given a quiver $Q = \bullet \rightarrow \bullet$, we can define two representations $V_1 \xrightarrow{f} V_2$ and $W_1 \xrightarrow{g} W_2$.

Now we can define a morphism between these two representations. A morphism of $V_1 \xrightarrow{f} V_2$ to $W_1 \xrightarrow{g} W_2$ is given by a commutative diagram

$$
\begin{array}{ccc}
V_1 & \xrightarrow{f} & V_2 \\
\downarrow{s_1} & & \downarrow{s_2} \\
W_1 & \xrightarrow{g} & W_2
\end{array}
$$

with $s_1, s_2$ linear maps.

In ([3]) a homotopy of quiver was developed and in ([2]) cyclic quiver ring was studies. Recently, the theory of projective representations was developed in ([4]) and the theory of injective representation was studied in ([5]).

**Definition 1.1.** ([7]) A left $R$-module $E$ is said to be injective if given any injective linear map $\sigma : M' \rightarrow M$ and any linear map $h : M' \rightarrow E$, there is a linear map $g : M \rightarrow E$ such that $g \circ \sigma = h$. That is

$$
\begin{array}{ccc}
0 & \rightarrow & M' & \xrightarrow{\sigma} & M \\
& \downarrow{h} & & \downarrow{g} & \\
& & E
\end{array}
$$

can always be completed to a commutative diagram.

**Definition 1.2.** ([7]) A left $R$-module $P$ is said to be projective if given any surjective linear map $\sigma : M' \rightarrow M$ and any linear map $h : P \rightarrow M$, there is a linear map $g : P \rightarrow M'$ such that $\sigma \circ g = h$. That is

$$
\begin{array}{ccc}
P & \xrightarrow{g} & M' \xrightarrow{\sigma} & M \\
& & \downarrow{h} & \\
& & M \rightarrow 0
\end{array}
$$

can always be completed to a commutative diagram.
Let \( G = G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_n \) be a direct product of groups. The projection map \( \pi_i : G \to G_i \) where \( \pi_i(g_1, g_2, \cdots, g_i, \cdots, g_n) = g_i \) is a homomorphism for each \( i = 1, 2, \cdots, n \). This follows immediately from the fact that the binary operation of \( G \) coincides in the \( i \)th component with the binary operation in \( G_i \). Let \( \phi_i : G_i \to G_1 \times G_2 \times \cdots \times G_i \times \cdots \times G_n \) be given by \( \phi_i(g_i) = (e_1, e_2, \cdots, g_i, \cdots, e_n) \) where \( g_i \in G_i \) and \( e_j \) is the identity of \( G_j \). This is an injection map. Let \( F = \{X_i|i \in I\} \) be an indexed family of left \( R \)-modules \( X_i \) and denote \( P = \prod_{i \in I} X_i \) the cartesian product of \( F \). Define an element of \( P \) as a function \( f : I \to \bigcup_{i \in I} X_i \) such that \( f(i) \in X_i \) for every \( i \in I \), and \( 0(i) = 0 \in X_i \) and \( (-f)(i) = -[f(i)] \). Then easily \( (P,+) \) is an abelian group. Define \( \mu : R \times P \to P \) by \( (r f)(i) = r[f(i)] \) for every \( i \in I \). Then easily \( P \) is a left \( R \)-module. We say \( P \) as the direct product of \( F \) over \( R \). Consider \( S \) the subset of \( P \) such that \( f(i) = 0 \) except only finite \( i \in I \). Then easily \( S \) is a submodule of \( P \). We say \( S \) as direct sum of \( F \) and is denoted by \( S = \bigoplus_{i \in I} X_i \).

**Remark 1.** \( p_j|_s : \bigoplus_{i \in I} X_i \to X_j \) is called the natural projection of \( S = \bigoplus_{i \in I} X_i \).

So we have morphisms

\[
X_j \xrightarrow{d_j} \bigoplus_{i \in I} X_i \xrightarrow{i} \prod_{i \in I} X_i \xrightarrow{p_k} X_k
\]

\( p_k \circ i \circ d_j : X_j \to X_k \) is trivial if \( j \neq k \), and is identity if \( j = k \).

**Remark 2.** The natural injection \( d_j : X_j \to \bigoplus_{i \in I} X_i \) is a monomorphism and the natural projection \( p_j : \prod_{i \in I} X_i \to X_j \) is an epimorphism.

Notation : In the representation of quivers we denote \( n \) arrows between two vertices as \( \Rightarrow \) and \( n \) maps as \( f_1 \sim f_n \) and \( E \oplus E \oplus \cdots \oplus E \) (\( n \) times) as \( \oplus_n E \).

**2. Injective representation of a quiver** \( Q = \bullet \Rightarrow \bullet \) with \( n \) edges

We define injective representation of a quiver with two vertices and multiple arrows. And consider their various injective representations as left \( R \)-modules.
**Definition 2.1.** A representation $M_1 \xrightarrow{f_1 \sim f_n} M_2$ of a quiver $Q = \bullet \Rightarrow \bullet$ is called an injective representation if for any representation $N_1 \xrightarrow{g_1 \sim g_n} N_2$ with a subrepresentation

\[
\begin{array}{c}
S_1 \xrightarrow{s_1 | s_2 \sim s_n | s_1} S_2
\end{array}
\]

and morphisms

\[
\begin{array}{c}
S_1 \xrightarrow{h} S_2 \\
M_1 \xrightarrow{f_1 \sim f_n} M_2
\end{array}
\]

there exist $H \in \text{Hom}_R(N_1, M_1)$ and $K \in \text{Hom}_R(N_2, M_2)$ such that the following diagram

\[
\begin{array}{c}
N_1 \xrightarrow{g_1 \sim g_n} N_2 \\
M_1 \xrightarrow{f_1 \sim f_n} M_2
\end{array}
\]

commutes and $H|_{S_1} = h$, $K|_{S_2} = k$.

In other words, every diagram of representations

\[
\begin{array}{c}
(0 \xrightarrow{0}) \xrightarrow{h} (S_1 \xrightarrow{s_1 | s_2 \sim s_n | s_1} S_2) \xrightarrow{k} (N_1 \xrightarrow{g_1 \sim g_n} N_2)
\end{array}
\]

can be completed to a commutative diagram as follows:

\[
\begin{array}{c}
(0 \xrightarrow{0}) \xrightarrow{h} (S_1 \xrightarrow{s_1 | s_2 \sim s_n | s_1} S_2) \xrightarrow{k} (N_1 \xrightarrow{g_1 \sim g_n} N_2)
\end{array}
\]

**Theorem 2.2.** If $E$ is an injective left $R$-module, then

\[
E \xrightarrow{} 0
\]

is an injective representation of $Q = \bullet \Rightarrow \bullet$. 
Proof. Let $M_1, M_2$ be left $R$-modules, $S_1$ be a submodule of $M_1$, $S_2$ be a submodule of $M_2$ and $g : S_1 \rightarrow E$ be an $R$-linear map. Consider the following diagram

\[
(0 \longrightarrow 0) \longrightarrow (S_1 \longrightarrow S_2) \longrightarrow (M_1 \overset{g_1 \sim g_n}{\longrightarrow} M_2)
\]

Then since $E$ is an injective left $R$-module, we can complete the following commutative diagram by $h$.

\[
0 \longrightarrow S_1 \longrightarrow M_1
\]

Then $0(h(m)) = 0 = 0(g_1(m)), 0(h(m)) = 0 = 0(g_2(m)), \cdots, 0(h(m)) = 0 = 0(g_n(m))$. Thus, we can complete the following diagram

\[
M_1 \overset{g_1 \sim g_n}{\longrightarrow} M_2
\]

as a commutative diagram by $0 : M_2 \rightarrow 0$.

Therefore, we can complete the diagram

\[
(0 \longrightarrow 0) \longrightarrow (S_1 \longrightarrow S_2) \longrightarrow (M_1 \overset{g_1 \sim g_n}{\longrightarrow} M_2)
\]

as a commutative diagram. Hence, $E \longrightarrow 0$ is an injective representation.

Theorem 2.3. If $E$ is an injective left $R$-module, then

\[
\bigoplus_n E \overset{p_1 \sim p_n}{\longrightarrow} E
\]

is an injective representation of $Q = \bullet \rightarrow \bullet$ where $p_i(a_1, a_2, \cdots, a_n) = a_i$, $i \in \{1, 2, \cdots, n\}$. 


**Proof.** Let $M_1, M_2$ be a left $R$-module, $S_1$ be a submodule of $M_1$, $S_2$ be a submodule of $M_2$ and $g : S_2 \to E$ be an $R$-linear map. Consider the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & (S_1 & \longrightarrow & S_2) & \longrightarrow & (M_1 & \overset{f_1 \sim f_n}{\longrightarrow} & M_2) \\
& & & & g & \downarrow & \downarrow & & \downarrow & \\
& & & & (\oplus_n E & \overset{p_1 \sim p_n}{\longrightarrow} & E) & & & \\
\end{array}
$$

Then since $E$ is an injective left $R$-module, we can consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & S_2 & \longrightarrow & M_2 \\
& & g & \downarrow & \downarrow & h \\
& & E & & \\
\end{array}
$$

by $h$. Define $H : M_1 \to \oplus_n E$ by $H(m) = (h(f_1(m)), h(f_2(m)), \ldots, h(f_n(m)))$. Then $p_i(H(m)) = h(f_i(m)), i = 1, 2, \ldots, n$ and we can complete the following diagram

$$
\begin{array}{ccccccccc}
M_1 & \overset{f_1 \sim f_n}{\longrightarrow} & M_2 \\
H & \downarrow & \downarrow & h \\
(\oplus_n E & \overset{p_1 \sim p_n}{\longrightarrow} & E) & & & & & & & & \\
\end{array}
$$

as a commutative diagram. Hence, we can complete the diagram

$$
\begin{array}{ccccccccc}
(0 & \longrightarrow & 0 & \longrightarrow & (S_1 & \longrightarrow & S_2) & \longrightarrow & (M_1 & \overset{f_1 \sim f_n}{\longrightarrow} & M_2) \\
& & & & H & \downarrow & \downarrow & h & \downarrow & \\
& & & & (\oplus_n E & \overset{p_1 \sim p_n}{\longrightarrow} & E) & & & \\
\end{array}
$$

as a commutative diagram. Therefore $\oplus_n E \overset{p_1 \sim p_n}{\longrightarrow} E$ is an injective representation. \qed

**Theorem 2.4.** If $M_1 \overset{f_1 \sim f_n}{\longrightarrow} M_2$ is an injective representation of a quiver $Q = \bullet \rightarrow \bullet$ then $M_1$ and $M_2$ are injective left $R$-modules.
Proof. First we show that $M_2$ is an injective left $R$-module. Let $S$ be a submodule of $N$ and $g : S \to M_2$ be an $R$-linear map and we consider the following diagram

$$
\begin{array}{c}
0 & \rightarrow & S & \rightarrow & N \\
\downarrow{g} & & \downarrow & & \\
& & M_2 & & \\
\end{array}
$$

Then since $M_1 \xrightarrow{f_1 \sim f_n} M_2$ is an injective representation, there exist $h : N \to M_2$ which completes the following

$$
\begin{array}{c}
(0 \rightarrow 0) & \rightarrow & (0 \rightarrow S) & \rightarrow & (0 \rightarrow N) \\
\downarrow & & \downarrow{h} & & \\
(M_1 \xrightarrow{f_1 \sim f_n} M_2) & & & & \\
\end{array}
$$

as a commutative diagram. Thus, $h : N \to M_2$ completes the above diagram as a commutative diagram. Therefore, $M_2$ is an injective left $R$-module.

Let $g : S \to M_1$ be an $R$-linear map and we consider the following diagram

$$
\begin{array}{c}
0 & \rightarrow & S & \rightarrow & N \\
\downarrow{g} & & \downarrow & & \\
& & M_1 & & \\
\end{array}
$$

Consider the following diagram

$$
\begin{array}{c}
(0 \rightarrow 0) & \rightarrow & (S \xrightarrow{i_1 \sim i_n} \oplus_n S) & \rightarrow & (N \xrightarrow{j_1 \sim j_n} \oplus_n N) \\
\downarrow{g} & & \downarrow{G} & & \\
(M_1 \xrightarrow{f_1 \sim f_n} M_2) & & & & \\
\end{array}
$$

where $G((s_1, s_2, \ldots, s_n)) = \sum_{k=1}^{n} f_k(g(s_k))$, and $i_k(s)$ is the $k$th injection.
Then since $M_1 \xrightarrow{f_1 \sim f_n} M_2$ is an injective representation, there exist $h : N \to M_1$ and $\alpha : \bigoplus_n E \to M_2$ such that the following diagram

\[
\begin{array}{ccc}
N & \xrightarrow{j_1 \sim j_n} & \bigoplus_n N \\
\downarrow h & & \downarrow \alpha \\
(M_1 \xrightarrow{f_1 \sim f_n} M_2) & & \\
\end{array}
\]

as a commutative diagram. Thus, $h : N \to M_1$ completes the following diagram

\[
\begin{array}{ccc}
0 & \to & S & \to & N \\
\downarrow g & & \downarrow h & & \\
M_1 & & \\
\end{array}
\]

as a commutative diagram. Therefore, $M_1$ is an injective left $R$-module.

\[
\square
\]

3. Projective representation of a quiver $Q = \bullet \Rightarrow \bullet$ with $n$ edges

**Definition 3.1.** A representation $P_1 \xrightarrow{f_1 \sim f_n} P_2$ of a quiver $Q = \bullet \Rightarrow \bullet$ is called a projective representation if every diagram of representations

\[
\begin{array}{ccc}
(P_1 \xrightarrow{f_1 \sim f_n} P_2) & \to & (M_1 \xrightarrow{g_1 \sim g_n} M_2) \\
\downarrow & & \downarrow \\
(N_1 \xrightarrow{h_1 \sim h_n} N_2) & \to & (0 \to 0) \\
\end{array}
\]

can be completed to a commutative diagram as follows:

\[
\begin{array}{ccc}
(P_1 \xrightarrow{f_1 \sim f_n} P_2) & \xrightarrow{H} & (M_1 \xrightarrow{g_1 \sim g_n} M_2) \\
\downarrow & & \downarrow \\
(N_1 \xrightarrow{h_1 \sim h_n} N_2) & \to & (0 \to 0) \\
\end{array}
\]
Theorem 3.2. If $P$ is a projective left $R$-module, then

$$0 \to P$$

is a projective representation of $Q = \bullet \to \bullet$.

Proof. Let $M_1, M_2, N_1, N_2$ be left $R$-modules, and $k : P \to N_2$ be an $R$-linear map. Consider the following diagram

$$
\begin{array}{ccc}
(M_1 & \xrightarrow{g_1 \sim g_n} & M_2) & \xrightarrow{k} & (N_1 & \xrightarrow{h_1 \sim h_n} & N_2) & \xrightarrow{0} & (0 & \to & 0) \\
M_1 & \xrightarrow{g_1 \sim g_n} & M_2 & & & & & & \end{array}
$$

Then since $P$ is a projective left $R$-module, we can complete the following commutative diagram by $h$.

Then $0 : 0 \to M_1$ completes the following diagram

$$
\begin{array}{ccc}
(0 & \to & P) & \xrightarrow{h} & (0 & \to & 0) \\
M_1 & \xrightarrow{g_1 \sim g_n} & M_2 & & \xrightarrow{h} & (N_1 & \xrightarrow{h_1 \sim h_n} & N_2) & \xrightarrow{0} & (0 & \to & 0) \\
M_1 & \xrightarrow{g_1 \sim g_n} & M_2 & & & & & & \end{array}
$$

as a commutative diagram. Hence, $0 \to P$ is a projective representation.

Theorem 3.3. If $P$ is a projective left $R$-module, then

$$P \xrightarrow{i_1 \sim i_n} \oplus_n P$$

is a projective representation of $Q = \bullet \to \bullet$ where $i_k$ is the $k$th injection.
**Proof.** Let \( M_1, M_2, N_1, N_2 \) be left \( R \)-modules and \( g : P \to N_1 \) be a \( R \)-linear map. Consider the following diagram

\[
\begin{array}{ccc}
(P \xrightarrow{i_1 \sim i_n} \oplus_n P) & \rightarrow & (N_1 \rightarrow N_2 \rightarrow 0) \\
g & & \\
(M_1 \xrightarrow{f_1 \sim f_n} M_2) & \rightarrow & (0 \rightarrow 0)
\end{array}
\]

Since \( P \) is a projective left \( R \)-module we can complete the following diagram by \( h \).

\[
\begin{array}{ccc}
P & \rightarrow & M_1 \rightarrow N_1 \rightarrow 0 \\
\nearrow & & \downarrow h & \\
& & g & \downarrow
\end{array}
\]

Define \( H((a_1, a_2, \cdots, a_n)) = f_1(h(a_1)) + f_2(h(a_2)) + \cdots + f_n(h(a_n)) \). Then \( f_1(h(a)) = H(i_1(a)), f_2(h(a)) = H(i_2(a)), \ldots, f_n(h(a)) = H(i_n(a)) \).

Thus we can complete the following diagram

\[
\begin{array}{ccc}
(P \xrightarrow{i_1 \sim i_n} \oplus_n P) & \rightarrow & (N_1 \rightarrow N_2 \rightarrow 0) \\
\nearrow h & & \\
(M_1 \xrightarrow{f_1 \sim f_n} M_2) & \rightarrow & (0 \rightarrow 0)
\end{array}
\]

as a commutative diagram. Hence, \( P \xrightarrow{i_1 \sim i_n} \oplus_n P \) is a projective representation.

**Theorem 3.4.** If \( M_1 \xrightarrow{f_1 \sim f_n} M_2 \) is an projective representation of a quiver \( Q = \bullet \Rightarrow \bullet \) then \( M_1 \) and \( M_2 \) are projective left \( R \)-modules.

**Proof.** First we show that \( M_1 \) is a projective left \( R \)-module. Let \( S \) and \( N \) be left \( R \)-modules and \( g : M_1 \to S \) be an \( R \)-linear map and we consider the following diagram

\[
\begin{array}{ccc}
M_1 & \rightarrow & N \rightarrow S \rightarrow 0 \\
g & & \downarrow
\end{array}
\]
Then since $M_1 \to M_2$ is a projective representation, there exist $h : M_1 \to N$ which completes the following diagram:

\[
\begin{array}{c}
(M_1 \xrightarrow{f_1 \sim f_n} M_2) \\
\downarrow h \\
(N \xrightarrow{0} S \xrightarrow{0} (0 \to 0))
\end{array}
\]

Therefore, $M_1$ is a projective left $R$-module. Let $g : M_2 \to S$ be an $R$-linear map and we consider the following diagram:

\[
\begin{array}{c}
M_2 \\
\downarrow g \\
N \xrightarrow{g} S \xrightarrow{0}
\end{array}
\]

Define $G : M_1 \to \bigoplus_{i=1}^n S_i$ where $p_k(G(m)) = g(f_k(m))$, for $k = 1, \ldots, n$ and $G(m) = (g(f_1(m)), g(f_2(m)), \ldots, g(f_n(m)))$, and $p_k((a_1, a_2, \ldots, a_n)) = a_k$, and consider the following diagram:

\[
\begin{array}{c}
(M_1 \xrightarrow{f_1 \sim f_n} M_2) \\
\downarrow G \\
(\bigoplus_{i=1}^n N_i \xrightarrow{q_i \sim q_n} N) \xrightarrow{g} (\bigoplus_{n} S \xrightarrow{p_1 \sim p_n} S) \xrightarrow{0}
\end{array}
\]

Then since $M_1 \xrightarrow{f_1 \sim f_n} M_2$ is a projective representation, there exist $h : M_1 \to \bigoplus_n N$ and $\alpha : M_2 \to N$ such that the following diagram:

\[
\begin{array}{c}
M_1 \xrightarrow{f_1 \sim f_n} M_2 \\
\downarrow h \\
(\bigoplus_n N \xrightarrow{i_1 \sim i_n} N)
\end{array}
\]

\[\alpha\]

\[\alpha\]
as a commutative diagram. Thus, $\alpha : M_2 \to N$ completes the following diagram

\[
\begin{array}{ccc}
M_2 & \xrightarrow{\alpha} & N \\
\downarrow{g} & & \downarrow{g} \\
N & \xrightarrow{} & S & \xrightarrow{} & 0
\end{array}
\]

Therefore, $M_2$ is a projective left $R$-module.

References


Department of Mathematics,
Dong-A University,
Pusan, Korea 604-714
E-mail: swpark@donga.ac.kr