ON PROPERTIES OF QUASI-CLASS A OPERATORS

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Abstract. Let the set of all quasi-class A operators for which \( \ker(A) \subseteq \ker(A^*) \) be denoted by \( A \in QA^* \). In this paper it is proved that an operator \( T \in QA^* \) is normal if and only if the Duggal transform of \( T \) is normal.

1. Introduction

Let \( L(H) \) denote the algebra of bounded linear operators on a complex infinite dimensional Hilbert space \( H \). Recall ([1], [3], [7]) that \( T \in L(H) \) is called \( p \)-hyponormal if \( (T^*T)^p \geq (TT^*)^p \) for \( p \in (0, 1] \), and \( T \) is called paranormal if \( ||T^2x|| \geq ||T^2||^2 \) for all unit vector \( x \in H \). Following [6] and [5] we say that \( T \in L(H) \) belongs to class A if \( ||T^2|| \geq ||T||^2 \). Recall ([9], [11]) that \( T \) is called \( p \)-quasihyponormal if \( T^*(T^*T)^pT \geq T^*(TT^*)^pT \) for \( p \in (0, 1] \). For brevity, we shall denote classes of \( p \)-hyponormal operators, \( p \)-quasihyponormal operators, paranormal operators, and class A operators by \( H(p) \), \( QH(p) \), \( PN \), and \( A \), respectively. It is well known that

\[ H(p) \subset A \subset PN \quad \text{and} \quad H(p) \subset QH(p) \subset PN. \]

In [8] Jeon and Kim considered an extension of the notion of class A operators, similar in spirit to the extension of the notion of \( p \)-hyponormality to \( p \)-quasihyponormality; we say that \( T \in L(H) \) is quasi-class A if

\[ T^*|T^2|T \geq T^*|T|^2T. \]

For brevity, we shall denote the set of quasi-class A operators by \( QA \). As shown in [8], the class of quasi-class A operators properly contains


2000 Mathematics Subject Classification: 47B20.

Key words and phrases: quasi-class A operator, the property (\( \beta \)).

This paper was supported by Research Fund, Kumoh National Institute of Technology.

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classes of class $A$ operators and $p$-quasihyponormal operators, i.e., the following inclusions holds;

\begin{equation}
\mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{QA} \quad \text{and} \quad \mathcal{H}(p) \subset A \subset \mathcal{QA}
\end{equation}

In view of inclusions (1.1) and (1.2), it seems reasonable to expect that the operators in class $QA$ are paranormal. But there exists an example [8] that one would be wrong in such an expectation.

Throughout this paper, restricting ourselves to those $A \in QA$ for which $\ker(A) \subseteq \ker(A^*)$, denoted $A \in QA^*$, we prove that an operator $T \in QA^*$ is normal if and only if the Duggal transform of $T$ is normal. From this result it is also proved that every operator $A \in QA^*$ satisfies property $(\beta)$, and for densely intertwined operators $QA^*$, their spectra, essential spectra, and indices are preserved.

2. Main results

If $T \in \mathcal{L}(\mathcal{H})$ has the polar decomposition $T = U|T|$, then $T^\sim = |T|U$ is called to be Duggal transform of $T$. It is well known that Duggal transform is one of very useful tools to study properties of operators ([4]). As an essential tool, to prove our main results below, we will use Duggal transforms of $QA^*$ operators.

**Theorem 2.1.** The operator $A \in QA^*$ is normal if and only if $A^\sim$ is normal, and then $A = A^\sim$.

**Proof.** First, we claim that

$A \in QA^*$ if and only if $A^\sim \in A$.

Let $A$ have a decomposition $A = A_1 \oplus A_2$ with respect to some decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of $\mathcal{H}$, such that $A_1 = A|_{\mathcal{H}_1}$ is normal and $A_2 = A|_{\mathcal{H}_2}$ is pure. Let $A_i$ have the polar decomposition $A_i = U_i|A_i|$. Then the partial isometry $U_2$ is an isometry, and we may choose the partial isometry $U_1$ to be a unitary such that the commutator

$$[[A_1], U_1] = |A_1|U_1 - U_1|A_1| = 0.$$ 

Define the Duggal transform $A^\sim$ of $A$ by

$$A^\sim = A_1 \oplus A_2^\sim = (|A_1| \oplus |A_2|)(U_1 \oplus U_2).$$
Since $A_2 \in QA^*$, $A_2$ is injective and $|A_2|$ is a quasiaffinity, it follows from the equivalence
\[ A_2^*(-|A_2|^2)A_2 \geq 0 \iff U_2^*(-|A_2|^2)U_2 \geq 0 \]
that
\[ |A_2^2|^2 = U_2^*|A_2|^2U_2 \leq U_2^*|A_2|^2U_2 \leq (U_2^*A_2^*A_2U_2)^2 = |A_2^2|, \]
i.e., $A_2^* = |A_2|U_2 \in \mathcal{A}$. Hence $A^*$ is a class $A$ operator.

Now, if $A$ is normal, then $A_2$ acts on the trivial space $\{0\}$, and so $A = A_1 = A^*$. If, instead, $A^*$ is normal, then the normality of $A_2^*$ implies that
\[ U_2^*|A_2|^2U_2 = |A_2|^2 = |A_2|U_2U_2^*|A_2|. \]
Since $|A_2|$ is a quasiaffinity, $U_2U_2^* = I|_{H_2}$, so that $U_2$ is a unitary. But then $U_2^*|A_2|^2U_2 = |A_2|^2$, which implies that $[|A_2|, U_2] = 0$, i.e., $A_2$ is normal. Since $A_2$ is pure, $A_2$ acts on the trivial space $\{0\}$. Hence $A = A^* = A_1$. \[\Box\]

Recall that a $T \in \mathcal{L}(\mathcal{H})$ has (Bishop’s) property $(\beta)$ if, for every open subset $U$ of $\mathbb{C}$ and every sequence of analytic functions $f_n : U \rightarrow \mathcal{H}$ with the property that
\[(T - \lambda)f_n(\lambda) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty\]
uniformly on all compact subsets of $U$, it follows that $f_n(\lambda) \rightarrow 0$ as $n \rightarrow \infty$ locally uniformly on $U$ [10, Definition 1.2.5].

**Corollary 2.2.** Every operator $A \in QA^*$ satisfies property $(\beta)$.

**Proof.** Let $A$ and $A^*$ be defined as in the proof of Theorem 2.1. Then observe that $A$ has property $(\beta)$ if and only if $A_2$ has property $(\beta)$. Recall that if $R, S \in \mathcal{L}(\mathcal{H})$ are injective, then $RS$ satisfies property $(\beta)$ if and only if $SR$ satisfies property $(\beta)$ [3, Theorem 5]. Since the operators $|A_2|$ and $U_2$ are injective, and the operator $A_2^* = |A_2|U_2 \in \mathcal{A}$ satisfies property $(\beta)$ [2], it follows that $A_2 = U_2|A_2| \in \mathcal{A}$ satisfies property $(\beta)$.

Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is Fredholm if $T(\mathcal{H})$ is closed, $\alpha(T) = \dim T^{-1}(0) < \infty$ and $\beta(T) = \dim T^*^{-1}(0) < \infty$. If $T \in \mathcal{L}(\mathcal{H})$ is Fredholm, then the index of $T$, denoted $i(T)$, is defined by $i(T) = \alpha(T) - \beta(T)$. The essential spectrum of $T$, denoted $\sigma_e(T)$, is the set of
all \( \lambda \in \mathbb{C} \) such that \( T - \lambda I \) is not Fredholm. The operators \( S, T \in \mathcal{L}(\mathcal{H}) \) are said to be densely intertwined if there exist operators \( X, Y \in \mathcal{L}(\mathcal{H}) \) with dense range such that \( SX = XT \) and \( TY = YS \).

**Corollary 2.3.** If \( A, B \in \mathcal{QA}^* \) are densely intertwined, then \( \sigma(A) = \sigma(B) \), \( \sigma_e(A) = \sigma_e(B) \) and \( i(A - \lambda) = i(B - \lambda) \) for all \( \lambda \in \mathbb{C} \setminus \sigma_e(A) \).

**Proof.** Both \( A \) and \( B \) have property \((\beta)\), and [10, Theorem 3.7.15] applies.

We conclude with comments on Putnam-Fuglede type commutativity theorem. In facts, we have tried to get a Putnam-Fuglede type commutativity theorem for operators in \( \mathcal{QA}^* \). But, first of all, we must find the proof of the following conjecture on Putnam-Fuglede type commutativity theorem for operators in \( \mathcal{A} \).

**Conjecture 2.4.** If \( A, C^* \in \mathcal{A} \) are such that \( AX = XC \) for some non-trivial \( X \in \mathcal{L}(\mathcal{H}) \), then \( A^*X = XC^* \), \( \text{ran}X \) reduces \( A \), \( \ker^\perp(X) \) reduces \( C \), and \( A|_{\text{ran}X} \) and \( C|_{\ker^\perp(X)} \) are unitarily equivalent normal operators.

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