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# ON THE STABILITY OF PEXIDER TYPE TRIGONOMETRIC FUNCTIONAL EQUATIONS

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Abstract. The aim of this paper is to study the stability problem for the pexider type trigonometric functional equation $f(x+y) - f(x-y) = 2g(x)h(y)$, which is related to the d'Alembert, the Wilson, the sine, and the mixed trigonometric functional equations.

## 1. Introduction

J. Baker, J. Lawrence and F. Zorzitto in [4] , and Bourgin [5] introduced the following: if $f$ satisfies the inequality $|f(x+y)-f(x)f(y)| \le \delta$, then either $|f(x)| \le \max(4, 4\delta)$, or $f(x+y) = f(x)f(y)$. This is frequently referred to as superstability.

In next year, J. Baker [3] proved the superstability of the cosine functional equation (also called the d'Alembert equation)

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{A}$$

which is improved by P. Găvruta [7].

And also the sine functional equation

$$f(x)f(y) = f\Big(\frac{x+y}{2}\Big)^2 - f\Big(\frac{x-y}{2}\Big)^2 \tag{S}$$

is investigated by P.W. Cholewa [6].

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The cosine functional equation (A) is generalized to the following functional equations

$$(A_{fg}) \qquad f(x+y)+f(x-y)=2f(x)g(y),$$
$$(A_{gf}) \qquad f(x+y)+f(x-y)=2g(x)f(y),$$
$$(A_{gg}) \qquad f(x+y)+f(x-y)=2g(x)g(y),$$

where the two unknown functions $f, g$ are to be determined. The equation $(A_{fg})$ introduced by Wilson, is sometimes referred to as the Wilson equation. Their stability have been investigated by Badora, Ger, Kannappan, Kim ([1], [2], [8], [10]) and others.

Motivated by some trigonometric identities (for example, $\sin(\alpha+\beta)-\sin(\alpha-\beta)=2\cos\alpha\sin\beta$), we consider the following trigonometric functional equation

$$(T_{gh}) \qquad f(x+y)-f(x-y)=2g(x)h(y),$$

which has special cases as follow :

$$(T) \qquad f(x+y)-f(x-y)=2f(x)f(y),$$
$$(T_{fg}) \qquad f(x+y)-f(x-y)=2f(x)g(y),$$
$$(T_{gf}) \qquad f(x+y)-f(x-y)=2g(x)f(y),$$
$$(T_{gg}) \qquad f(x+y)-f(x-y)=2g(x)g(y).$$

The aim of this paper is to investigate stability problem for the pexider type trigonometric functional equations $(T_{gh})$.

As a consequence, we obtain the stability of the above trigonometric type equations $(T)$, $(T_{fg})$, $(T_{gf})$, $(T_{gg})$ as corollaries, and also extend the obtained results to the Banach algebra.

In this paper, let $(G,+)$ be an Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers. Whenever we deal with (S), we need to assume additionally that $(G,+)$ is a uniquely 2-divisible group. We will write then "under 2-divisibility", for short. We may assume that $f, g$ and $h$ are non-zero functions and $\varepsilon$ is a nonnegative real constant.

## 2. Stability of the equation $(T_{gh})$

In this section, we investigate the stability of the pexider type trigonometric functional equation $(T_{gh})$ related to the d'Alembert (A), the Wilson types $(A_{fg})$ and $(A_{gf})$ the sine (S), and the mixed type functional equations $(T_{fg})$ and $(T_{gf})$.

Theorem 1. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2g(x)h(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.1}$$

*If $h$ fails to be bounded, then*

*(i) $g$ satisfies (S) under 2-divisibility and one of the cases $g(0) = 0$, $f(x) = f(-x)$,*

*(ii) if, additionally, $h$ satisfies (A) or (T), $g$ and $h$ are solutions of Wilson type :*

$$g(x+y) + g(x-y) = 2g(x)h(y). \tag{$A_{ggh}$}$$

*Proof.* (i) Let $h$ be unbounded. Then we can choose a sequence $\{y_n\}$ in $G$ such that

$$0 \ne |h(y_n)| \to \infty, \quad \text{as} \quad n \to \infty \tag{2.2}$$

Taking $y = y_n$ in (2.1) we obtain

$$\Big|\frac{f(x+y_n) - f(x-y_n)}{2h(y_n)} - g(x)\Big| \le \frac{\varepsilon}{|2h(y_n)|},$$

that is,

$$g(x) = \lim_{n\to\infty} \frac{f(x+y_n) - f(x-y_n)}{2h(y_n)} \quad \forall\ y \in G. \tag{2.3}$$

Using (2.1), we have

$$\begin{aligned}
&\big|f\left(x+(y+y_n)\right) - f\left(x-(y+y_n)\right) - 2g(x)h(y+y_n) \\
&\qquad - f\left(x+(y-y_n)\right) + f\left(x-(y-y_n)\right) + 2g(x)h(y-y_n)\big| \\
&\le 2\varepsilon
\end{aligned}$$

so that

$$\begin{aligned}&\left|\frac{f((x+y)+y_n)-f\left((x+y)-y_n\right)}{2h(y_n)}\right.\\&\left.+\frac{f((x-y)+y_n)-f\left((x-y)-y_n\right)}{2h(y_n)}-2g(x)\cdot\frac{h(y+y_n)-h(y-y_n)}{2h(y_n)}\right|\\&\leq\frac{\varepsilon}{|h(y_n)|}\end{aligned}\tag{2.4}$$

for all $x, y \in G$.

Taking the limit as $n \longrightarrow \infty$ with the use of (2.2) and (2.3), we conclude that, for every $y \in G$, there exists the limit

$$k_1(y) := \lim_{n\to\infty}\frac{h(y+y_n)-h(y-y_n)}{h(y_n)},\tag{2.5}$$

where the function $k_1 : G \to \mathbb{C}$ obtained in that way has to satisfy the equation

$$g(x+y)+g(x-y)=g(x)k_1(y) \quad \forall x, y \in G.\tag{2.6}$$

Applying the case $g(0) = 0$ in (2.6), it implies that $g$ is an odd function. Keeping this in mind, by means of (2.6), we infer the equality

$$\begin{aligned}g(x+y)^2-g(x-y)^2&=[g(x+y)+g(x-y)][g(x+y)-g(x-y)]\\&=g(x)k_1(y)[g(x+y)-g(x-y)]\\&=g(x)\big[g(x+2y)-g(x-2y)\big]\\&=g(x)\big[g(2y+x)+g(2y-x)\big]\\&=g(x)g(2y)k_1(x).\end{aligned}\tag{2.7}$$

Putting $y = x$ in (2.6) we get the equation

$$g(2x)=g(x)k_1(x), \quad x \in G.$$

This (2.7), in return, leads to the equation

$$g(x+y)^2-g(x-y)^2=g(2x)g(2y)$$

valid for all $x, y \in G$ which, in the light of the unique 2-divisibility of $G$, states nothing else but (S).

Next, in particular case $f(x) = f(-x)$, it is enough to show that $g(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (2.1), from the above assumption and a given condition, we obtain the inequality

$$|h(y)| \le \frac{\varepsilon}{2|g(0)|}, \quad y \in G.$$

This inequality means that $h$ is globally bounded – a contradiction. Thus the claimed $g(0) = 0$ holds.

(ii) if $h$ satisfies $(T)$, the defined limit $k_1$ of (2.5) states nothing else but $2h$, so (2.6) validates $(A_{ggh})$.

Finally, an obvious slight change in the steps of the proof applied after (2.3) gives us the inequality

$$\begin{aligned}&\big|f\left(x + (y + y_n)\right) - f\left(x - (y + y_n)\right) - 2g(x)h(y + y_n)\\ &\qquad + f\left(x + (-y + y_n)\right) - f\left(x - (-y + y_n)\right) - 2g(x)h(-y + y_n)\big|\\ &\le 2\varepsilon\end{aligned}$$

so that

$$\begin{aligned}&\Bigg|\frac{f((x + y) + y_n) - f\left((x + y) - y_n)\right)}{2h(y_n)}\\ &+ \frac{f((x - y) + y_n) - f\left((x - y) - y_n)\right)}{2h(y_n)}\\ &- 2g(x)\cdot\frac{h(y + y_n) + h(-y + y_n)}{2h(y_n)}\Bigg|\\ &\le \frac{\varepsilon}{|h(y_n)|}\end{aligned} \tag{2.8}$$

for all $x, y \in G$. Taking the limit as $n \longrightarrow \infty$ with the use of (2.3), and since $h$ satisfies (A), so (2.8) implies $(A_{ggh})$. □

Theorem 2. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality* (2.1) *for all $x, y \in G$.*

*If $g$ fails to be bounded, then*

*(i) $h$ satisfies (S) under 2-divisibility,*

*(ii) if, additionally, $g$ satisfies (A), $g$ and $h$ are solutions of $h(x + y) - h(x - y) = 2g(x)h(y)$ and $h(x + y) + h(x - y) = 2h(x)g(y)$.*

*Proof.* Let $g$ be unbounded solution of the inequality (2.1). Then, there exists a sequence $\{x_n\}$ in $G$ such that $0 \ne |g(x_n)| \to \infty$ as $n \to \infty$.

Taking $x = x_n$ in the inequality (2.1), dividing both sides by $|2g(x_n)|$ and passing to the limit as $n \to \infty$ we obtain that

$$h(y) = \lim_{n\to\infty} \frac{f(x_n + y) - f(x_n - y)}{2g(x_n)}, \quad x \in G. \tag{2.9}$$

Replacing $x$ by $x_n + x$ and $x_n - x$ in (2.1), we can, with an application of (2.9), state the existence of a limit function

$$k_2(x) := \lim_{n\to\infty} \frac{g(x_n + x) + g(x_n - x)}{g(x_n)},$$

where the function $k_2 : G \to \mathbb{C}$ satisfies the equation

$$h(x + y) - h(x - y) = k_2(x)h(y) \quad \forall x, y \in G. \tag{2.10}$$

From the definition of $k_2$, we get the equality $k_2(0) = 2$, which jointly with (2.10) implies that $h$ has to be odd.

(i) Run along the same lines applied after (2.6), which states nothing else but (S).

(ii) if $g$ satisfies (A), the defined limit function $k_2$ is simply $2g$, so (2.10) implies $h(x + y) - h(x - y) = 2g(x)h(y)$.

Secondly, as above, replacing $x$ by $x_n + y$ and $x_n - y$, and replacing $y$ by $x$ in (2.1), respectively. Then, since $g$ satisfies (A), we obtain, with an application of (2.9), the equation $h(x+y)+h(x-y) = 2h(x)g(y)$. □

By replacing $h$ by $f$, $g$ by $f$, $h$ by $g$ in Theorem 1 and Theorem 2, we obtain the following corollaries.

Corollary 1. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x + y) - f(x - y) - 2g(x)f(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.11}$$

*Then either $f$ is bounded or $g$ satisfies (A).*

*Proof.* Replacing $h$ by $f$ in (2.1) of Theorem 1. An obvious slight change in the steps of the proof of Theorem 1, gives that $k_1$ of (2.5) states nothing else but $2g$. Hence, from (2.6), $g$ satisfies the equation (A). □

Corollary 2. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality* (2.11)

*If $g$ fails to be bounded, then*

*(i) $g$ satisfies (A),*

*(ii) $f$ and $g$ are solutions of $(T_{gf})$ and $(A_{fg})$.*

*Proof.* Replacing $h$ by $f$ in (2.1) of Theorem 2.

(i) It is enough from Corollary 1 to show that the boundedness of $f$ implies the boundedness of $g$. Namely, If $f$ is bounded, choose $y_0 \in G$ such that $f(y_0) \neq 0$, and then by (2.11) we obtain

$$\begin{aligned} |g(x)| - \Big|\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)}\Big| \\ &\le \Big|\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)} - g(x)\Big| \\ &\le \frac{\varepsilon}{2|f(y_0)|}, \end{aligned}$$

from which follows that $g$ is also bounded on $G$. Since $f$ is nonzero, the unboundedness of $g$ implies the unboundedness of $f$. Hence, from Corollary 1, $g$ satisfies (A).

(ii) Since we known that $g$ satisfies (A) by (i), Following steps applied in Theorem 2, then the equation $(T_{gf})$ holds from (2.10).

For the case $(A_{gf})$, replacing $x$ by $x_n + y$ and $x_n - y$, and $y$ by $x$ in (2.11), respectively. other step runs same as above. □

Corollary 3. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)h(y)| \le \varepsilon \quad \forall\, x, y \in G. \tag{2.12}$$

*If $f$ fails to be bounded, then*

*(i) $h$ satisfies (S) under 2-divisibility,*

*(ii) If, additionally, $f$ satisfies (A), $f$ and $h$ are solutions of $h(x+y) - h(x-y) = 2f(x)h(y)$ and $h(x+y) + h(x-y) = 2h(x)f(y)$.*

Corollary 4. *Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)h(y)| \le \varepsilon \quad \forall\, x, y \in G.$$

*If $h$ fails to be bounded, then*

*(i) $f$ satisfies (S) under 2-divisibility and one of the cases $f(0) = 0$, $f(x) = f(-x)$,*

*(ii) if, additionally, $h$ satisfies (A) or (T), $f$ and $h$ are solutions of $f(x+y) + f(x-y) = 2f(x)h(y)$,*

*(iii) $h$ satisfies (S) under 2-divisibility,*

*(iv) if, additionally, $f$ satisfies (A), $f$ and $h$ are solutions of $h(x+y) - h(x-y) = 2f(x)h(y)$ and $h(x+y) + h(x-y) = 2h(x)f(y)$.*

*Proof.* It is trivial from Theorem 2 except for (iii) and (iv).

As a boundedness proof of (ii) in Corollary 1, we can see that the unboundedness of $h$ implies the unboundedness of $f$, so (v) and (vi) follows from (i) and (ii) immediately . □

From Theorem 2, we obtain easily the following result :

Corollary 5. *Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2g(x)g(y)| \le \varepsilon \quad \forall\, x, y \in G.$$

*Then either $g$ is bounded or $g$ satisfies (S) under 2-divisibility.*

*Proof.* It is trivial from (i) of Theorem 2. □

Corollary 6. *([12]) Suppose that a non-zero function $f : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \le \varepsilon \qquad \forall\, x, y \in G.$$

*Then $f$ is bounded.*

*Proof.* Assume that $f$ is not bounded. Then, by applying $g = h = f$ in Corollary 1 and Corollary 2, $f$ satisfies simultaneously (A) and $(T)$. This forces that $f$ is a zero function. But we know that there exists the cosine function which satisfies (A) as a non-zero. Hence we arrive the result by a contradiction. □

## 3. Extension to the Banach algebra

The obtained results for the functional equations $(T_{gh})$ in section 2 can be extended to the Banach algebra. For simplify, we only will represent one of them, and the applications to the other theorems and corollaries will be omitted.

Given mappings $f, g, h : G \to \mathbb{C}$, for above equations, we will denote a difference for each equation by an operator $DT_{gh} : G \times G \to \mathbb{C}$ as

$$DT_{gh}(x, y) := f(x+y) - f(x-y) - 2g(x)h(y).$$

Theorem 3. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : G \to E$ satisfy the inequality*

$$\|f(x+y) - f(x-y) - 2g(x)h(y)\| \le \varepsilon \quad \forall\, x, y \in G. \tag{3.1}$$

*For an arbitrary linear multiplicative functional $x^* \in E^*$,*

*if the superposition $x^* \circ g$ fails to be bounded, then*
*(i) $h$ satisfies (S) under 2-divisibility,*
*(ii) if, additionally, $x^* \circ g$ satisfies (A), $g$ and $h$ are solutions of the equation $h(x+y) - h(x-y) = 2g(x)h(y)$.*

*Proof.* (i) Assume that (3.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have

$$\begin{aligned}\varepsilon &\geq \|f(x+y) - f(x-y) - 2g(x)h(y)\| \\ &= \sup_{\|y^*\|=1} \left|y^*\big(f(x+y) - f(x-y) - 2g(x)h(y)\big)\right| \\ &\geq \left|x^*\big(f(x+y)\big) - x^*\big(f(x-y)\big) - 2x^*\big(g(x)\big)x^*\big(h(y)\big)\right|,\end{aligned}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$ and $x^* \circ h$ yield a solution of inequality (2.1).

Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 2 shows that the function $x^* \circ h$ solves the equation (S). In other words, bearing the linear multiplicativity of $x^*$ in mind, for all $x, y \in G$, the difference $DS_h(x,y) := h(x)h(y) - h\big(\frac{x+y}{2}\big)^2 + h\big(\frac{x-y}{2}\big)^2$ falls into the kernel of $x^*$. Therefore, in view of the unrestricted choice of $x^*$, we infer that

$$DS_h(x,y) \in \bigcap \{\ker x^* : x^* \in E^*\}$$

for all $x, y \in G$. Since the algebra $E$ has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.

$$h(x)h(y) - h\Big(\frac{x+y}{2}\Big)^2 + h\Big(\frac{x-y}{2}\Big)^2 = 0 \qquad \text{for all} \quad x, y \in G,$$

as claimed.

(ii) Under the assumption that the superposition $x^* \circ g$ satisfies (A), we know from Theorem 2 that the superpositions $x^* \circ h$ and $x^* \circ g$ are solutions of the equation

$$x^*\big(h(x+y)\big) - x^*\big(h(x-y)\big) = 2x^*\big(g(x)\big)x^*\big(h(y)\big).$$

Namely,

$$\begin{aligned}&h(x+y) - h(x-y) - 2g(x)h(y) \\ &\qquad \in \bigcap \{\ker x^* : x^* \in E^*\}.\end{aligned}$$

The other argument is similar. □

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