ON THE STABILITY OF PEXIDER TYPE
TRIGONOMETRIC FUNCTIONAL EQUATIONS

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Abstract. The aim of this paper is to study the stability problem for the pexider type trigonometric functional equation \( f(x + y) - f(x - y) = 2g(x)h(y) \), which is related to the d’Alembert, the Wilson, the sine, and the mixed trigonometric functional equations.

1. Introduction

J. Baker, J. Lawrence and F. Zorzitto in [4] , and Bourgin [5] introduced the following: if \( f \) satisfies the inequality \( |f(x + y) - f(x)f(y)| \leq \delta \), then either \( |f(x)| \leq \max(4, 4\delta) \), or \( f(x + y) = f(x)f(y) \). This is frequently referred to as superstability.

In next year, J. Baker [3] proved the superstability of the cosine functional equation (also called the d’Alembert equation)

\[(A) \quad f(x + y) + f(x - y) = 2f(x)f(y),\]

which is improved by P. Găvruta [7].

And also the sine functional equation

\[(S) \quad f(x)f(y) = f\left(\frac{x + y}{2}\right)^2 - f\left(\frac{x - y}{2}\right)^2\]

is investigated by P.W. Cholewa [6].
The cosine functional equation (A) is generalized to the following functional equations

\begin{align*}
(A_{fg}) & \quad f(x + y) + f(x - y) = 2f(x)g(y), \\
(A_{gf}) & \quad f(x + y) + f(x - y) = 2g(x)f(y), \\
(A_{gg}) & \quad f(x + y) + f(x - y) = 2g(x)g(y),
\end{align*}

where the two unknown functions \( f, g \) are to be determined. The equation \((A_{fg})\) introduced by Wilson, is sometimes referred to as the Wilson equation. Their stability have been investigated by Badora, Ger, Kannappan, Kim ([1], [2], [8], [10]) and others.

Motivated by some trigonometric identities (for example, \( \sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos \alpha \sin \beta \)), we consider the following trigonometric functional equation

\begin{align*}
(T_{gh}) & \quad f(x + y) - f(x - y) = 2g(x)h(y),
\end{align*}

which has special cases as follow:

\begin{align*}
(T) & \quad f(x + y) - f(x - y) = 2f(x)f(y), \\
(T_{fg}) & \quad f(x + y) - f(x - y) = 2f(x)g(y), \\
(T_{gf}) & \quad f(x + y) - f(x - y) = 2g(x)f(y), \\
(T_{gg}) & \quad f(x + y) - f(x - y) = 2g(x)g(y).
\end{align*}

The aim of this paper is to investigate stability problem for the pexider type trigonometric functional equations \((T_{gh})\).

As a consequence, we obtain the stability of the above trigonometric type equations \((T), (T_{fg}), (T_{gf}), (T_{gg})\) as corollaries, and also extend the obtained results to the Banach algebra.

In this paper, let \((G, +)\) be an Abelian group, \( \mathbb{C} \) the field of complex numbers, and \( \mathbb{R} \) the field of real numbers. Whenever we deal with \((S)\), we need to assume additionally that \((G, +)\) is a uniquely 2-divisible group. We will write then “under 2-divisibility”, for short. We may assume that \( f, g \) and \( h \) are non-zero functions and \( \epsilon \) is a nonnegative real constant.
2. Stability of the equation \((T_{gh})\)

In this section, we investigate the stability of the pexider type trigonometric functional equation \((T_{gh})\) related to the d’Alembert \((A)\), the Wilson types \((A_{fg})\) and \((A_{gf})\) the sine \((S)\), and the mixed type functional equations \((T_{fg})\) and \((T_{gf})\).

**Theorem 1.** Suppose that \(f, g, h : G \to \mathbb{C}\) satisfy the inequality

\[
|f(x + y) - f(x - y) - 2g(x)h(y)| \leq \varepsilon \quad \forall \ x, y \in G.
\]

If \(h\) fails to be bounded, then

(i) \(g\) satisfies \((S)\) under 2-divisibility and one of the cases \(g(0) = 0\), \(f(x) = f(-x)\),

(ii) if, additionally, \(h\) satisfies \((A)\) or \((T)\), \(g\) and \(h\) are solutions of Wilson type :

\((A_{gh})\) \hspace{1cm} g(x + y) + g(x - y) = 2g(x)h(y).

**Proof.** (i) Let \(h\) be unbounded. Then we can choose a sequence \(\{y_n\}\) in \(G\) such that

\[
0 \neq |h(y_n)| \to \infty, \quad \text{as} \quad n \to \infty
\]

Taking \(y = y_n\) in (2.1) we obtain

\[
\left| \frac{f(x + y_n) - f(x - y_n)}{2h(y_n)} - g(x) \right| \leq \frac{\varepsilon}{|2h(y_n)|},
\]

that is,

\[
g(x) = \lim_{n \to \infty} \frac{f(x + y_n) - f(x - y_n)}{2h(y_n)} \quad \forall \ y \in G.
\]

Using (2.1), we have

\[
\left| f(x + (y + y_n)) - f(x - (y + y_n)) - 2g(x)h(y + y_n) - f(x + (y - y_n)) + f(x - (y - y_n)) + 2g(x)h(y - y_n) \right| \leq 2\varepsilon
\]
so that
\[
\frac{f((x+y)+y_n) - f((x+y)-y_n))}{2h(y_n)}
+ \frac{f((x-y)+y_n) - f((x-y)-y_n))}{2h(y_n)} - 2g(x) \cdot \frac{h(y+y_n) - h(y-y_n)}{2h(y_n)}
\leq \frac{\varepsilon}{|h(y_n)|}
\]
(2.4)
for all \(x, y \in G\).

Taking the limit as \(n \to \infty\) with the use of (2.2) and (2.3), we conclude that, for every \(y \in G\), there exists the limit
\[
(2.5)
k_1(y) := \lim_{n \to \infty} \frac{h(y+y_n) - h(y-y_n)}{h(y_n)},
\]
where the function \(k_1 : G \to \mathbb{C}\) obtained in that way has to satisfy the equation
\[
(2.6)\quad g(x+y) + g(x-y) = g(x)k_1(y) \quad \forall x, y \in G.
\]

Applying the case \(g(0) = 0\) in (2.6), it implies that \(g\) is an odd function. Keeping this in mind, by means of (2.6), we infer the equality
\[
g(x+y)^2 - g(x-y)^2 = [g(x+y) + g(x-y)][g(x+y) - g(x-y)]
= g(x)k_1(y)[g(x+y) - g(x-y)]
= g(x)[g(x+2y) - g(x-2y)]
= g(x)[g(2y+x) + g(2y-x)]
= g(x)g(2y)k_1(x).
\]
(2.7)
Putting \(y = x\) in (2.6) we get the equation
\[
g(2x) = g(x)k_1(x), \quad x \in G.
\]

This (2.7), in return, leads to the equation
\[
g(x+y)^2 - g(x-y)^2 = g(2x)g(2y)
\]
valid for all \(x, y \in G\) which, in the light of the unique 2-divisibility of \(G\), states nothing else but (S).

Next, in particular case \(f(x) = f(-x)\), it is enough to show that \(g(0) = 0\). Suppose that this is not the case.
Putting \( x = 0 \) in (2.1), from the above assumption and a given condition, we obtain the inequality

\[
|h(y)| \leq \frac{\varepsilon}{2|g(0)|}, \quad y \in G.
\]

This inequality means that \( h \) is globally bounded – a contradiction. Thus the claimed \( g(0) = 0 \) holds.

(ii) if \( h \) satisfies \((T)\), the defined limit \( k_1 \) of (2.5) states nothing else but \( 2h \), so (2.6) validates \((A_{gh})\).

Finally, an obvious slight change in the steps of the proof applied after (2.3) gives us the inequality

\[
\begin{align*}
&|f(x + (y + y_n)) - f(x - (y + y_n)) - 2g(x)h(y + y_n) \\
&\quad + f(x + (-y + y_n)) - f(x - (-y + y_n)) - 2g(x)h(-y + y_n)| \\
&\leq 2\varepsilon
\end{align*}
\]

so that

\[
\begin{align*}
&\left|\frac{f((x + y) + y_n) - f((x + y) - y_n)}{2h(y_n)} + \frac{f((x - y) + y_n) - f((x - y) - y_n)}{2h(y_n)} \\
&\quad - 2g(x) \cdot \frac{h(y + y_n) + h(-y + y_n)}{2h(y_n)}\right| \\
&\leq \frac{\varepsilon}{|h(y_n)|}
\end{align*}
\]

for all \( x, y \in G \). Taking the limit as \( n \to \infty \) with the use of (2.3), and since \( h \) satisfies \((A)\), so (2.8) implies \((A_{gh})\).

**Theorem 2.** Suppose that \( f, g, h : G \to \mathbb{C} \) satisfy the inequality (2.1) for all \( x, y \in G \).

If \( g \) fails to be bounded, then

(i) \( h \) satisfies \((S)\) under 2-divisibility,

(ii) if, additionally, \( g \) satisfies \((A)\), \( g \) and \( h \) are solutions of \( h(x + y) - h(x - y) = 2g(x)h(y) \) and \( h(x + y) + h(x - y) = 2h(x)g(y) \).

**Proof.** Let \( g \) be unbounded solution of the inequality (2.1). Then, there exists a sequence \( \{x_n\} \) in \( G \) such that \( 0 \neq |g(x_n)| \to \infty \) as \( n \to \infty \).
Taking $x = x_n$ in the inequality (2.1), dividing both sides by $|2g(x_n)|$ and passing to the limit as $n \to \infty$ we obtain that
\begin{equation}
(2.9) \quad h(y) = \lim_{n \to \infty} \frac{f(x_n + y) - f(x_n - y)}{2g(x_n)}, \quad x \in G.
\end{equation}

Replacing $x$ by $x_n + x$ and $x_n - x$ in (2.1), we can, with an application of (2.9), state the existence of a limit function
\[ k_2(x) := \lim_{n \to \infty} \frac{g(x_n + x) + g(x_n - x)}{g(x_n)}, \]
where the function $k_2 : G \to \mathbb{C}$ satisfies the equation
\begin{equation}
(2.10) \quad h(x + y) - h(x - y) = k_2(x)h(y) \quad \forall x, y \in G.
\end{equation}

From the definition of $k_2$, we get the equality $k_2(0) = 2$, which jointly with (2.10) implies that $h$ has to be odd.

(i) Run along the same lines applied after (2.6), which states nothing else but (S).

(ii) if $g$ satisfies (A), the defined limit function $k_2$ is simply $2g$, so (2.10) implies $h(x + y) - h(x - y) = 2g(x)h(y)$.

Secondly, as above, replacing $x$ by $x_n + y$ and $x_n - y$, and replacing $y$ by $x$ in (2.1), respectively. Then, since $g$ satisfies (A), we obtain, with an application of (2.9), the equation $h(x + y) + h(x - y) = 2h(x)g(y)$. \hfill \square

By replacing $h$ by $f$, $g$ by $f$, $h$ by $g$ in Theorem 1 and Theorem 2, we obtain the following corollaries.

**Corollary 1.** Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality
\begin{equation}
(2.11) \quad |f(x + y) - f(x - y) - 2g(x)f(y)| \leq \varepsilon \quad \forall x, y \in G.
\end{equation}

Then either $f$ is bounded or $g$ satisfies (A).

**Proof.** Replacing $h$ by $f$ in (2.1) of Theorem 1. An obvious slight change in the steps of the proof of Theorem 1, gives that $k_1$ of (2.5) states nothing else but $2g$. Hence, from (2.6), $g$ satisfies the equation (A). \hfill \square

**Corollary 2.** Suppose that $f, g : G \to \mathbb{C}$ satisfy the inequality
\begin{equation}
(2.11)
\end{equation}

If $g$ fails to be bounded, then

(i) $g$ satisfies (A),

(ii) $f$ and $g$ are solutions of $(T_{gf})$ and $(Af_g)$.  

Proof. Replacing $h$ by $f$ in (2.1) of Theorem 2.

(i) It is enough from Corollary 1 to show that the boundedness of $f$ implies the boundedness of $g$. Namely, if $f$ is bounded, choose $y_0 \in G$ such that $f(y_0) \neq 0$, and then by (2.11) we obtain

$$|g(x)| - \left|\frac{f(x + y) - f(x - y)}{2f(y)} \right| \leq \left|\frac{f(x + y) - f(x - y)}{2f(y)} - g(x)\right| \leq \frac{\varepsilon}{2|f(y_0)|},$$

from which follows that $g$ is also bounded on $G$. Since $f$ is nonzero, the unboundedness of $g$ implies the unboundedness of $f$. Hence, from Corollary 1, $g$ satisfies (A).

(ii) Since we known that $g$ satisfies (A) by (i). Following steps applied in Theorem 2, then the equation ($T_{gf}$) holds from (2.10).

For the case ($A_{gf}$), replacing $x$ by $x_n + y$ and $x_n - y$, and $y$ by $x$ in (2.11), respectively. other step runs same as above.

Corollary 3. Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality

(2.12) $$|f(x + y) - f(x - y) - 2f(x)h(y)| \leq \varepsilon \quad \forall x, y \in G.$$

If $f$ fails to be bounded, then

(i) $h$ satisfies (S) under 2-divisibility,

(ii) If, additionally, $f$ satisfies (A), $f$ and $h$ are solutions of $h(x + y) - h(x - y) = 2f(x)h(y)$ and $h(x + y) + h(x - y) = 2h(x)f(y)$.

Corollary 4. Suppose that $f, h : G \to \mathbb{C}$ satisfy the inequality

$$|f(x + y) - f(x - y) - 2f(x)h(y)| \leq \varepsilon \quad \forall x, y \in G.$$

If $h$ fails to be bounded, then

(i) $f$ satisfies (S) under 2-divisibility and one of the cases $f(0) = 0$, $f(x) = f(-x)$,

(ii) if, additionally, $h$ satisfies (A) or (T), $f$ and $h$ are solutions of $f(x + y) + f(x - y) = 2f(x)h(y)$,

(iii) $h$ satisfies (S) under 2-divisibility,

(iv) if, additionally, $f$ satisfies (A), $f$ and $h$ are solutions of $h(x + y) - h(x - y) = 2f(x)h(y)$ and $h(x + y) + h(x - y) = 2h(x)f(y)$.
Proof. It is trivial from Theorem 2 except for (iii) and (iv).

As a boundedness proof of (ii) in Corollary 1, we can see that the unboundedness of \( h \) implies the unboundedness of \( f \), so (v) and (vi) follows from (i) and (ii) immediately.

From Theorem 2, we obtain easily the following result:

**Corollary 5.** Suppose that \( f, g : G \to \mathbb{C} \) satisfy the inequality
\[
|f(x + y) - f(x - y) - 2g(x)g(y)| \leq \varepsilon \quad \forall \ x, y \in G.
\]
Then either \( g \) is bounded or \( g \) satisfies (S) under 2-divisibility.

**Proof.** It is trivial from (i) of Theorem 2.

**Corollary 6.** ([12]) Suppose that a non-zero function \( f : G \to \mathbb{C} \) satisfy the inequality
\[
|f(x + y) - f(x - y) - 2f(x)f(y)| \leq \varepsilon \quad \forall \ x, y \in G.
\]
Then \( f \) is bounded.

**Proof.** Assume that \( f \) is not bounded. Then, by applying \( g = h = f \) in Corollary 1 and Corollary 2, \( f \) satisfies simultaneously (A) and (T). This forces that \( f \) is a zero function. But we know that there exists the cosine function which satisfies (A) as a non-zero. Hence we arrive the result by a contradiction.

3. Extension to the Banach algebra

The obtained results for the functional equations \((T_{gh})\) in section 2 can be extended to the Banach algebra. For simplify, we only will represent one of them, and the applications to the other theorems and corollaries will be omitted.

Given mappings \( f, g, h : G \to \mathbb{C} \), for above equations, we will denote a difference for each equation by an operator \( DT_{gh} : G \times G \to \mathbb{C} \) as
\[
DT_{gh}(x, y) := f(x + y) - f(x - y) - 2g(x)h(y).
\]

**Theorem 3.** Let \((E, \| \cdot \|)\) be a semisimple commutative Banach algebra. Assume that \( f, g, h : G \to E \) satisfy the inequality
\[
\|f(x + y) - f(x - y) - 2g(x)h(y)\| \leq \varepsilon \quad \forall \ x, y \in G.
\]
For an arbitrary linear multiplicative functional \( x^* \in E^* \),
if the superposition $x^* \circ g$ fails to be bounded, then
(i) if $h$ satisfies (S) under 2-divisibility,
(ii) if, additionally, $x^* \circ g$ satisfies (A), $g$ and $h$ are solutions of the equation $h(x + y) - h(x - y) = 2g(x)h(y)$.

Proof. (i) Assume that (3.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$ whence, for every $x, y \in G$, we have
\[\varepsilon \geq \|f(x + y) - f(x - y) - 2g(x)h(y)\|\]
\[= \sup_{\|y^*\| = 1} |y^*(f(x + y) - f(x - y) - 2g(x)h(y))|\]
\[\geq |x^*(f(x + y)) - x^*(f(x - y)) - 2x^*(g(x))x^*(h(y))|,
\]
which states that the superpositions $x^* \circ f$, $x^* \circ g$ and $x^* \circ h$ yield a solution of inequality (2.1).

Since, by assumption, the superposition $x^* \circ g$ is unbounded, an appeal to Theorem 2 shows that the function $x^* \circ h$ solves the equation (S). In other words, bearing the linear multiplicativity of $x^*$ in mind, for all $x, y \in G$, the difference $DS_h(x, y) := h(x)h(y) - h\left(\frac{x+y}{2}\right)^2 + h\left(\frac{x-y}{2}\right)^2$ falls into the kernel of $x^*$. Therefore, in view of the unrestricted choice of $x^*$, we infer that
\[DS_h(x, y) \in \bigcap \{\ker x^* : x^* \in E^*\}\]
for all $x, y \in G$. Since the algebra $E$ has been assumed to be semisimple, the last term of the above formula coincides with the singleton $\{0\}$, i.e.
\[h(x)h(y) - h\left(\frac{x+y}{2}\right)^2 + h\left(\frac{x-y}{2}\right)^2 = 0 \quad \text{for all} \quad x, y \in G,
\]
as claimed.

(ii) Under the assumption that the superposition $x^* \circ g$ satisfies (A), we know from Theorem 2 that the superpositions $x^* \circ h$ and $x^* \circ g$ are solutions of the equation
\[x^*(h(x + y)) - x^*(h(x - y)) = 2x^*(g(x))x^*(h(y)).
\]
Namely,
\[h(x + y) - h(x - y) - 2g(x)h(y) \in \bigcap \{\ker x^* : x^* \in E^*\}.
\]
The other argument is similar. \qed
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