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LINDELÖFICATION OF BIFRAMES

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Abstract. We introduce countably strong inclusions $\triangleleft = (\triangleleft_1, \triangleleft_2)$ on a biframe $\mathrm{L} = (\mathrm{L}_0, \mathrm{L}_1, \mathrm{L}_2)$ and i-strongly regular σ-ideals ($i = 1, 2$) and then using them, we construct biframe Lindelöfication of L. Furthermore, we obtain a sufficient condition for which L has a unique countably strong inclusion.

1. Introduction and preliminaries

This section is a collection of basic definitions and results on frames. For general notions and facts concerning frames, we refer to Johnstone[9] and Khang[10], and for compactifications to [1], [2], [3], [5].

1.1. Frames.

Definition 1.1. (1) *A frame is a complete lattice L in which binary meet distributes over arbitrary join, that is,* $x \wedge \bigvee S = \bigvee\{x \wedge s \in S\}$ *for any* x *in L and any subset* S *of L.*

(2) *A frame homomorphism is a map* $h : L \to M$ *between frames L and M preserving all finitary meets and binary joins.*

We will denote the bottom element of a frame L by 0 or 0_L and the top element by e or e_L.

For any element a of a frame L, the map $a \wedge \_ : \mathrm{L} \to \mathrm{L}$ preserves arbitraty joins; hence it has a right adjont, which will be denoted by $a \to \_ : \mathrm{L} \to \mathrm{L}$. In particular, $a \to 0$ exists for any a in L and we write $a \to 0 = a^*$, called the *pseudocomplement* of a.

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Definition 1.2. (1) *Let L be a frame and a, b in L. We say that a is rather below b if there exists c in L such that $a \wedge c = 0$ and $b \vee c = e$, equivalently, $a^* \vee b = e$. In this case, we write $a \prec b$.*

(2) *A frame L is said to be regular if for any a in L, $a = \bigvee\{b \in L | b \prec a\}$.*

We note that $u \prec v$ in $\Omega(\mathrm{X})$ means $\overline{u} \subseteq v$, for a topological space $(\mathrm{X}, \Omega(\mathrm{X}))$ and it is clear that a topological space $(\mathrm{X}, \Omega(\mathrm{X}))$ is regular if and only if $\Omega(\mathrm{X})$ is a regular frame.

Definition 1.3. ([7], [11]) *Let L be a complete lattice and a, b in L. We say that a is way below (countably way below, resp.) b and write $a \ll b$ ($a \ll_c b$, resp.) if for any subset S of L, $b \leq \bigvee S$ implies $a \leq \bigvee C$ for some finite (countable, resp.) subset C of S.*

Example 1.4. (1) *Let A and B be subsets of a set X. Then $A \ll_c B$ in the frame $\wp(X)$ of the power set of X if and only if there is a countable subset C of X with $A \subseteq C \subseteq B$.*

(2) *In $\Omega(X)$ of a topological space $(X, \Omega(X))$, $u \ll_c v$ if there is a Lindelöf subset w of X with $u \subseteq w \subseteq v$. If X is locally Lindelöf, then the converse also holds.*

Proposition 1.5. *Let L be a frame and a, b, x, y in L. Then*

(1) $0 \ll_c a$.
(2) $a \ll_c b$ *implies* $a \leq b$.
(3) *If* $x \leq a \ll_c b \leq y$, *then* $x \ll_c y$.
(4) *If* $a_n \ll_c b$ *for all* $n \in N$, *then* $\bigvee\limits_{n \in N} a_n \ll_c b$.
(5) *If* $a \ll b$, *then* $a \ll_c b$.

Definition 1.6. ([11]) *A complete lattice L is said to be countably approximating if for any x in L, $x = \bigvee\{a \in L \mid a \ll_c x\}$.*

The following definition is a natural generalization of compact frames.

Definition 1.7. *A frame L is said to be a Lindelöf frame if for any subset S of L with $\bigvee S = e$, there is a countable subset C of S with $\bigvee C = e$.*

A 1-1 frame homomorphism is clearly codense and therefore the following is immedate :

Proposition 1.8. *If $h : L \to M$ is a 1-1 frame homomorphism and M is a Lindelöf frame, then L is a Lindelöf frame.*

Definition 1.9. ([6]) *A frame L is said to be a $D(\aleph_1)$ frame if for any a in L and any sequence $(b_n)_{n\in N}$ in L, $a \vee (\bigwedge_{n\in N} b_n) = \bigwedge_{n\in N}(a \vee b_n)$.*

Proposition 1.10. *If $x_n \prec y$ for all n in N in a $D(\aleph_1)$ frame L, then $\bigvee_{n\in N} x_n \prec y$ in L.*

1.2. Lindelöfication of a Frame.

Using a concept of countably strong inclusions, we have obtained a Lindelöfication of a frame L, i.e. a dense, onto frame homomorphism $h : \mathrm{L} \to \mathrm{M}$ such that M is a Lindelöf regular frame.([10])

Definition 1.11. *A binary relation $\lhd$ on a frame L is said to be a countably strong inclusion, if it satisfies :*

(1) *if $x \le a \lhd b \le y$, then $x \lhd y$.*
(2) *$\lhd$ is closed under finite meets and countable joins.*
(3) *$a \lhd b$ implies $a \prec b$.*
(4) *$\lhd$ interpolates.*
(5) *$a \lhd b$ implies $b^* \lhd a^*$.*
(6) *$a = \bigvee\{x \in L \mid x \lhd a\}$ for any a in L.*

Proposition 1.12. ([10]) *If L is a Lindelöf regular $D(\aleph_1)$ frame, then $\prec$ is a countably strong inclusion.*

Definition 1.13. *A subset I of a frame L is said to be a σ-ideal if it is a countably directed lower set, equivalently, it is a lower set and closed under countable joins.*

Let σIdL denote the set of all σ-ideals in L. Then σIdL is clearly closed under arbitrary intersections in the power set lattice $\wp(\mathrm{L})$ of L and therefore it is a complete lattice.

Using the fact that for $(\mathrm{I}_\lambda)_{\lambda\in\Lambda} \subseteq \sigma\mathrm{IdL}$, $\bigvee_{\lambda\in\Lambda} \mathrm{I}_\lambda = \{\bigvee_{k\in N} x_k \mid (x_k)_{k\in N}$ is a sequence in $\bigcup_{\lambda\in\Lambda} \mathrm{I}_\lambda\}$ in σIdL, one has :

Proposition 1.14. *σIdL is a Lindelöf frame.*

Definition 1.15. *Let $\lhd$ be a countably strong inclusion on a frame L. Then a σ-ideal I is said to be a $\lhd$ -σ-ideal if for any a in I, there is b in I such that $a \lhd b$.*

Let $S(\lhd)$ denote the subframe of σIdL determined by $\lhd$ -σ-ideals, then the join map $j_0 : S(\lhd) \to \mathrm{L}$ is indeed a Lindelöfication of L ([10]).

2. Lindelöfication of Biframes

In order to set up a frame version for bitopological spaces, a concept of biframes was introduced ([4]).

In this chapter, we deal with Lindelöfications of biframes.

DEFINITION 2.1. (1) *A biframe is a triple* $L = (L_0, L_1, L_2)$ *where* L_1 *and* L_2 *are subframes of a frame* L_0 *such that* L_0 *is generated by* $L_1 \cup L_2$.

(2) *Let* $L = (L_0, L_1, L_2)$ *and* $M = (M_0, M_1, M_2)$ *be biframes. A map* $h : L \to M$ *is said to be a biframe homomorphism, if* $h : L_0 \to M_0$ *is a frame homomorphism and satisfies* $h(L_i) \subseteq M_i$ *for* $i = 1, 2$.

EXAMPLE 2.2. (1) *Let* $L_0 = \Omega(R)$ *the open set lattice of the real line* R, L_1 *all open downsets and* L_2 *all open upsets in* R. *Then* $L = (L_0, L_1, L_2)$ *is a biframe.*

(2) *If we let* $L_0 = L_1 = L_2 = L$ *for a frame* L, *then* $L = (L_0, L_1, L_2)$ *is a biframe.*

DEFINITION 2.3. *Let* $L = (L_0, L_1, L_2)$ *be a biframe.*

(1) L *is said to be a Lindelöf biframe if* L_0 *is a Lindelöf frame.*

(2) *Let* $i, k = 1, 2$ *and* $i \neq k$,

  a) $x \prec_i y$ *if* $x, y \in L_i$ *and there is* c *in* L_k *with* $x \wedge c = 0$ *and* $y \vee c = e$.

  b) L *is said to be regular if for any* x *in* L_i, $x = \bigvee\{y | y \prec_i x\}$.

  c) *For any* $x \in L_i$ $(i = 1, 2)$, *let* $x^\bullet = \bigvee\{z \in L_k | z \wedge x = 0\}$.

(3) L *is said to be* $D(\aleph_1)$ *if* L_0 *is a* $D(\aleph_1)$ *frame.*

In the above example (1), $u \prec_i v$ if and only if $u \subseteq v$ such that one of the following holds :

i) $u \neq v$,

ii) $u = v = \varnothing$,

iii) $u = v = R$.

REMARK. Let L be a biframe and $a, b \in \mathrm{L}_0$. For any $i = 1, 2$,

(1) $a \prec_i b$ implies $a \leq b$.

(2) $a \prec_i b$ if and only if $a^\bullet \vee b = e$.

DEFINITION 2.4. *A biframe homomorphism* $h : L \to M$ *is said to be :*

(1) *dense if* $h : L_0 \to M_0$ *is dense.*
(2) *onto if* $h|_{L_1}$ *and* $h|_{L_2}$ *are both onto.*

Definition 2.5. *A Lindelöfication of a biframe L is a dense, onto biframe homomorphism* $h : M \to L$ *such that* M *is a Lindelöf regular biframe.*

We now introduce a concept of countably strong inclusion on a biframe.

Definition 2.6. *Let* $L = (L_0, L_1, L_2)$ *be a biframe and* $\lhd_i \subseteq L_i \times L_i$, *for* $i = 1, 2$. *Then* $\lhd = (\lhd_1, \lhd_2)$ *is said to be a countably strong inclusion on* L *if* $\lhd$ *satisfies the following, where* $i, k = 1, 2$ *and* $i \neq k$.

1) *If* $x \leq a \lhd_i b \leq y$, *then* $x \lhd_i y$.
2) $\lhd_i$ *is closed under finite meets and countable joins.*
3) *If* $a \lhd_i b$, *then* $a \prec_i b$.
4) $\lhd_i$ *interpolates.*
5) *If* $a \lhd_i b$, *then there are* u, v *in* L_k *such that* $u \lhd_k v$, $a \wedge v = 0$ *and* $b \vee u = e$.
6) *For any* $a \in L_i$, $a = \bigvee\{x \in L_i | x \lhd_i a\}$.

Remark. (1) The condition 5) in the above definition may be replaced by the following : $a \lhd_i b$ implies $b^\bullet \lhd_k a^\bullet$.
Indeed, suppose $a \lhd_i b$, then there are u, v in L$_k$ such that $u \lhd_k v$, $a \wedge v = 0$ and $b \vee u = e$. Thus $v \leq a^\bullet$ and $b^\bullet = b^\bullet \wedge e = b^\bullet \wedge (b \vee u) = b^\bullet \wedge u$; hence $b^\bullet \leq u$. Therefore $b^\bullet \leq u \lhd_k v \leq a^\bullet$, so that $b^\bullet \lhd_k a^\bullet$. Conversely, suppose $a \lhd_i b$ then by 4), there is x in L$_i$ such that $a \lhd_i x \lhd_i b$. Thus $x^\bullet \lhd_k a^\bullet$, so that $x^\bullet \lhd_k a^\bullet$, $a \wedge a^\bullet = 0$. Moreover $x^\bullet \vee b = e$, for $x \lhd_i b$ implies $x \prec_i b$.

(2) By the exactly same arguments as those in Proposition 1.12, $\prec = (\prec_1, \prec_2)$ in a Lindelöf regular $D(\aleph_1)$ biframe L is a countably strong inclusion on L.

Proof for the following lemma is straightforward and hence we omit it.

Lemma 2.7. *Let* $h : N \to L$ *be an onto biframe homomorphism. If* $\lhd = (\lhd_1, \lhd_2)$ *is a countably strong inclusion on* N, *then* $\hat{\lhd} = (\overset{2}{h}(\lhd_1), \overset{2}{h}(\lhd_2))$ *is a countably strong inclusion on* L.

We now have the following by the above Lemma and Remark.

Corollary 2.8. *If a biframe L has a* $D(\aleph_1)$ *Lindelöfication, then it has a countably strong inclusion.*

For a biframe $\mathrm{L} = (\mathrm{L}_0, \mathrm{L}_1, \mathrm{L}_2)$, let $j_i : \mathrm{L}_i \to \mathrm{L}_0$ be the inclusion homomorphism and let $\tilde{j}_i : \sigma\mathrm{IdL}_i \to \sigma\mathrm{IdL}_0$ be the frame homomorphism induced by j_i between the frames of σ-ideals of L_i and L_0 respectively $(i = 1, 2)$. Then for any $\mathrm{J} \in \sigma\mathrm{IdL}_i$, $\tilde{j}_i(\mathrm{J}) = \downarrow \mathrm{J}$. Moreover, $\tilde{j}_i(\sigma\mathrm{IdL}_i) = \{\downarrow \mathrm{J} | \mathrm{J} \in \sigma\mathrm{IdL}_i\}$ is a subframe of $\sigma\mathrm{IdL}_0$, which will be denoted by $\sigma\mathrm{Id}_b\mathrm{L}_i$.

Definition 2.9. *Let $\lhd = (\lhd_1, \lhd_2)$ be a countably strong inclusion on a biframe $L = (L_0, L_1, L_2)$ and $i = 1, 2$. A σ-ideal J on L_0 is said to be i-strongly regular if $J \in \sigma Id_b L_i$ and for any x in $J \cap L_i$, there is y in $J \cap L_i$ with $x \lhd_i y$.*

Let $\Re_i$ denote the set of all i-strongly regular σ-ideals in a biframe $\mathrm{L} = (\mathrm{L}_0, \mathrm{L}_1, \mathrm{L}_2)$ $(i = 1, 2)$.

Using these notions, we now have the following immediately :

Proposition 2.10. *$\Re_i$ is a subframe of σIdL_0.*

Now let $\Re_0$ be the subframe of $\sigma\mathrm{IdL}_0$ generated by $\Re_1 \cup \Re_2$, then $\Re = (\Re_0, \Re_1, \Re_2)$ is a biframe.

Since $\sigma\mathrm{IdL}_0$ is a Lindelöf frame, so is $\Re_0$. Thus $\Re$ is a Lindelöf biframe.

Since $j : \sigma\mathrm{IdL}_0 \to \mathrm{L}_0$ defined by $j(\mathrm{J}) = \bigvee \mathrm{J}$ is dense, the restriction $j_0 : \Re_0 \to \mathrm{L}_0$ of j to $\Re_0$ is also dense, so that the biframe homomorphism $j_0 : \Re \to \mathrm{L}$ is dense.

Consider $\gamma_i : \mathrm{L}_i \to \Re_i$ defined by $\gamma_i(a) = \downarrow \{x \in \mathrm{L}_i | x \lhd_i a\}$. Then γ_i is well-defined, because $\{x \in \mathrm{L}_i | x \lhd_i a\}$ is a $\lhd_i$-σ-ideal in L by the definition of countably strong inclutions. Furthermore, for any a in L_i, $a = \bigvee \gamma_i(a) = j_0(\gamma_i(a))$; therefore j_0 is onto.

Lemma 2.11. *If $a \lhd_i b$, then $\gamma_i(a) \prec_i \gamma_i(b)$ $(i = 1, 2)$.*

Proof. Since $\lhd_i$ interpolates, there is c in L_i such that $a \lhd_i c \lhd_i b$. Since $a \lhd_i c$ there are u,v in L_k such that $v \lhd_k u$, $a \wedge u = 0$ and $c \vee v = e$. For any z in $\gamma_i(a) \wedge \gamma_k(u) = \gamma_i(a) \cap \gamma_k(u)$, $z \lhd_i a$ and $z \lhd_k u$ and hence $z \leq a \wedge u = 0$. Thus $\gamma_i(a) \wedge \gamma_k(u) = \{0\}$. Since $c \vee v = e \in \gamma_i(b) \cap \gamma_k(u)$, $\gamma_i(b) \wedge \gamma_k(u) = \mathrm{L}_0$. So $\gamma_i(a) \prec_i \gamma_i(b)$. □

Lemma 2.12. *For any J in $\Re_i$, $J = \bigvee_{a \in J \cap L_i} \gamma_i(a)$, for $i = 1, 2$.*

Proof. Since $\mathrm{J} \in \sigma\mathrm{Id}_b\mathrm{L}_i$, $\mathrm{J} =\downarrow (\mathrm{J} \cap \mathrm{L}_i)$ and therefore $x \in \mathrm{J}$ if and only if there are a,b in $\mathrm{J} \cap \mathrm{L}_i$, such that $x \leq a \lhd_i b$. Thus we have $\mathrm{J} = \bigvee_{a \in \mathrm{J} \cap \mathrm{L}_i} \gamma_i(a)$. □

PROPOSITION 2.13. *$\Re$ is regular.*

Proof. For any J in $\Re_i$ and any a in $\mathrm{J} \cap \mathrm{L}_i$, there is b in $\mathrm{J} \cap \mathrm{L}_i$ with $a \lhd_i b$, so that $\gamma_i(a) \prec \gamma_i(b) \leq \mathrm{J}$. Hence $\mathrm{J} = \bigvee_{a \in \mathrm{J} \cap \mathrm{L}_i} \gamma_i(a) \leq \bigvee\{\mathrm{I} \in \Re_i | \mathrm{I} \prec_i \mathrm{J}\} \leq \mathrm{J}$. Thus $\mathrm{J} = \bigvee\{\mathrm{I} \in \Re_i | \mathrm{I} \prec_i \mathrm{J}\}$. □

Collecting the above results, we have :

THEOREM 2.14. *If $\lhd$ is a countably strong inclusion on a biframe L, then $j_0 : \Re \to L$ is a Lindelöfication of L.*

Let $\mathrm{CS}_b(\mathrm{L})$ be the set of all countably strong inclusions on a biframe L. Then $(\mathrm{CS}_b(\mathrm{L}), \subseteq)$ is a poset.

DEFINITION 2.15. *Let $f : M \to L$ and $g : N \to L$ be Lindelöfications of a biframe L. If there is a biframe homomorphism $h : M \to N$ with $g \circ h = f$, then we say that f is smaller than g and write $f \leq g$.*

Clearly, $\leq$ is a preoder on the class of Lindelöfications of a biframe L and the relation $\leq \cap \leq^{\mathrm{op}}$ is an equivalence relation on the class and let $\mathrm{Lind}_b(\mathrm{L})$ be the set of all equivalence classes of Lindelöfications of a biframe L. Then $(\mathrm{Lind}_b(\mathrm{L}), \leq)$ is a poset, where $[f] \leq [g]$ in $\mathrm{Lind}_b(\mathrm{L})$ means $f \leq g$.

Define $\varphi : \mathrm{Lind}^*(\mathrm{L}) \to \mathrm{CS}_b(\mathrm{L})$ by $\varphi(h : \mathrm{M} \to \mathrm{L}) = (\overset{2}{h}(\prec_1), \overset{2}{h}(\prec_2))$ and $\psi : \mathrm{CS}_b(\mathrm{L}) \to \mathrm{Lind}(\mathrm{L})$ by $\psi(\lhd) = (j_0 : \Re \to \mathrm{L})$, where $\mathrm{Lind}^*(\mathrm{L})$ denotes the set of all $D(\aleph_1)$ Lindelöfications of a biframe L. Then φ and ψ are isotones. Using the exactly same arguments as those in section 2 in [10], we have the following :

THEOREM 2.16.
1) *Suppose that $\lhd$ is a countably strong inclusion on a biframe L such that $\Re$ is $D(\aleph_1)$. Then $\varphi(\psi(\lhd)) = \lhd$.*
2) *For a $D(\aleph_1)$ Lindelöfication $h : M \to L$ of a biframe L, $\psi(\varphi(h)) \cong M$.*

We will introduce stably countably approximating frames and we will then find smallest countably strong inclusion.

Definition 2.17. *A frame M is said to be stably countably approximating if M is countably approximating and $\ll_c$ is closed under finite meets in M.*

Example 2.18. (1) *If M is Lindelöf regular $D(\aleph_1)$, then M is stably countably approximating, since $\ll_c$ and $\prec$ are same in a Lindelöf regular $D(\aleph_1)$ frame.([10])*

(2) *It is known that $I \ll_c J$ in σIdL if and only if $I \subseteq \downarrow a \subseteq J$, for some a in L ([11]). Thus σIdL is stably countably approximating.*

Lemma 2.19. *Let $L = (L_0, L_1, L_2)$ be a regular $D(\aleph_1)$ biframe. Then each L_i is stably countably approximating and $\ll_{ci}$ satisfies the condition 5) in Definition 2.6 of countably strong inclusion if and only if $(\ll_{c1}, \ll_{c2})$ is a countably strong inclusion on L.*

Proof. ($\Leftarrow$) By the condition 2), 5) and 6) of countably strong inclusion, it is trivial.

($\Rightarrow$)

1) It follows from (3) in Proposition 1.5.
2) Since each L_i is stably countably approximating, each $\ll_{ci}$ is closed under finite meets and by (4) in Proposition 1.5, each $\ll_{ci}$ is closed under countably joins.
3) Since L is regular $D(\aleph_1)$, $x \ll_{ci} y$ implies $x \prec_i y$.
4) Since each L_i is countably approximating, each $\ll_{ci}$ interpolates.
5) It is trivial by the assumption.
6) If follows from the fact that each L_i is countably approximating.

□

Proposition 2.20. *If L is a regular $D(\aleph_1)$ biframe such that each L_i is stably countably approximating and $\ll_{ci}$ satisfies the condition 5) in Definition 2.6 of countably strong inclusion, then $(\ll_{c1}, \ll_{c2})$ is the smallest countably strong inclusion on L.*

Proof. Let $(\triangleleft_1, \triangleleft_2)$ be any countably strong inclusion on L. If $x \ll_{ci} y$, then $x \ll_{ci} y = \bigvee\{z \in L_i | z \triangleleft_i y\}$. Thus there is a countable subset $\{z_n | n \in N\}$ of L_i such that for any $n \in N$, $z_n \triangleleft_i y$ and $x \leq \bigvee_{n\in N} z_n$ and hence $x \leq \bigvee_{n\in N} z_n \triangleleft_i y$, so $x \triangleleft_i y$. In all, $(\ll_{c1}, \ll_{c2}) \subseteq (\triangleleft_1, \triangleleft_2)$. □

Lemma 2.21. *Let L be a regular $D(\aleph_1)$ biframe in which each L_i is countably approximating and $a \prec_i b$ implies that $a \ll_{ci} b$ whenever $a < e$. Then $(\prec_1, \prec_2)$ is a countably strong inclusion on L.*

Proof. Conditions 1), 2), 3) are trivial.

4) We note that for $a < e$, $a \prec_i b$ if and only if $a \ll_{ci} b$ since L is regular $D(\aleph_1)$. Since L_i is countably approximating, there is z in L_i such that $a \ll_{ci} z \ll_{ci} b$ and hence $a \prec_i z \prec_i b$. For $a = e$, there is nothing to prove.
5) Suppose that $a \prec_i b$. Then by 4), there is c in L_i such that $a \prec_i c \prec_i b$. So there are s, t in L_k such that $a \wedge s = 0$, $c \vee s = e$, $c \wedge t = 0$ and $b \vee t = e$. Thus $a \wedge s = 0$, $t \prec_k s$ and $b \vee t = e$.
6) It follows from the regularity of L_i.

□

THEOREM 2.22. *Let L be a regular $D(\aleph_1)$ biframe in which each L_i is countably approximating and $a \prec_i b$ implies that $a \ll_{ci} b$ whenever $a < e$. Then L has a unique countably strong inclusion.*

Proof. By Lemma 2.21, $(\prec_1, \prec_2)$ is a countably strong inclusion on L. Let $(\triangleleft_1, \triangleleft_2)$ be any countably strong inclusion on L. Then $(\triangleleft_1, \triangleleft_2) \subseteq (\prec_1, \prec_2)$ by the condition 3) of countably strong inclusion. Note that for $a < e$, $a \prec_i b$ if and only if $a \ll_{ci} b$. Thus by Proposition 2.20, $(\ll_{c1}, \ll_{c2})$ is the smallest countably strong inclusion, that is, $(\prec_1, \prec_2)$ is the smallest countably strong inclusion. Hence $(\triangleleft_1, \triangleleft_2) = (\prec_1, \prec_2)$. □

References

[1] B. Banaschewski, *Compactification of frames*, Math. Nach., **149**, 1990, pp. 105-116.

[2] B. Banaschewski, *Frames and compactifications, Extension theory of topological structures and its applications*, Deutscher Verlag der Wissenschaften, Berlin, 1969, pp. 29-33.

[3] B. Banaschewski, and G. C. L. Brümmer, *Stably continuous frames*, Math. Proc. Camb. Phil. Soc., **104**, 1988, no. 1, pp. 7-19.

[4] B. Banaschewski, and G. C. L. Brümmer, K. A. Hardie, *Biframes and bispaces*, Quaestiones Math., **6**, 1983, pp. 13-25.

[5] B. Banaschewski, and C. J. Mulvey, *Stone-Čech compactification of locales I*, Houston J. Math., **6**, 1980, pp. 301-312.

[6] X. Chen, *Closed homomorphism*, Doctoral Dissertation, McMaster University, Hamilton, 1991.

[7] G. Gierz, K. H. Hoffman, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *A compendium of continuous lattices*, Springer-Verlag, Berlin, 1980.

[8] J. R. Isbell, *Atomless parts of locales*, Math. Scand., 31, 1972, pp. 5-32.

[9] P. T. Johnstone, *Stone spaces*, Cambridge Univ. Press, Cambridge, 1982.

[10] M. K. Khang, *Lindelöfication of Frames*, Kangweon-Kyungki Math. Jour.,15(2007), No. 2, pp. 87-100.
[11] S. O. Lee, *On countably approximating lattices*, J. Korean Math, Soc., **25**, no. 1, 1988, pp. 11-23.

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