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FUZZY LATTICES

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Abstract. We define the operations $\vee$ and $\wedge$ for fuzzy sets in a lattice, characterize fuzzy sublattices in terms of $\vee$ and $\wedge$, develop some properties of the distributive fuzzy sublattices, and find the fuzzy ideal generated by a fuzzy subset in a lattice and the fuzzy dual ideal generated by a fuzzy subset in a lattice.

1. Introduction

The concept of a fuzzy set was first introduced by Zadeh ([6]) and this concept was adapted by Yuan and Wu ([5]) to introduce the concepts of fuzzy sublattices and fuzzy ideals of a lattice. Ying ([4]) defined a L-fuzzy semilattice and established its properties. Ajmal and Thomas ([1]) defined a fuzzy sublattice as a fuzzy algebra and characterized fuzzy sublattices. In this note, as a continuation of these studies, we define the operations $\vee$ and $\wedge$ for fuzzy sets in a lattice and develop some properties of fuzzy sublattices based on those operations.

In section 2, we give some definitions and develop some basic properties of fuzzy sublattices which will be used in next section. In section 3, we characterize a fuzzy sublattice in terms of the operations $\vee$ and $\wedge$ for fuzzy sets in a lattice, develop some properties of the distributive fuzzy lattices, and find the fuzzy ideal generated by a fuzzy subset in a lattice and the fuzzy dual ideal generated by a fuzzy subset in a lattice.

2.Preliminaries

In this section, we give some definitions and develop some basic properties of fuzzy sublattices which will be used in next section.

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Definition 2.1. A function B from a set X to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy set* in X. For every $x \in B$, $B(x)$ is called a *membership grade* of x in B. A fuzzy set in X is called a *fuzzy point* iff it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$, we denote this fuzzy point by x_α, where the point x is called its *support*. The fuzzy point x_α is said to be contained in a fuzzy set A, denoted by $x_\alpha \in A$, iff $\alpha \leq A(x)$.

Remark. The crisp set L itself is a fuzzy subset of L such that $L(x) = 1$ for all $x \in L$ (see Lemma 2.4 of [3]).

Throughout this note, we shall denote by L a lattice $(L, +, \cdot)$, where $+$ is the join operation and $\cdot$ is the meet operation. The following definition is due to Ajmal and Thomas ([1]).

Definition 2.2. A function H from a lattice $(L, +, \cdot)$ to the closed unit interval [0, 1] in $\mathbb{R}$ is called a *fuzzy sublattice* in L iff $H(x + y) \geq \min\ (H(x), H(y))$ and $H(x \cdot y) \geq \min\ (H(x), H(y))$.

We define operations $\vee$ and $\wedge$ for fuzzy sets in a lattice which play important roles in this note and develop some properties of these operations.

Definition 2.3. Let $(L, +, \cdot)$ be a lattice and let U and V be fuzzy subsets of L. $U \vee V$ is defined by

$$(U \vee V)(x) = \begin{cases} \sup\limits_{a+b=x} \ \min\ (U(a), V(b)) & \text{if } \ a + b = x \\ 0 & \text{if } \ a + b \neq x. \end{cases}$$

$U \wedge V$ is defined by

$$(U \wedge V)(x) = \begin{cases} \sup\limits_{a \cdot b=x} \ \min\ (U(a), V(b)) & \text{if } \ a \cdot b = x \\ 0 & \text{if } \ a \cdot b \neq x. \end{cases}$$

Proposition 2.4. *Let A, B be fuzzy sets in a lattice $(L, +.\cdot)$ and let x_p, y_q be fuzzy points in X. Then*

(1) $x_p \vee y_q = (x + y)_{\min\ (p,q)}$ *and* $x_p \wedge y_q = (x \cdot y)_{\min\ (p,q)}$.
(2) $A \vee B = \bigcup\limits_{x_p \in A, y_q \in B} x_p \vee y_q$, *where*

$(x_p \vee y_q)(z) = \sup_{c+d=z} \min\ (x_p(c), y_q(d)).$
$A \wedge B = \bigcup_{x_p\in A, y_q\in B} x_p \wedge y_q$, *where*
$(x_p \wedge y_q)(z) = \sup_{c\cdot d=z} \min\ (x_p(c), y_q(d)).$

Proof. (1) If $z \neq x+y$, $(x_p \vee y_q)(z) = 0$. If $z = x+y$, $(x_p \vee y_q)(z) = (x_p \vee y_q)(x+y) = \sup_{a+b=x+y} \min\ (x_p(a), y_q(b)) = \min\ (x_p(x), y_q(y)) = \min\ (p,q)$. Thus $x_p \vee y_q = (x+y)_{\min(p,q)}$. Similarly we may prove $x_p \wedge y_q = (x \cdot y)_{\min\ (p,q)}$.
(2) Since $s_{A(s)} \in A$ and $t_{B(t)} \in B$,

$$\begin{aligned}(\bigcup_{x_p\in A, y_q\in B} x_p \vee y_q)(z) &= \sup_{x_p\in A, y_q\in B} \sup_{s+t=z} \min\ (x_p(s), y_q(t)) \\ &\geq \sup_{s+t=z} \min\ (s_{A(s)}(s),\ t_{B(t)}(t)) \\ &= \sup_{s+t=z} \min\ (A(s),\ B(t)) \\ &= (A \vee B)(z).\end{aligned}$$

For $x_p \in A$ and $y_q \in B$, $A(s) \geq x_p(s)$ and $B(t) \geq y_q(t)$. Thus

$$\begin{aligned}(A \vee B)(z) &= \sup_{s+t=z} \min\ (A(s), B(t)) \\ &\geq \sup_{s+t=z} \min(x_p(s), y_q(t))\end{aligned}$$

for all $x_p \in A$ and all $y_q \in B$. Let $C = \{c \in \mathbb{R} : c \leq \sup_{s+t=z} \min(A(s), B(t))\}$, and $D = \{\sup_{s+t=z} \min(x_p(s), y_q(t)) :\ x_p \in A,\ y_q \in B\}$. Then $D \subseteq C$ and $\sup_{x_p\in A, y_q\in B} D \in \overline{D} \subseteq \overline{C}$. Since C is closed, $\sup_{x_p\in A, y_q\in B} D \in C$. Thus

$$\begin{aligned}(A \vee B)(z) &\geq \sup_{x_p\in A, y_q\in B} \sup_{s+t=z} \min\ (x_p(s),\ y_q(t)) \\ &= (\bigcup_{x_p\in A, y_q\in B} x_p \vee y_q)(z).\end{aligned}$$

Similarly we may prove $A \wedge B = \bigcup_{x_p\in A, y_q\in B} x_p \wedge y_q$ □

PROPOSITION 2.5. *Let A, B, and C be fuzzy sets in a lattice $(L, +, \cdot)$. Then $(A \vee B) \vee C = A \vee (B \vee C)$ and $(A \wedge B) \wedge C = A \wedge (B \wedge C)$.*

Proof. Let $S = \{\min\ (A(p), B(q)) : p + q = a\} \subseteq \mathbb{R}$ and let $\sup\ S = \alpha$. Then α is an upper bound of S and there exists a sequence $s_n \in S$ such that $s_n \to \alpha$. Since α is an upper bound of S, $\min\ (s, r) \leq \min\ (\alpha, r)$ for all $s \in S$. Since min is a continuous function (see [2]), $\lim_{n \to \infty} \min\ (s_n, r) = \min\ (\alpha, r)$. Since $\min\ (\alpha, r)$ is an upper bound of $\min\ (S, r)$ and there exists $\min\ (s_n, r) \in \min\ (S, r)$ such that $\lim_{n \to \infty} \min\ (s_n, r) = \min\ (\alpha, r)$, $\sup\ \min\ (S, r) = \min\ (\alpha, r) = \min\ (\sup\ S, r)$. That is,

$$\sup_{p+q=a} \min\, [\min\, (A(p), B(q)), C(b)] = \min\, [\sup_{p+q=a} \min\, (A(p), B(q)), C(b)].$$

Thus

$$\begin{aligned}
&\sup_{a+b=z} \min[\sup_{p+q=a} \min(A(p), B(q)), C(b)] \\
&= \sup_{a+b=z} \sup_{p+q=a} \min[\min(A(p), B(q)), C(b)] \\
&= \sup_{(p+q)+b=z} \min\, [\min\, (A(p), B(q)), C(b)].
\end{aligned}$$

Similarly we may show that

$$\begin{aligned}
&\sup_{p+a=z} \min\, [A(p), \sup_{q+b=a} \min\, (B(q), C(b))] \\
&= \sup_{p+(q+b)=z} \min\, [A(p), \min\, (B(q), C(b))].
\end{aligned}$$

Since $p + (q + b) = (p + q) + b$ in a lattice L,

$$\begin{aligned}
[(A \vee B) \vee C]\ (z) &= \sup_{a+b=z} \min\, [\sup_{p+q=a} \min\, (A(p), B(q)), C(b)] \\
&= \sup_{(p+q)+b=z} \min\, [\min\, (A(p), B(q)),\ C(b)] \\
&= \sup_{p+(q+b)=z} \min\, [A(p),\ \min\, (B(q), C(b))] \\
&= \sup_{p+a=z} \min\, [A(p), \sup_{q+b=a} \min\, (B(q), C(b))] \\
&= \sup_{p+a=z} \min\, [A(p), (B \vee C)(a)] = [A \vee (B \vee C)](z).
\end{aligned}$$

Similarly we may show $(A \wedge B) \wedge C = A \wedge (B \wedge C)$. □

Proposition 2.6. *Let A be a fuzzy sublattice in a lattice L and let $x_p, y_q \in A$. Then*

(1) $x_p \vee x_p = x_p$ *and* $x_p \wedge x_p = x_p$.
(2) $x_p \vee y_q = y_q \vee x_p$ *and* $x_p \wedge y_q = y_q \wedge x_p$.
(3) $(x_p \vee y_q) \vee z_r = x_p \vee (y_q \vee z_r)$ *and* $(x_p \wedge y_q) \wedge z_r = x_p \wedge (y_q \wedge z_r)$.
(4) $(x_p \vee y_q) \wedge x_p = x_{\min(p,q)}$ *and* $(x_p \wedge y_q) \vee x_p = x_{\min(p,q)}$.

Proof. (1) $x_p \vee x_p = (x \vee x)_{\min(p,p)} = x_p$ and $x_p \wedge x_p = (x \wedge x)_{\min(p,p)} = x_p$.
(2) Straightforward.
(3) Straightforward.
(4) $(x_p \vee y_q) \wedge x_p = (x \vee y)_{\min(p,q)} \wedge x_p = [(x \vee y) \wedge x]_{\min(p,q)} = x_{\min(p,q)}$. Similarly we may prove $(x_p \wedge y_q) \vee x_p = x_{\min(p,q)}$.

□

Definition 2.7. Let A be a fuzzy sublattice in a lattice L. Then A is *distributive* iff $x_p \wedge (y_q \vee z_r) = (x_p \wedge y_q) \vee (x_p \wedge z_r)$ and $x_p \vee (y_q \wedge z_r) = (x_p \vee y_q) \wedge (x_p \vee z_r)$ for all $x_p, y_q, z_r \in A$.

Proposition 2.8. *Let $A_1, A_2, \dots, A_n$ be fuzzy subsets in a lattice $(L, +, \cdot)$. Then*

(1) $L \wedge (A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq (L \wedge A_1) \cup (L \wedge A_2) \cup \cdots \cup (L \wedge A_n)$
(2) $(A_1 \cup A_2 \cup \cdots \cup A_n) \wedge L \subseteq (A_1 \wedge L) \cup (A_2 \wedge L) \cup \cdots \cup (A_n \wedge L)$.

Proof. (1) Since $L(a) = 1$,

$$\begin{aligned}[L \wedge (A_1 \cup A_2 \cup \cdots \cup A_n)](x) &= \sup_{a \cdot b = x} \min\,[L(a), (A_1 \cup A_2 \cup \cdots \cup A_n)(b)] \\ &= \sup_{a \cdot b = x} \max[A_1(b), A_2(b), \dots, A_n(b)].\end{aligned}$$

Since $L(a) = 1$,

$$\begin{aligned}&[(L \wedge A_1) \cup \cdots \cup (L \wedge A_n)](x) \\ &\qquad = \max[(L \wedge A_1)(x), (L \wedge A_2)(x), \dots, (L \wedge A_n)(x)] \\ &\qquad = \max[\sup_{a \cdot b = x} A_1(b), \sup_{a \cdot b = x} A_2(b), \dots, \sup_{a \cdot b = x} A_n(b)].\end{aligned}$$

Thus $L \wedge (A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq (L \wedge A_1) \cup (L \vee A_2) \cup \cdots \cup (L \wedge A_n)$.

(2) Similarly, we may prove

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \wedge L \subseteq (A_1 \wedge L) \cup (A_2 \wedge L) \cup \cdots \cup (A_n \wedge L).$$

□

Proposition 2.9. *Let $A_1, A_2, \ldots, A_n$ be fuzzy subsets in a lattice $(L, +, \cdot)$. Then*

(1) $L \vee (A_1 \cup A_2 \cup \cdots \cup A_n) \subseteq (L \vee A_1) \cup (L \vee A_2) \cup \cdots \cup (L \vee A_n)$
(2) $(A_1 \cup A_2 \cup \cdots \cup A_n) \vee L \subseteq (A_1 \vee L) \cup (A_2 \vee L) \cup \cdots \cup (A_n \vee L)$.

Proof. The proof is similar to that of Proposition 2.8. □

The following definition is due to Ajmal and Thomas ([1]).

Definition 2.10. Let A be a fuzzy sublattice in a lattice $(L, +, \cdot)$. Then A is called a *fuzzy ideal* if $x \leq y$ in L implies $A(x) \geq A(y)$. Let B be a fuzzy sublattice in a lattice $(L, +, \cdot)$. Then B is called a *fuzzy dual ideal* if $x \leq y$ in L implies $B(x) \leq B(y)$.

3.Fuzzy lattices

In this section, we characterize a fuzzy sublattice in terms of the operations $\vee$ and $\wedge$, develop some properties of the distributive fuzzy sublattices, and find the fuzzy ideal generated by a fuzzy subset in a lattice and the fuzzy dual ideal generated by a fuzzy subset in a lattice.

Theorem 3.1. *Let A be an non-empty fuzzy set of a lattice $(L, +, \cdot)$. Then the followings are equivalent.*

(1) *A is a fuzzy sublattice.*
(2) *For any $x_p, y_q \in A$, $x_p \vee y_q \in A$ and $x_p \wedge y_q \in A$.*
(3) *$A \vee A \subseteq A$ and $A \wedge A \subseteq A$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $A(x + y) \geq \min(A(x), A(y))$ and $A(x \cdot y) \geq \min(A(x), A(y))$. By Proposition 2.4,

$$(x_p \vee y_q)(z) = [(x + y)_{\min(p,q)}](z) = \begin{cases} \min(p, q), & \text{if } z = x + y \\ 0, & \text{if } z \neq x + y. \end{cases}$$

Let $x_p, y_q \in A$. Then $A(x) \geq p$ and $A(y) \geq q$. If $z = x + y$, $A(z) = A(x+y) \geq \min(A(x), A(y)) \geq \min(p,q) = (x_p \vee y_q)(z)$, and hence $x_p \vee y_q \in A$. If $z \neq x+y$, $A(z) \geq (x_p \vee y_q)(z) = 0$, and hence $x_p \vee y_q \in A$. Similarly we may show that $x_p \wedge y_q \in A$.
(2) $\Rightarrow$ (3). Suppose that for any $x_p, y_q \in A$, $x_p \vee y_q \in A$ and $x_p \wedge y_q \in A$. By Proposition 2.4,

$$(A \vee A)(z) = [\bigcup_{x_p \in A, y_q \in A} x_p \vee y_q](z) = \sup_{x_p \in A, y_q \in A} (x_p \vee y_q)(z).$$

Let $C = \{c \in \mathbb{R} : c \leq A(z)\}$ and $D = \{(x_p \vee y_q)(z) : x_p, y_q \in A\}$. Then $D \subseteq C$ and $\sup_{x_p \in A, y_q \in A} (x_p \vee y_q)(z) \in \overline{D} \subseteq \overline{C} = C$. Thus $(A \vee A)(z) = \sup_{x_p \in A, y_q \in A} (x_p \vee y_q)(z) \leq A(z)$. Similarly we may show that $A \wedge A \subseteq A$.
(3) $\Rightarrow$ (1). Suppose $A \vee A \subseteq A$ and $A \wedge A \subseteq A$. Then $A(x+y) \geq (A \vee A)(x+y) = \sup_{a+b=x+y} \min(A(a), A(b)) \geq \min(A(x), A(y))$ and $A(x \cdot y) \geq (A \wedge A)(x \cdot y) = \sup_{a \cdot b = x \cdot y} \min(A(a), A(b)) \geq \min(A(x), A(y))$
Thus A is a fuzzy sublattice. □

We now turn to the distributive law of a fuzzy sublattice.

Proposition 3.2. *Let A be a fuzzy sublattice in a lattice $(L, +, \cdot)$. If $\min(p,q,r) = p$ for $x_p, y_q, z_r \in A$, then*

$$x_p \wedge (y_q \vee z_r) = (x_p \wedge y_q) \vee (x_p \wedge z_r) \Longleftrightarrow x_p \vee (y_q \wedge z_r) = (x_p \vee y_q) \wedge (x_p \vee z_r).$$

Proof. ($\Rightarrow$) By Proposition 2.5 and Proposition 2.6, $(x_p \vee y_q) \wedge (x_p \vee z_r) = [(x_p \vee y_q) \wedge x_p] \vee [(x_p \vee y_q) \wedge z_r] = x_{\min(p,q)} \vee [z_r \wedge (x_p \vee y_q)] = x_{\min(p,q)} \vee [(z_r \wedge x_p) \vee (z_r \wedge y_q)] = [x_{\min(p,q)} \vee (z_r \wedge x_p)] \vee (z_r \wedge y_q) = x_{\min(p,q,r)} \vee (z_r \wedge y_q) = x_p \vee (y_q \wedge z_r)$.
($\Leftarrow$) By Proposition 2.5 and Proposition 2.6, $(x_p \wedge y_q) \vee (x_p \wedge z_r) = [(x_p \wedge y_q) \vee x_p] \wedge [(x_p \wedge y_q) \vee z_r] = x_{\min(p,q)} \wedge [z_r \vee (x_p \wedge y_q)] = x_{\min(p,q)} \wedge [(z_r \vee x_p) \wedge (z_r \vee y_q)] = [x_{\min(p,q)} \wedge (z_r \vee x_p)] \wedge (z_r \vee y_q) = x_{\min(p,q,r)} \wedge (z_r \vee y_q) = x_p \wedge (y_q \vee z_r)$. □

Proposition 3.3. *Let A be a fuzzy sublattice on a distributive lattice $(L, +, \cdot)$. Then A is a distributive fuzzy sublattice.*

Proof. Let $x_p, y_q, z_r \in A$. Since L is distributive, $x \cdot (y+z) = x \cdot y + x \cdot z$. $[x_p \wedge (y_q \vee z_r)](x \cdot (y+z)) = \sup\limits_{a \cdot (b+c) = x \cdot (y+z)} \min\,[x_p(a), (y_q \vee z_r)(b+c)] = \min\,[p, (y_q \vee z_r)(y+z)] = \min\,[p, \sup\limits_{l+m=y+z} \min\,(y_q(l), z_r(m))] = \min\,[p, \min\,(q,r)] = \min(p,q,r)$. $[(x_p \wedge y_q) \vee (x_p \wedge z_r)](x \cdot y + x \cdot z) = \sup\limits_{s \cdot t + v \cdot w = x \cdot y + x \cdot z} \min\,[(x_p \wedge y_q)(s \cdot t), (x_p \wedge z_r)(v \cdot w)] = \min\,[(x_p \wedge y_q)(x \cdot y), (x_p \wedge z_r)(x \cdot z)] = \min\,[\min(p,q),\ \min(p,r)] = \min\,(p,q,r)$. If $u \neq x \cdot (y+z)$, $[x_p \wedge (y_q \vee z_r)](u) = 0$. If $u \neq x \cdot y + x \cdot z$, $[(x_p \wedge y_q) \vee (x_p \wedge z_r)](u) = 0$. Thus $x_p \wedge (y_q \vee z_r) = (x_p \wedge y_q) \vee (x_p \wedge z_r)$. Similarly we may prove $x_p \vee (y_q \wedge z_r) = (x_p \vee y_q) \wedge (x_p \vee z_r)$ □

We now turn to the characterization of the fuzzy ideal generated by a fuzzy subset in a lattice and the fuzzy dual ideal generated by a fuzzy subset in a lattice. Proposition 3.4 and Proposition 3.5 are due to Ajmal and Thomas ([1]).

PROPOSITION 3.4. *Let A be a fuzzy set in a lattice $(L, +, \cdot)$. Then the followings are equivalent.*

(1) *$x \leq y$ implies $A(x) \geq A(y)$.*
(2) *$A(x \cdot y) \geq \max\,(A(x), A(y))$.*
(3) *$A(x+y) \leq \min\,(A(x), A(y))$.*

PROPOSITION 3.5. *Let A be a fuzzy set in a lattice $(L, +, \cdot)$. Then the followings are equivalent.*

(1) *$x \leq y$ implies $A(x) \leq A(y)$.*
(2) *$A(x+y) \geq \max\,(A(x), A(y))$.*
(3) *$A(x \cdot y) \leq \min\,(A(x), A(y))$.*

THEOREM 3.6. *Let A be a fuzzy subset in a lattice $L(+, \cdot)$. Then the fuzzy ideal I generated by A is $A \cup (L \wedge A)$. That is, $I(x) = \max[A(x), \sup\limits_{a \cdot b = x} A(b)]$.*

Proof. Let $\{J_i : i \in I\}$ be the collection of all fuzzy ideals of L containing A. Then $\bigcap\limits_{i \in I} J_i$ is a fuzzy ideal (see Theorem 3.17 of [1]).

Since $J_i(\alpha \cdot \beta) \geq \max\ (J_i(\alpha), J_i(\beta))$ by Proposition 3.4 and $L(a) = 1$,

$$\begin{aligned}(L \wedge J_i)(x) &= \sup_{a \cdot b = x} \min\ (L(a), J_i(b)) \\ &\leq \sup_{a \cdot b = x} \min\ (L(a), J_i(a \cdot b)) \\ &= J_i(x)\end{aligned}$$

for each $i \in I$. Thus $L \wedge A \subseteq \bigcap_{i \in I} J_i$. Hence $A \cup (L \wedge A) \subseteq \bigcap_{i \in I} J_i$. By Proposition 2.8, $L \wedge (A \cup (L \wedge A)) \subseteq (L \wedge A) \cup (L \wedge (L \wedge A))$. By Proposition 2.5, $L \wedge (A \cup (L \wedge A)) \subseteq (L \wedge A) \cup ((L \wedge L) \wedge A)$. Since L is a crisp set, $L(x) = 1$ for all $x \in L$, and hence $L \wedge L \subseteq L$. Thus

$$L \wedge (A \cup (L \wedge A)) \subseteq (L \wedge A) \cup (L \wedge A) \subseteq L \wedge A \subseteq A \cup (L \wedge A).$$

Since $L(x) = 1$,

$$\begin{aligned}[A \cup (L \wedge A)](x \cdot y) &\geq [L \wedge (A \cup (L \wedge A))](x \cdot y) \\ &= \sup_{a \cdot b = x \cdot y} \min\ [L(a), (A \cup (L \wedge A))(b)] \\ &\geq \min\ [L(x), (A \cup (L \wedge A))(y)] \\ &= [A \cup (L \wedge A)](y).\end{aligned}$$

Since $x \cdot y = y \cdot x$, we may show $[A \cup (L \wedge A)](x \cdot y) \geq [A \cup (L \wedge A)](x)$. Thus

$$[A \cup (L \wedge A)](x \cdot y) \geq \max\ [(A \cup (L \wedge A))(x),\ (A \cup (L \wedge A))(y)].$$

Similarly, we may show $[A \cup (L \wedge A)](x + y) \geq \max\ [(A \cup (L \wedge A))(x), (A \cup (L \wedge A))(y)]$. Then $A \cup (L \wedge A)$ is a fuzzy ideal of L containing A by Proposition 3.4, that is, $\cap J_i \subseteq A \cup (L \wedge A)$. Hence $\cap J_i = A \cup (L \wedge A)$. Also $(L \wedge A)(x) = \sup_{a \cdot b = x} \min\ (L(a), A(b) = \sup_{a \cdot b = x} A(b)$. □

THEOREM 3.7. *Let A be a fuzzy subset in a lattice $L(+, \cdot)$. Then the fuzzy dual ideal D generated by A is $A \cup (L \vee A)$. That is, $D(x) = \max[A(x), \sup_{a + b = x} A(b)]$.*

Proof. Let $\{J_i : i \in I\}$ be the collection of all fuzzy dual ideals of L containing A. Then $\underset{i\in I}{\cap} J_i$ is a fuzzy dual ideal (see Theorem 3.17 of [1]) and $J_i(\alpha+\beta) \geq \max\ (J_i(\alpha), J_i(\beta))$ by Proposition 3.5. We may show $A \cup (L \vee A) \subseteq \underset{i\in I}{\cap} J_i$ by the same way as shown in Theorem 3.6. By Proposition 2.9, $L \vee (A \cup (L \vee A)) \subseteq (L \vee A) \cup (L \vee (L \vee A))$. By Proposition 2.5, $L \vee (A \cup (L \vee A)) \subseteq (L \vee A) \cup ((L \vee L) \vee A)$. By the same way as shown in Theorem 3.6, we may show $[A \cup (L \vee A)](x+y) \geq \max\ [(A \cup (L \vee A))(x), (A \cup (L \vee A))(y)]$ and $[A \cup (L \vee A)](x \cdot y) \geq \max\ [(A \cup (L \vee A))(x), (A \cup (L \vee A))(y)]$. Thus $A \cup (L \vee A)$ is a fuzzy dual ideal of L containing A by Proposition 3.5, that is, $\cap J_i \subseteq A \cup (L \vee A)$. Hence $\cap J_i = A \cup (L \vee A)$. Also $(L \vee A)(x) = \sup_{a+b=x} \min\ (L(a), A(b) = \sup_{a+b=x} A(b)$. □

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