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# STABILITY OF FUNCTIONAL EQUATIONS ASSOCIATED WITH INNER PRODUCT SPACES: A FIXED POINT APPROACH

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Abstract. In [21], Th.M. Rassias proved that the norm defined over a real vector space $V$ is induced by an inner product if and only if for a fixed integer $n \geq 2$

$$n\left\|\frac{1}{n}\sum_{i=1}^{n} x_i\right\|^2 + \sum_{i=1}^{n}\left\|x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right\|^2 = \sum_{i=1}^{n}\|x_i\|^2$$

holds for all $x_1, \cdots, x_n \in V$. We consider the functional equation

$$nf\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} f\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right) = \sum_{i=1}^{n} f(x_i).$$

Using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional equation

$$(1)\quad 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) = f(x) + f(y).$$

## 1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [30] concerning the stability of group homomorphisms. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive

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mappings and by Th.M. Rassias [20] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [20] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [29] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. Czerwik [6] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [10], [13], [16]–[18], [22]–[28]).

We recall two fundamental results in fixed point theory.

THEOREM 1.1. [2] *Let $(X, d)$ be a complete metric space and let $J : X \to X$ be strictly contractive, i.e.,*

$$d(Jx, Jy) \le Lf(x, y), \qquad \forall x, y \in X$$

*for some Lipschitz constant $L < 1$. Then*

(1) *the mapping $J$ has a unique fixed point $x^* = Jx^*$;*

(2) *the fixed point $x^*$ is globally attractive, i.e.,*

$$\lim_{n\to\infty} J^n x = x^*$$

*for any starting point $x \in X$;*

(3) *one has the following estimation inequalities:*

$$\begin{aligned} d(J^n x, x^*) &\leq L^n d(x, x^*), \\ d(J^n x, x^*) &\leq \frac{1}{1-L} d(J^n x, J^{n+1} x), \\ d(x, x^*) &\leq \frac{1}{1-L} d(x, Jx) \end{aligned}$$

*for all nonnegative integers $n$ and all $x \in X$.*

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on $X$ if $d$ satisfies
(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

THEOREM 1.2. [7] *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*
(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

This paper is organized as follows: In Section 2, using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional equation (1) in real Banach spaces: an even case.

In Section 3, using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional equation (1) in real Banach spaces: an odd case.

Throughout this paper, let $X$ be a real normed vector space with norm $||\cdot||$, and $Y$ a real Banach space with norm $\|\cdot\|$.

In 1996, G. Isac and Th.M. Rassias [11] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4], [12], [14], [15], [19]).

## 2. Fixed points and generalized Hyers-Ulam stability of the functional equation (1): an even case

One can easily show that an even mapping $f : X \to Y$ satisfies (1) if and only if the even mapping $f : X \to Y$ is a Jensen type quadratic mapping, i.e.,

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y),$$

and that an odd mapping $f : X \to Y$ satisfies (1) if and only if the odd mapping mapping $f : X \to Y$ is a Jensen additive mapping, i.e.,

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y).$$

For a given mapping $f : X \to Y$, we define

$$Cf(x,y) := 2f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) + f\left(\frac{y-x}{2}\right) - f(x) - f(y)$$

for all $x, y \in X$.

Using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional equation $Cf(x,y) = 0$: an even case.

THEOREM 2.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $L < 1$ such that $\varphi(x,0) \le \frac{1}{4}L\varphi(2x,0)$ for all $x \in X$, and*

$$\sum_{j=0}^{\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{2}$$

$$\|Cf(x,y)\| \le \varphi(x,y) \tag{3}$$

*for all $x, y \in X$. Then there exists a unique Jensen type quadratic mapping $Q : X \to Y$ satisfying*

$$\|f(x) + f(-x) - Q(x)\| \le \frac{1}{1-L}(\varphi(x,0) + \varphi(-x,0)) \tag{4}$$

*for all $x \in X$.*

*Proof.* Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on $S$:

$$d(g,h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \le K(\varphi(x,0) + \varphi(-x,0)), \quad \forall x \in X\}.$$

It is easy to show that $(S,d)$ is complete. (See the proof of Theorem 2.5 of [3].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of [2] that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

Letting $y = 0$ in (3), we get

$$\left\|2f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right) - f(x)\right\| \leq \varphi(x,0) \tag{5}$$

for all $x \in X$. Replacing $x$ by $-x$ in (5), we get

$$\left\|2f\left(-\frac{x}{2}\right) + f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right) - f(-x)\right\| \leq \varphi(-x,0) \tag{6}$$

for all $x \in X$. Let $g(x) := f(x) + f(-x)$ for all $x \in X$. Then $g : X \to Y$ is an even mapping. It follows from (5) and (6) that

$$\left\|g(x) - 4g\left(\frac{x}{2}\right)\right\| \leq \varphi(x,0) + \varphi(-x,0)$$

for all $x \in X$. Hence $d(g, Jg) \leq 1$.

By Theorem 1.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) $Q$ is a fixed point of $J$, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{7}$$

for all $x \in X$. Then $Q : X \to Y$ is an even mapping. The mapping $Q$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that $Q$ is a unique mapping satisfying (7) such that there exists a $K \in (0,\infty)$ satisfying

$$\|g(x) - Q(x)\| \leq K(\varphi(x,0) + \varphi(-x,0))$$

for all $x \in X$;

(2) $d(J^n g, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n g\left(\frac{x}{2^n}\right) = Q(x) \tag{8}$$

for all $x \in X$;

(3) $d(g,Q) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g,Q) \leq \frac{1}{1-L}.$$

This implies that the inequality (4) holds.

It follows from (2), (3) and (8) that

$$\begin{aligned} \|CQ(x,y)\| &= \lim_{n\to\infty} 4^n \left\| Cg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} 4^n \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right) = 0 \end{aligned}$$

for all $x, y \in X$. So $CQ(x,y) = 0$ for all $x, y \in X$. Since $Q : X \to Y$ is even, the mapping $Q : X \to Y$ is a Jensen type quadratic mapping.

Therefore, there exists a unique Jensen type quadratic mapping $Q : X \to Y$ satisfying (4), as desired. □

COROLLARY 2.2. *Let $p > 2$ and $\theta \geq 0$ be real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|Cf(x,y)\| \leq \theta(||x||^p + ||y||^p) \tag{9}$$

*for all $x, y \in X$. Then there exists a unique Jensen type quadratic mapping $Q : X \to Y$ satisfying*

$$\|f(x) + f(-x) - Q(x)\| \leq \frac{2^{p+1}\theta}{2^p - 4} ||x||^p$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. □

REMARK 2.3. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (3) and $f(0) = 0$ such that

$$\sum_{j=0}^{\infty} \frac{1}{4^j} \varphi(2^j x, 2^j y) < \infty \tag{10}$$

for all $x, y \in X$. By a similar method to the proof of Theorem 2.1, one can show that if there exists an $L < 1$ such that $\varphi(x,0) \le 4L\varphi(\frac{x}{2},0)$ for all $x \in X$, then there exists a unique Jensen type quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \le \frac{L}{1-L}(\varphi(x,0) + \varphi(-x,0))$$

for all $x \in X$.

For the case $p < 2$, one can obtain a similar result to Corollary 2.2: Let $p < 2$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying (9). Then there exists a unique Jensen type quadratic mapping $Q : X \to Y$ satisfying

$$\|f(x) + f(-x) - Q(x)\| \le \frac{2^{p+1}\theta}{4-2^p}||x||^p$$

for all $x \in X$.

## 3. Fixed points and generalized Hyers-Ulam stability of the functional equation (1): an odd case

Using fixed point methods, we prove the generalized Hyers-Ulam stability of the functional equation $Cf(x,y) = 0$: an odd case.

THEOREM 3.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that there exists an $L < 1$ such that $\varphi(x,0) \le \frac{1}{2}L\varphi(2x,0)$ for all $x \in X$, and*

$$\sum_{j=0}^{\infty} 2^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \tag{11}$$

$$\|Cf(x,y)\| \le \varphi(x,y) \tag{12}$$

*for all $x, y \in X$. Then there exists a unique Jensen additive mapping $A : X \to Y$ satisfying*

$$\|f(x) - f(-x) - A(x)\| \le \frac{1}{1-L}(\varphi(x,0) + \varphi(-x,0)) \tag{13}$$

*for all $x \in X$.*

*Proof.* Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on $S$:

$$d(g,h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \le K(\varphi(x,0) + \varphi(-x,0)), \quad \forall x \in X\}.$$

It is easy to show that $(S, d)$ is complete. (See the proof of Theorem 2.5 of [3].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

It follows from the proof of Theorem 3.1 of [2] that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

Letting $y = 0$ in (12), we get

$$\left\| 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right) - f(x) \right\| \le \varphi(x,0) \tag{14}$$

for all $x \in X$. Replacing $x$ by $-x$ in (14), we get

$$\left\| 2f\left(-\frac{x}{2}\right) + f\left(\frac{x}{2}\right) + f\left(\frac{-x}{2}\right) - f(-x) \right\| \le \varphi(-x,0) \tag{15}$$

for all $x \in X$. Let $g(x) := f(x) - f(-x)$ for all $x \in X$. Then $g : X \to Y$ is an odd mapping. It follows from (14) and (15) that

$$\left\| g(x) - 2g\left(\frac{x}{2}\right) \right\| \le \varphi(x,0) + \varphi(-x,0)$$

for all $x \in X$. Hence $d(g, Jg) \le 1$.

By Theorem 1.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) $A$ is a fixed point of $J$, i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \tag{16}$$

for all $x \in X$. Then $A : X \to Y$ is an odd mapping. The mapping $A$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that $A$ is a unique mapping satisfying (16) such that there exists a $K \in (0, \infty)$ satisfying

$$\|g(x) - A(x)\| \le K(\varphi(x,0) + \varphi(-x,0))$$

for all $x \in X$;

(2) $d(J^n g, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n g\left(\frac{x}{2^n}\right) = A(x) \tag{17}$$

for all $x \in X$;

(3) $d(g, A) \le \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, A) \le \frac{1}{1-L}.$$

This implies that the inequality (13) holds.

It follows from (11), (12) and (17) that

$$\begin{aligned} \|CA(x,y)\| &= \lim_{n\to\infty} 2^n \left\| Cg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \\ &\le \lim_{n\to\infty} 2^n \left(\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + \varphi\left(-\frac{x}{2^n}, -\frac{y}{2^n}\right)\right) = 0 \end{aligned}$$

for all $x, y \in X$. So $CA(x,y) = 0$ for all $x, y \in X$. Since $A : X \to Y$ is odd, the mapping $A : X \to Y$ is a Jensen additive mapping.

Therefore, there exists a unique Jensen additive mapping $A : X \to Y$ satisfying (13), as desired. □

COROLLARY 3.2. *Let $p > 1$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying* (9). *Then there exists a unique Jensen additive mapping $A : X \to Y$ satisfying*

$$\|f(x) - f(-x) - A(x)\| \le \frac{2^{p+1}\theta}{2^p - 2}||x||^p$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{1-p}$ and we get the desired result. □

Combining Corollaries 2.2 and 3.2 yields the following.

THEOREM 3.3. *Let $p > 2$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying* (9). *Then there exist a unique Jensen type* *quadratic mapping* $Q : X \to Y$ *and a unique Jensen additive mapping* $A : X \to Y$ *satisfying*

$$\|2f(x) - Q(x) - A(x)\| \le \left(\frac{2^{p+1}}{2^p - 4} + \frac{2^{p+1}}{2^p - 2}\right)\theta||x||^p$$

*for all* $x \in X$.

REMARK 3.4. Let $f : X \to Y$ be a mapping for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (12) and $f(0) = 0$ such that

$$\sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. By a similar method to the proof of Theorem 3.1, one can show that if there exists an $L < 1$ such that $\varphi(x, 0) \le 2L\varphi(\frac{x}{2}, 0)$ for all $x \in X$, then there exists a unique Jensen additive mapping $A : X \to Y$ satisfying

$$\|f(x) - f(-x) - A(x)\| \le \frac{L}{1 - L}(\varphi(x, 0) + \varphi(-x, 0))$$

for all $x \in X$.

For the case $p < 1$, one can obtain a similar result to Corollary 3.2: Let $p < 1$ and $\theta \ge 0$ be real numbers, and let $f : X \to Y$ be a mapping satisfying (9). Then there exists a unique Jensen additive mapping $A : X \to Y$ satisfying

$$\|f(x) - f(-x) - A(x)\| \le \frac{2^{p+1}\theta}{2 - 2^p}||x||^p$$

for all $x \in X$.

Combining Remarks 2.3 and 3.4 yields the following.

THEOREM 3.5. *Let* $p < 1$ *and* $\theta \ge 0$ *be real numbers, and let* $f : X \to Y$ *be a mapping satisfying* (9). *Then there exist a unique Jensen type quadratic mapping* $Q : X \to Y$ *and a unique Jensen additive mapping* $A : X \to Y$ *satisfying*

$$\|2f(x) - Q(x) - A(x)\| \le \left(\frac{2^{p+1}}{4 - 2^p} + \frac{2^{p+1}}{2 - 2^p}\right)\theta||x||^p$$

*for all* $x \in X$.

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