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COMBINATORIAL PROOF FOR e-POSITIVITY OF THE POSET OF RANK 1

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Abstract. Let P be a poset and $G = G(P)$ be the incomparability graph of P. Stanley [7] defined the chromatic symmetric function $X_{G(P)}$ which generalizes the chromatic polynomial χ_G of G, and showed all coefficients are nonnegative in the e-expansion of $X_{G(P)}$ for a poset P of rank 1. In this paper, we construct a sign reversing involution on the set of special rim hook P-tableaux with some conditions. It gives a combinatorial proof for $(\mathbf{3}+\mathbf{1})$-free conjecture of a poset P of rank 1.

1. Introduction

Let G be a simple graph with d vertices. In [7], Stanley defined a homogeneous symmetric function X_G of degree d which generalizes the chromatic polynomial χ_G of G. Let P be a poset and $G(P)$ be the incomparability graph of P. Then the symmetric function $X_{G(P)}$ can be expanded in terms of various symmetric function bases. In particular, if we use the elementary symmetric function basis $\{e_\mu\}$, we have

$$X_{G(P)} = \sum_{\mu} c_\mu e_\mu.$$

Through their work on immanants of Jacobi-Trudi matrices, Stanley and Stembridge [9] were led to the following conjecture.

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Conjecture 1.1 (($\mathbf{3}$+$\mathbf{1}$)-free conjecture). *If P is a ($\mathbf{3}$+$\mathbf{1}$)-free poset, $X_{G(P)}$ is e-positive, i.e., if*

$$X_{G(P)} = \sum_{\mu} c_\mu e_\mu,$$

then all $c_\mu \geq 0$.

Using the acyclic orientation of the incomparability graph $G(P)$ of P, Stanley [7] proved that ($\mathbf{3}$+$\mathbf{1}$)-free conjecture is true for a poset P of rank 1.

On the other hand, Eğecioğlu and Remmel [2] gave a combinatorial interpretation for the entries of the inverse of Kostka matrix and Chow [1] used Eğecioğlu and Remmel's interpretation to get a combinatorial object for c_μ appeared in Conjecture 1.1.

Using Chow's combinatorial object for c_μ, we construct a sign reversing involution on the set of special rim hook P-tableaux with some conditions. It gives a combinatorial proof for ($\mathbf{3}$+$\mathbf{1}$)-free conjecture of a poset P of rank 1. In Section 2 we describe basic definitions from the theory of Young tableaux. A sign reversing involution to prove the main result with an example is given in Section 3.

2. Definitions and combinatorial interpretation for $K^{-1}_{\mu,\lambda}$

In this section we describe some definitions necessary for later. See [3], [6] or [8] for definitions and notations not described here.

Definition 2.1. A *partition* λ of a positive integer n is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

(i) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$,
(ii) $\sum_{i=1}^{\ell} \lambda_i = n$.

We write $\lambda \vdash n$, or $|\lambda| = n$. We say each term λ_i is a *part* of λ and the number of nonzero parts is called the *length* of λ and is written $\ell = \ell(\lambda)$. In addition, we will use the notation $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ which means that the integer j appears m_j times in λ.

Definition 2.2. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition. The *Ferrers diagram* D_λ of λ is the array of cells or boxes arranged in rows and columns, λ_1 in the first row, λ_2 in the second row, etc., with each row left-justified. That is,

$$D_\lambda = \{(i,j) \in \mathbf{Z}^2 \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\},$$

where we regard the elements of D_λ as a collection of boxes in the plane with matrix-style coordinates.

Definition 2.3. If λ, μ are partitions with $D_\lambda \supseteq D_\mu$, the *skew shape* $D_{\lambda/\mu}$ or just λ/μ is defined as the set-theoretic difference $D_\lambda \setminus D_\mu$. Thus

$$D_{\lambda/\mu} = \{(i,j) \in \mathbf{Z}^2 \mid 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\}.$$

Figure 2.1 shows the Ferrers diagram D_λ and skew shape $D_{\lambda/\mu}$, respectively, when $\lambda = (5,4,2,1) \vdash 12$ and $\mu = (2,2,1) \vdash 5$.

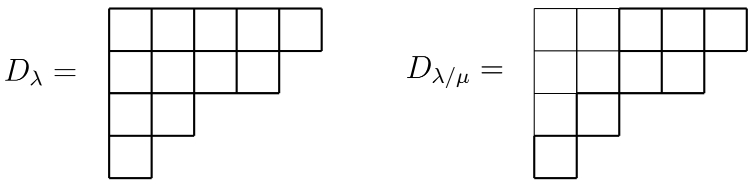


Figure 2.1

Definition 2.4. Let λ be a partition. A *tableau* T of shape λ is an assignment $T : D_\lambda \to \mathbf{P}$ of positive integers to the cells of λ. The *content* of the tableau T, denoted by content(T), is the finite nonnegative vector whose ith component is the number of entries i in T.

A tableau T of shape λ is said to be *column strict* if it satisfies the following two conditions:

(i) $T(i,j) \leq T(i,j+1)$, i.e., the entries increase weakly along the rows of λ from left to right.
(ii) $T(i,j) < T(i+1,j)$, i.e., the entries increase strictly along the columns of λ from top to bottom.

In Figure 2.2, T is a tableau of shape $(5,4,2,1)$ and S is a column strict tableau of shape $(5,4,2,1)$ and of content $(3,3,1,2,2,1)$.

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 1 & 8 & 3 \\ \hline 5 & 2 & 2 & 5 & \\ \cline{1-4} 6 & 7 & & & \\ \cline{1-2} 4 & & & & \\ \cline{1-1}\end{array} \qquad S = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 4 & 6 \\ \hline 2 & 2 & 2 & 5 & \\ \cline{1-4} 3 & 5 & & & \\ \cline{1-2} 4 & & & & \\ \cline{1-1}\end{array}$$

Figure 2.2

Definition 2.5. For partitions λ and μ such that $|\lambda| = |\mu|$, the *Kostka number* $K_{\lambda,\mu}$ is the number of column strict tableaux of shape λ and content μ.

If we use the reverse lexicographic order on the set of partitions of a fixed positive integer n, the *Kostka matrix* $K = (K_{\lambda,\mu})$ becomes upper unitriangular so that K is non-singular.

Definition 2.6. A *rim hook* H is a skew shape which is connected and contains no 2×2 square of cells. The *size of* H is the number of cells it contains. The *leg length of rim hook* H, $\ell(H)$, is the number of vertical edges in H when viewed as in Figure 2.3. We define the *sign* of a rim hook H to be $\epsilon(H) = (-1)^{\ell(H)}$.

Figure 2.3 shows the rim hook H of size 6 with $\ell(H) = 2$ and $\epsilon(H) = (-1)^2 = 1$.

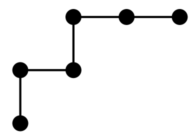
$H =$ 

Figure 2.3

Definition 2.7. A *rim hook tableau* T of shape λ is a partition of the diagram of λ into rim hooks. The *type* of T is $\text{type}(T) = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ where m_k is the number of rim hooks in T of size k. We now define the *sign* of a rim hook tableau T as

$$\epsilon(T) = \prod_{H \in T} \epsilon(H).$$

A rim hook tableau S is called *special* if each of the rim hooks contains a cell from the first column of λ. We use nodes for the Ferrers diagram and connect them if they are adjacent in the same rim hook as S in Figure 2.4.

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 2 & 5 \\ \cline{1-4} 2 & 2 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad S =$$

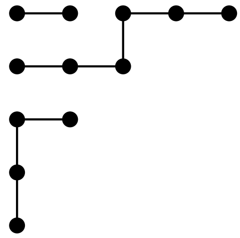


Figure 2.4

In Figure 2.4, T is a rim hook tableau of shape $(5, 4, 2, 1)$, type$(T) = (1^2, 2, 4^2)$ and $\epsilon(T) = (-1)^1 \cdot (-1)^1 \cdot (-1)^0 \cdot (-1)^0 \cdot (-1)^0 = 1$, while S is a special rim hook tableau with shape $(5, 3, 2, 1, 1)$, type$(S) = (2, 4, 6)$ and $\epsilon(S) = (-1)^0 \cdot (-1)^1 \cdot (-1)^2 = -1$.

We can now state Eğecioğlu and Remmel's interpretation for the entries of the inverse of Kostka matrix.

Theorem 2.8 (Eğecioğlu and Remmel[2]). *The entries of the inverse Kostka matrix are given by*

$$K^{-1}_{\mu,\lambda} = \sum_S \epsilon(S)$$

where the sum is over all special rim hook tableaux S with shape λ and type μ. □

3. A sign reversing involution

We begin with Stanley's chromatic symmetric functions in this section.

Definition 3.1. Let $G = G(V, E)$ be a graph with a finite set of vertices V and edges E. A *proper coloring* of G is a function $\kappa : V \to \mathbb{P}$ such that $uv \in E$ implies $\kappa(u) \neq \kappa(v)$. Now consider a countably infinite set of variables $\mathbf{x} = \{x_1, x_2, \ldots\}$. The *chromatic symmetric function* X_G associated with a graph G is a formal power series

$$X_G = X_G(\mathbf{x}) = \sum_{\kappa: V \to \mathbb{P}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}$$

where κ is a proper coloring.

Note that if one sets $x_1 = x_2 = \ldots = x_n = 1$ and $x_i = 0$ for $i > n$, denoted $\mathbf{x} = 1^n$, then X_G reduces to the number of proper colorings of G from a set with n elements. So under this substitution, $X_G(1^n) = \chi_G(n)$ where $\chi_G(n)$ is the chromatic polynomial of Whitney [10]. Also, because permuting the colors of a proper coloring keeps the coloring proper, $X_G(\mathbf{x})$ is a symmetric function in $\mathbf{x}$ over the rationals. In [7], Stanley derived many interesting properties of the chromatic symmetric function $X_G(\mathbf{x})$ some of which generalize those of the chromatic polynomial.

Definition 3.2. Let $(P, \leq)$ be a finite partially ordered set(poset). We say that P is $(\mathbf{a}+\mathbf{b})$-*free* if it contains no induced subposet isomorphic to a disjoint union of an a-element chain and a b-element chain. Also, given any poset P, *incomparability graph* $G(P)$ of P is a graph having vertices $V = P$ and an edge between u and v in $G(P)$ if and only if u and v are incomparable in P.

Figure 3.1 shows a poset P and its incomparability graph $G(P)$.

Figure 3.1

Although $(\mathbf{3}+\mathbf{1})$-free conjecture introduced in Section 1 still remains open, a weak result proved by Gasharov [4]. He gave a combinatorial interpretation to the coefficients in the s-expansion of $X_{G(P)}$ and proved that if P is $(\mathbf{3}+\mathbf{1})$-free then $X_{G(P)}$ is s-positive, where s_λ is the Schur function corresponding to λ.

Definition 3.3. Let P be a poset. A *P-tableau T of shape* λ is a bijection $D_\lambda \to P$ such that for all $(i, j) \in \lambda$:

(i) $T_{i,j} < T_{i+1,j}$, and
(ii) $T_{i,j} \not> T_{i,j+1}$,

where a condition is considered vacuously true if subscripts refer to a cell outside of λ. We denote the number of P-tableaux of shape λ by f_P^λ.

Note that when P is a chain, then a P-tableau is just a standard Young tableau and $f_P^\lambda = f^\lambda$. Figure 3.2 shows all P-tableaux of shape $\lambda = (3,1)$ when P is a poset given in Figure 3.1.

$$\begin{array}{ccc} a & b & d \\ c & & \end{array} \qquad \begin{array}{ccc} b & a & d \\ c & & \end{array} \qquad \begin{array}{ccc} b & d & a \\ c & & \end{array} \qquad \begin{array}{ccc} b & a & c \\ d & & \end{array}$$

Figure 3.2

Using P-tableaux, Gasharov proved the following result which immediately implies s-positivity of $X_{G(P)}$, where P is a $(\mathbf{3}+\mathbf{1})$-free poset.

THEOREM 3.4 (Gasharov [4]). *If P is $(\mathbf{3}+\mathbf{1})$-free then*

$$X_{G(P)} = \sum_\lambda f_P^\lambda s_{\lambda'} \tag{1}$$

where λ' is the conjugate of λ. □

Chow [1] pointed out that (1) could be combined with Eğecioğlu and Remmel's result to obtain a combinatorial interpretation of the coefficients c_μ in Conjecture 1.1. First note that the change of basis matrix between the Schur and elementary symmetric functions is

$$s_{\lambda'} = \sum_\mu K_{\mu,\lambda}^{-1} e_\mu$$

Combining this with (1) we get

$$X_{G(P)} = \sum_{\lambda,\mu} K_{\mu,\lambda}^{-1} f_P^\lambda e_\mu.$$

Since the e_μ are a basis, we have

$$c_\mu = \sum_\lambda K_{\mu,\lambda}^{-1} f_P^\lambda.$$

Finally we apply Theorem 2.8 to get the desired interpretation.

Corollary 3.5 (Chow [1]). *Let P be a finite poset and let*

$$X_{G(P)} = \sum_{\mu} c_\mu e_\mu.$$

Then, the coefficients c_μ satisfy

$$c_\mu = \sum_{(S,T)} \epsilon(S)$$

where the sum is over all pairs of a special rim hook tableau S of type μ and a P-tableau T with the same shape as S. □

Note that a column of a P-tableau T must be a chain in P and the number of rim hooks in S is at most the length of its first column because they are special. So the previous corollary implies that $c_\mu = 0$ whenever μ has more parts than the height of P, $h(P)$ (which is defined as the number of elements in the longest chain of P).

To present pairs (S,T) described in Corollary 3.5 economically, we will combine each pair (S,T) into a single tableau S_T, called *special rim hook P-tableau*, with elements in the same places as in T and edges between pairs of elements which are adjacent in a hook of S. See Figure 3.3 for an example of special rim hook P-tableau.

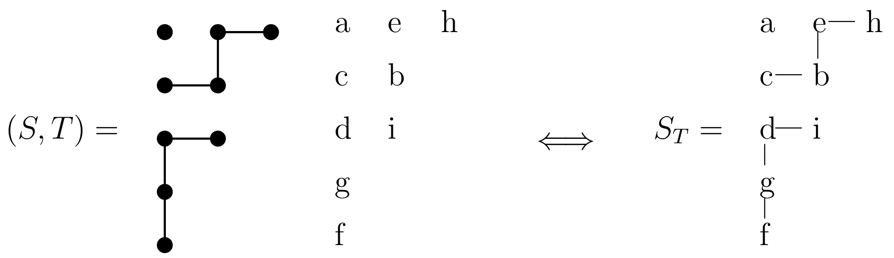


Figure 3.3

Using special rim hook P-tableaux Corollary 3.5 can be rewritten as follows.

Corollary 3.6. *Let P be a finite poset. Then the coefficients c_μ in the e-expansion of $X_G(P)$ are*

$$c_\mu = \sum_{S} \epsilon(S)$$

where the sum is over all special rim hook P-tableaux S of type μ. $\square$

We can now state the main result and give a sign reversing involution to prove it.

THEOREM 3.7. *Let P be a poset with n elements of rank* 1. *Then*

$$\sum_{S} \epsilon(S)$$

is non-negative, where the sum is over all special rim hook P-tableaux S of type $\mu \vdash n$.

Proof. Let μ be a fixed partition of n and Γ_μ be the set of all special rim hook P-tableaux of type μ. We divide the set Γ_μ into two disjoint subsets Γ_μ^+ and Γ_μ^- as follows.

$$\Gamma_\mu^+ = \{\, S \in \Gamma_\mu \mid \epsilon(S) = 1 \,\}$$
$$\Gamma_\mu^- = \{\, S \in \Gamma_\mu \mid \epsilon(S) = -1 \,\}$$

Note that P cannot have a chain of three elements and a column of a P-tableau T in Γ_μ must be a chain in P. This fact implies that the shape of T has at most two rows, and the number of rim hooks in T is at most two because it is special. This means that either $\mu = (n^1)$ or $\mu = (r^1, s^1)$ with $r \geq s$.

Suppose first $\mu = (n^1)$. Since T contains only one rim hook, T is a special rim hook P-tableaux of form

$$a_1 — a_2 — \cdots — a_{n-1} — a_n \qquad \text{or} \qquad \begin{array}{l} b_1 — b_2 — \cdots — b_{n-1} \\ | \\ b_n \end{array}$$

Define

$$I\left(a_1 — a_2 — \cdots — a_{n-1} — a_n\right) = \begin{cases} a_1 — a_2 — \cdots — a_{n-1} — a_n & \text{if } a_1 \not< a_n \\ \begin{array}{l} a_1 — a_2 — \cdots — a_{n-1} \\ | \\ a_n \end{array} & \text{otherwise} \end{cases}$$

and

$$I\begin{pmatrix} b_1 — b_2 — \cdots — b_{n-1} \\ | \\ b_n \end{pmatrix} = b_1 — b_2 — \cdots — b_{n-1} — b_n$$

If $b_{n-1} > b_n$ in the above, $\{b_1, b_n, b_{n-1}\}$ forms a chain of three elements in P. Hence we have $b_{n-1} \not> b_n$ and I is well-defined on Γ_μ.

Suppose now $\mu = (r^1, s^1)$ with $r \geq s$. Since there are two rim hooks in each P-tableau in Γ_μ, such tableau is of form

$$T_1 = \begin{array}{l} a_1 - a_2 - \cdots - a_{s-1} - a_s - a_{s+1} - \cdots - a_r \\ b_1 - b_2 - \cdots - b_{s-1} - b_s \end{array}$$

or

$$T_2 = \begin{array}{lll} c_1 - c_2 - \cdots - c_{s-1} - c_s & d_{s+2} & - \cdots - d_r \\ & \quad | & \\ d_1 - d_2 - \cdots - d_{s-1} - d_s - & d_{s+1} & \end{array}$$

Define

$$I(T_1) = \begin{cases} T_1 & \text{if } a_{s+1} \not< a_r \\ T_3 & \text{otherwise} \end{cases}$$

and

$$I(T_2) = T_4$$

where

$$T_3 = \begin{array}{lll} a_1 - a_2 - \cdots - a_{s-1} - a_s & a_{s+1} & - \cdots - a_{r-1} \\ & \quad | & \\ b_1 - b_2 - \cdots - b_{s-1} - b_s - & a_r & \end{array}$$

and

$$T_4 = \begin{array}{l} c_1 - c_2 - \cdots - c_s - d_{s+2} - \cdots - d_r - d_{s+1} \\ d_1 - d_2 - \cdots - d_s \end{array}$$

If $d_r > d_{s+1}$ or $b_s > a_r > a_{s+1}$ in the above, P contains a chain of three elements $\{d_{s+2}, d_{s+1}, d_r\}$ or $\{a_{s+1}, a_r, b_s\}$. Thus we have $d_r \not> d_{s+1}$, $b_s \not> a_r$ and I is well-defined.

In either case, we can check easily that I is a sign reversing involution on Γ_μ, i.e., $I \circ I = 1_{\Gamma_\mu}$ and

$$\epsilon(I(S)) = \begin{cases} 1 & \text{if } S \in \Gamma_\mu^-, \\ -1 & \text{if } S \in \Gamma_\mu^+ \text{ and } I(S) \neq S \\ 1 & \text{if } S \in \Gamma_\mu^+ \text{ and } I(S) = S. \end{cases}$$

Class	shape	type	sign	# of special rim hook P-tableaux
I	(5)	(5)	1	42
II	$(4,1)$	(5)	-1	12
III	$(4,1)$	$(4,1)$	1	12
IV	$(3,2)$	$(3,2)$	1	6
V	$(3,2)$	$(4,1)$	-1	6

TABLE 1

Using the above involution I, we finally have

$$\sum_{S} \epsilon(S) = \sum_{S \in \Gamma_\mu} \epsilon(S) = \sum_{S \in \Gamma_\mu^+,\, I(S)=S} \epsilon(S) \geq 0$$

which immediately implies our theorem. □

Combining Corollary 3.6 and Theorem 3.7, we get the following facts.

COROLLARY 3.8 (Stanley[7]). *Let P be a finite poset of rank 1. For any partition μ, the coefficient c_μ of e_μ in the e-expansion of $X_{G(P)}$ is non-negative.* □

EXAMPLE 3.9. Consider the poset P as in Figure 3.4.

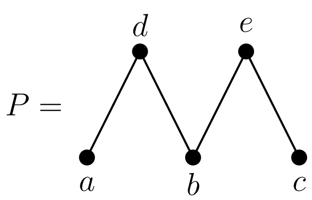


Figure 3.4

Then there are 78 special rim hook P-tableaux. Table 1 shows all possible shapes and types of special rim hook P-tableaux, and the number of special rim hook P-tableaux with given shape and type. Examples of a special rim hook P-tableaux contained in each class of Table 1 are given in Figure 3.5.

Class I: $a - b - c - d - e$

Class II: $\begin{array}{ccccccc} a & - & b & - & c & - & e \\ | & & & & & & \\ d & & & & & & \end{array}$

Class III: $\begin{array}{ccccccc} a & - & b & - & c & - & e \\ & & & & & & \\ d & & & & & & \end{array}$

Class IV: $\begin{array}{ccccc} a & - & b & - & c \\ & & & & \\ d & - & e & & \end{array}$

Class V: $\begin{array}{ccccc} a & & b & - & c \\ & & | & & \\ d & - & e & & \end{array}$

Figure 3.5

Each special rim hook P-tableaux in Class II is matched to just one of tableaux in Class I, and each special rim hook P-tableaux in Class V is matched to one in Class III as follows;

$$\begin{array}{ccccccc} a & - & b & - & c & - & e \\ | & & & & & & \\ d & & & & & & \end{array} \quad \Longleftrightarrow \quad a - b - c - e - d$$

$$\begin{array}{ccccc} a & & b & - & c \\ & & | & & \\ d & - & e & & \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccccccc} a & - & b & - & c & - & e \\ & & & & & & \\ d & & & & & & \end{array}$$

Figure 3.6

30 unmatched tableaux in Class I, 6 unmatched tableaux in Class III and 6 unmatched tableaux in Class IV are fixed by the involution I described in Theorem 3.7. Hence, we have

$$X_{G(P)} = 30e_{(5)} + 6e_{(4,1)} + 6e_{(3,2)}.$$

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