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PERIODIC SOLUTIONS FOR NONLINEAR PARABOLIC SYSTEMS WITH SOURCE TERMS

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Abstract. We have a concern with the existence of solutions (ξ, η) for perturbations of the parabolic system with Dirichlet boundary condition

$$
\begin{aligned}
\xi_t &= -L\xi + \mu g(3\xi+\eta) - s\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\
\eta_t &= -L\eta + \nu g(3\xi+\eta) - s\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0, 2\pi).
\end{aligned} \tag{0.1}
$$

We prove the uniqueness theorem when the nonlinearity does not cross eigenvalues. We also investigate multiple solutions $(\xi(x,t), \eta(x,t))$ for perturbations of the parabolic system with Dirichlet boundary condition when the nonlinearity f' is bounded and $f'(-\infty) < \lambda_1, \lambda_n < (3\mu+\nu)f'(+\infty) < \lambda_{n+1}$.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^{\mathbf{n}}$ with smooth boundary $\partial\Omega$ and let L denote the differential operator

$$
L = \sum_{1 \le i,j \le n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}),
$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$. In [2, 4, 5, 7, 8] the authors investigate multiplicity of solutions of the nonlinear elliptic equation with Dirichlet boundary condition

$$
\begin{aligned}
&Lu + g(u) = f(x) \quad \text{in} \quad \Omega, \\
&u = 0 \quad \text{on} \quad \partial\Omega,
\end{aligned} \tag{1.1}
$$

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where the semilinear term $g(u) = bu^+ - au^-$ and L is a second order linear elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated according to its multiplicity,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 \leq \cdots \leq \lambda_i \leq \cdots \to \infty.$$

Here the source term f is generated by the eigenfunctions of the second order elliptic operator with Dirichlet boundary condition.

Equation (1.1) and the following type nonlinear equation with Dirichlet boundary condition was studied by many authors:

$$\begin{aligned} Lu &= bu^+ - au^- + f \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.2}$$

In [9] Lazer and McKenna point out that this kind of nonlinearity $bu^+ - au^-$ can furnish a model to study traveling waves in suspension bridges. So the nonlinear equation with jumping nonlinearity have been extensively studied by many authors. For fourth elliptic equation Tarantello [15] , Micheletti and Pistoia [12][13] proved the existence of nontrivial solutions used degree theory and critical points theory separately. For one-dimensional case Lazer and McKenna [10] proved the existence of nontrivial solution by the global bifurcation method. For this jumping nonlinearity we are interest in the multiple nontrivial solutions of the equation. Here we used variational reduction method to find the nontrivial solutions of problem (1.2).

In [6, 11] the authors investigate multiplicity of solutions of the nonlinear parabolic equation with Dirichlet boundary condition

$$\begin{aligned} u_t &= -Lu + f(u) - s\phi_1 - h(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi). \end{aligned} \tag{1.3}$$

In this paper we investigate the existence of solutions $\xi(x,t), \eta(x,t)$ for perturbations of the parabolic system with Dirichlet boundary condition

$$\begin{aligned} \xi_t &= -L\xi + \mu g(3\xi + \eta) - s_1\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ \eta_t &= -L\eta + \nu g(3\xi + \eta) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ \xi &= 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi), \end{aligned} \tag{1.4}$$

where we assume that $h \in H^*$ and f' is bounded, $f'(-\infty) < \lambda_1, \lambda_n < (3\mu + \nu)f'(+\infty) < \lambda_{n+1}$. We also assume that $s_1, s_2 > 0$.

The organization of this paper is as following. In section 2, we have a concern with the parabolic equation with Dirichlet boundary condition when the nonlinearity crosses eigenvalues. We investigate the multiplicity of solutions for the single nonlinear parabolic equation. In section 3, we investigate the uniqueness when the nonlinearity does not cross eigenvalues. We also investigate multiple solutions $(\xi(x,t), \eta(x,t))$ for perturbations of the parabolic system with Dirichlet boundary condition when the nonlinearity f' is bounded and $f'(-\infty) < \lambda_1, \lambda_n < (3\mu + \nu)f'(+\infty) < \lambda_{n+1}$.

2. Appendix: Single parabolic equations with source terms

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$ and let L denote the differential operator

$$L = \sum_{1\le i,j\le n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}),$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$. In this section we look for weak solutions of the parabolic equation with Dirichlet boundary condition

$$\begin{aligned} u_t &= -Lu + f(u) - s\phi_1 - h(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi). \end{aligned} \tag{2.1}$$

We assume that the eigenfunctions ϕ_i of L are an orthonormal basis for $L^2(\Omega)$ with eigenfunctions $-\lambda_i, \lambda_1 > 0, \lambda_i \to +\infty$, and that $\phi_1(x) > 0, x \in \Omega$. These are the assumptions of this section. For the more results for the parabolic equation we refer to [6, 11].

We shall work with the complex Hilbert space $H_T^* = L^2(\Omega \times (0,T))$, equipped with the usual inner product

$$\langle v, \omega\rangle^* = \int_0^{2\pi}\int_\Omega v(x,t)\overline{\omega}(x,t)dxdt$$

and norm $\|v\| = \langle v, v\rangle^{\frac{1}{2}}$. Later we shall switch to the real subspace H_T. The functions $\phi_{mn} = \frac{\phi_n(x)e^{imt}}{\sqrt{2\pi}}, n \ge 1, m = 0, \pm1, \pm2, \dots$ are a complete orthornormal basis for H^*. Let Σ^* denote sums over the indices m, n. Every $v \in H^*$ has a Fourier expansion

$$v = \Sigma^* v_{mn}\phi_{mn},$$

with $\Sigma|v_{mn}|^2 = \|v\|^2, v_{mn} = \langle v, \phi^*_{mn}\rangle$. A weak solution to the boundary value problem (2.1) is, by definition, a function $u \in H$ satisfying $Lu \in H$, i.e. $\Sigma^*|u_{mn}|^2(m^2 + \lambda_n^2) < \infty$ satisfying (2.1) in H.

For real $\alpha \neq \lambda_n$, the operator $R = (L + \alpha - D_t)^{-1}$ denoted by

$$u = Rh \leftrightarrow u_{mn} = \frac{h_{mn}}{-\lambda_n + \alpha + im}$$

is a compact linear operator on H^* and the operator norm of R, $\|R\| = \frac{1}{|\alpha-\lambda_n|}$, where λ_n is an eigenvalue of $-L$ closest to α.

From now on, we restrict ourselves to the real subspace H and observe that it is invariant under R.

Our first theorem is a non-self-adjoint problem.

THEOREM 2.1. *Assume that f' is bounded, that $f'(+\infty) = \alpha$ satisfies $\lambda_n < \alpha < \lambda_{n+1}$ and that $h \in H$. Then there exists $s_0 > 0, \epsilon > 0$ such that the Leray-Schauder degree*

$$\deg(u - (-L + D_t)^{-1}(f(u) - s\phi_1 - h), B^*_{s\epsilon}(s\theta), 0) = (-1)^n \tag{2.2}$$

for $s \geq s_0$. Here B^*_r denotes a ball of radius r in H and

$$\theta = -(-L - \alpha + D_t)^{-1}\phi_1 = \frac{\phi_1}{\alpha - \lambda_1}.$$

Proof. The first part of the proof, where we show there are no solutions on the boundary of the ball. We shall just indicate the changes, so we can be sure the degree is defined.

Let R be the operator $(-L - \alpha + D_t)^{-1}$. Let $A = (D_t - L)^{-1}$, and let $g(u) = \alpha u - f(u)$. Then the periodic problem (2.1) is equivalent to

$$u = s\theta + Rh + Rg(u) \equiv Su. \tag{2.3}$$

Let B^* be the open unit ball in H, let $K = \overline{R(B^*)}$. It follows that any solution $u \in s\theta + s \in \overline{B^*}$, of (2.3) belongs to $s\theta + \frac{3}{4}s\epsilon\overline{B^*}$ and this holds when $h + g(u)$ is replaced by $\lambda(h + g(u)), 0 \leq \lambda \leq 1$. Solutions of the corresponding equation (2.3) are solution of

$$u = A(-s\phi_1 + \alpha u\lambda(h + g(u)))$$

and it follows that if $G = B^*_{s\epsilon}(s\theta_1)$,

$$\deg(u - A(-s\phi_1 + \alpha_u - (h + g(u))), G, 0) = \deg(u - A(\alpha u - s\phi_1), G, 0).$$

Now by substituting $v = u - s\theta$, and using $u - A(\alpha u - s\phi_1) = u - s\theta + \alpha(Au - s\theta)$, we observe that

$$\deg(u - A(\alpha u - s\phi_1), G, 0) = \deg(v - \alpha Av, s\epsilon B^*, 0).$$

Thus, to prove the theorem, we have to show that this degree is $(-1)^n$. To do this, we calculate the degree on finite dimensional subspaces which we now choose. The functions

$$\phi_{on} = \frac{1}{\sqrt{2\pi}}\phi_n(x)$$

$$\phi^c_{mn} = \frac{1}{\sqrt{\pi}}\phi_n(x)\cos mt \quad m = 1, 2, 3\dots$$

$$\phi^s_{mn} = \frac{1}{\sqrt{\pi}}\phi_n(x)\sin mt$$

form a real orthonormal basis for H. If $h \in H$, then $h = \Sigma h_{mn}\phi_{mn}$ in H^* and h can be expanded in terms of $\phi_{on}, h^c_{mn}, h^s_{mn}$, with the identities

$$\|A - PA\|^2 = \Sigma \frac{1}{\lambda_n^2 + m^2}(|h_{mn}|^2 + |h_{-m,n}|^2).$$

It follows that

$$\|A - PA\|^2 \leq \min_{m,n>b} \frac{1}{\lambda_n^2 + m^2} \leq \max\left(\frac{1}{p+1}, \frac{1}{\lambda_{p+1}}\right)$$

and by the definition of degree

$$\deg(v - \alpha PAv, s\epsilon B^*, 0) = \deg(v - \alpha Av, s\epsilon B^*, 0)$$

for large p, since the operator PA is of finite rank, with its range contained in PH.

Taking the functions $\phi_{on}, \phi^c_{mn}, \phi^s_{mn}, 1 \leq m, n \leq p$, as a basis H_p, the equation $v + \alpha PAv$ becomes a matrix equation on the space H_p, of the form

$$(I_q + \alpha C)x = 0 \quad \text{for} \quad \mathrm{x} \in \mathrm{R}^\mathrm{q}, \quad \mathrm{q} = \mathrm{p}(2\mathrm{p}+1)$$

where I_q is the identity matrix of rank q, C is a $q \times q$ block diagonal matrix $C = diag(C_1, \cdots, C_p)$ and each C_n is a $2p+1$ by $2p+1$ block diagonal matrix given by

$$C_n = diag\ \left(-\frac{1}{\lambda_n}, A_{1n}, \cdots, A_{pn}\right)$$

Now let $D = I_q + \alpha C = diag(D_1, \cdots, D_n)$, where

$$D_n = diag\left(1 - \frac{\alpha}{\lambda_n}, I_2 - \alpha A_{1n}, \cdots, I_2 - \alpha A_{pn}\right).$$

Since $\det D_n = \left(1 - \frac{\alpha}{\lambda_n}\right) a_{1n}, \cdots, a_{pn}$ where $\det(I_2 - \alpha A_{mn} = a_{mn} = a_{mn} > 0$, we finally get for large p that

$$sign \det D = sign \left(1 - \frac{\alpha}{\lambda_1}\right) \cdots \left(1 - \frac{\alpha}{\lambda_p}\right) = (-1)^n.$$

Recalling that $\lambda_n < \alpha < \lambda_{n+1}$. Since $sign \det D$ is equal to $\deg(v + \alpha Pv, s\epsilon B^*, 0)$ for large p, the theorem is proved by letting $p \to +\infty$. $\square$

PROPOSITION 1. *If f' is bounded, and $\overline{\alpha} = f'(-\infty) < \lambda_1$, then there exist positive constants s_0, ϵ such that*

$$\deg(u - (D_t - L)^{-1}(f_(u) - s\phi_1 - h), B^*_{s\epsilon}(s\overline{\theta}), 0) = 1$$

for $s \geq s_0$, where $\overline{\theta} = \frac{\phi_1}{\overline{\alpha} - \lambda_1} < 0$.

LEMMA 2.1. *Assume that $|f(u)| \leq a + c|u|, f'(-\infty), f'(+\infty)$ exist, that $f(u) - \lambda_1 u \geq \epsilon|u| - b$, and that $h \in H$ satisfies $\|h\| \leq r$, where a, b, c, r, ϵ are positive constants. Then there exists C depending only on a, b, c, r, ϵ such that*

$$D_t u = Lu + f(u) - s\phi_1 - h$$
$$u(x, t + 2\pi) = u(x, t)$$

satisfies $\|u\| \leq c$.

Proof. Suppose not. Then there exist u_n with $\|u_n\| \to \infty$ which satisfy the equation. Now let $v_n = \frac{u_n}{\|u_n\|}$, and v_n satisfies

$$D_t v_n = Lv_n + \frac{1}{\|u_n\|} f(\|u_n\| v_n) - h_n(x, t).$$

Since $f_n(u) - \lambda_1 u \geq \epsilon|u| - b$, we can conclude, by multiplying across by ϕ_1 and itegrating, that

$$\langle D_t u_n - Lu_n - \lambda_1 u_n, \phi_1 \rangle = \langle f(u_n) - \lambda_1 u_n, \phi_1 \rangle - \langle (h_n, \phi_1) \rangle$$

and thus

$$0 \geq \int (\epsilon|u_n| - b)\phi_1 - \|h_n\| \geq \epsilon \int u\phi_1 - b \int \phi - r,$$

from which we conclude that if $u_n = c_n \phi - 1 + x_n$, then the c_n's are bounded. Now,

$$v_n = (D_t - L)^{-1} \left(\frac{1}{\|u_n\|} F(\|u_n\| v_n) - \frac{h_n}{\|u_n\|} \right)$$

and one can check that the v_n's are precompact in H since, by virtue of $|f(u)| \leq a + c|u|$, we have that $\frac{1}{\|u_n\|}(f(\|u_n\|v_n) - h_n)$ is bounded and $(D_t - L)^{-1}$ is a compact operator. Therefore, there exists a convergent subsequence, still called v_n, converging to v. Since $v_n = \frac{1}{\|u_n\|}(c_n\phi_1 + x_n)$ and the c_n's are bounded, it follows that $v \perp \phi_1$. Since $f(s) = f'(+\infty)s^+ - f'(-\infty)s^- + f_1(s)$ where $\frac{f_1(s)}{s} \to 0$ as $s \to +\infty$, we have that

$$\frac{1}{\|u_n\|}(f(\|u_n\|v) - h_n) \to f'(+\infty)v^+ - f'(-\infty)v^-$$

and

$$(D_t - L)v = f'(+\infty)v^+ - f'(-\infty)v^-$$

or

$$(D_t - L - \lambda_1)v = (f'(+\infty) - \lambda_1)v^+ - (f(-\infty) - \lambda_1)v^-.$$

Since $(f'(+\infty) - \lambda_1)v^+ - (f'(-\infty) - \lambda_1)v_1 \geq \epsilon|v|$ after multiplying across by ϕ_1 and integrating by parts, we obtain a contradiction. □

Lemma 2.2. *Let $s_1 \in R$ under the assumptions of the preceding lemma, there exists $C_1 > 0$, depending on s_1 and the constants of Lemma 1, such that*

$$\deg(u - (D_t - Lu)^{-1}(f(u) - (h + s\phi_1)), B^*_\beta(0), 0) = 0$$

for $s \leq s_1$ and $\beta > c_1$.

The proof of Lemma 2.2 is the same as those for the self-adjoint case, as done in Chapter I. T here are on solutions on the boundary of the ball for $s \leq s_1$, by the previous lemma. Therefore, by homotopy, the degree is the same for all $s \leq s_1$, and since it must be zero for large negative s, it must be zero for all $s \leq s_1$.

We have now assembled all the ingredients for our first existence theorem.

Theorem 2.2. *Let $h \in H^*$. Assume f' is bounded, $f'(-\infty) < \lambda_1, \lambda_n < f'(+\infty) < \lambda_{n+1}$. Then there exists s_0 so that if $s \geq s_0$, equation (2.1) has at least two 2π-periodic solutions if nis even, and at least three if n is odd.*

The proof is by now obvious. The degree on a large ball is zero. By Theorem 2.1, we can find a ball near $\overline{\theta}$, on which the degree of the map

$$u - (D_t - L)^{-1}(f(u) - (s\phi_1 + h(x)))$$

is 1, and a ball on which the degree is zero, we have two solutions if n is odd, and three if n is even. This concludes the proof.

3. Periodic solutions of the parabolic system

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$ and let L denote the differential operator

$$L = \sum_{1\le i,j\le n} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}),$$

where $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$. In this section we investigate the existence of solutions $(\xi(x,t), \eta(x,t))$ for perturbations of the parabolic system with Dirichlet boundary condition

$$\begin{aligned} &\xi_t = -L\xi + \mu g(3\xi+\eta) - s\phi_1 - h_1(x,t) \quad \text{in } \Omega\times(0,2\pi),\\ &\eta_t = -L\eta + \nu g(3\xi+\eta) - s\phi_1 - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi),\\ &\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi), \end{aligned} \tag{3.1}$$

where we assume that $h \in H^*$ and g' is bounded, $g'(-\infty) < \lambda_1, \lambda_n < (3\mu+\nu)g'(+\infty) < \lambda_{n+1}$. We also assume that $s > 0$.

THEOREM 3.1. *Let μ, ν be nonzero constants and $\frac{1}{3} + \frac{\mu}{\nu} \neq 0$. Assume that $(3\mu+\nu)A < \lambda_1$ and $h \in H^*$. Then the system the parabolic system with Dirichlet boundary condition*

$$\begin{aligned} &\xi_t = -L\xi + \mu A(3\xi+\eta)^+ - s_1\phi_1 - h_1(x,t) \quad \text{in } \Omega\times(0,2\pi),\\ &\eta_t = -L\eta + \nu A(3\xi+\eta)^+ - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi),\\ &\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi), \end{aligned} \tag{3.2}$$

has a unique solution (ξ, η).

Proof. From problem (3.2) we get that

$$(\xi - \frac{\mu}{\nu}\eta)_t = -L(\xi - \frac{\mu}{\nu}\eta) - (s_1 - \frac{\mu}{\nu}s_2)\phi_1 + (h_1 - \frac{\mu}{\nu}h_2).$$

By the contraction mapping principle, for any $F \in H_0$ the problem

$$\begin{aligned} &u_t + Lu = F \quad \text{in} \quad \Omega\times(0,2\pi),\\ &u = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi) \end{aligned} \tag{3.3}$$

has a unique solution. If $u_{1-\frac{\mu}{\nu}}$ is a solution of $L(\xi - \frac{\mu}{\nu}\eta) = (1 - \frac{\mu}{\nu})f$, then the solution (ξ, η) of problem (3.2) satisfies

$$\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}}. \tag{A}$$

On the other hand, from problem (3.2) we get the equation

$$\begin{aligned} &(3\xi + \eta)_t = -L(3\xi + \eta) + (3\mu + \nu)A(3\xi + \eta)^+ \\ &- (3s_1 + s_2)\phi_1 - 3h_1(x,t) - h_2(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ &\xi = 0, \quad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi). \end{aligned} \tag{3.4}$$

Put $w = 3\xi + \eta$. Then the above equation is equivalent to

$$\begin{aligned} &Lw + (\mu + 2\nu)g(\xi + 2\eta) = 3f \quad \text{in} \quad \Omega, \\ &w = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.5}$$

When $(3\mu+\nu)A < \lambda_1$, by the contraction mapping principle, the above equation has a unique solution, say w_1. Hence we get the solutions (ξ, η) of problem (3.2) from the following systems:

$$\begin{aligned} &\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}}, \\ &3\xi + \eta = w_1. \end{aligned} \tag{3.6}$$

Since $\frac{1}{3} + \frac{\mu}{\nu} \neq 0$, system (3.6) has a unique solution (ξ, η). □

THEOREM 3.2. *Let μ, ν be nonzero constants and $\frac{1}{3} + \frac{\mu}{\nu} \neq 0$. Assume that f' is bounded, $g'(-\infty) < \lambda_1, \lambda_n < g'(+\infty) < \lambda_{n+1}$. Then there exists s_0 so that if $s \geq s_0$, equation (3.1) has at least two 2π-periodic solutions if n is even, and at least three if n is odd.*

Proof. From problem (3.2) we get that

$$(\xi - \frac{\mu}{\nu}\eta)_t = -L(\xi - \frac{\mu}{\nu}\eta) - (s_1 - \frac{\mu}{\nu}s_2)\phi_1 + (h_1 - \frac{\mu}{\nu}h_2).$$

By the contraction mapping principle, for any $F \in H_0$ the problem

$$\begin{aligned} &u_t + Lu = F \quad \text{in} \quad \Omega \times (0, 2\pi), \\ &u = 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi) \end{aligned} \tag{3.7}$$

has a unique solution. If $u_{1-\frac{\mu}{\nu}}$ is a solution of $L(\xi - \frac{\mu}{\nu}\eta) = (1 - \frac{\mu}{\nu})f$, then the solution (ξ, η) of problem (3.2) satisfies

$$\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}}. \tag{A}$$

On the other hand, from problem (3.2) we get the equation

$$
\begin{aligned}
&(3\xi+\eta)_t = -L(3\xi+\eta) + (3\mu+\nu)A(3\xi+\eta)^+ \\
&-(3s_1+s_2)\phi_1 - 3h_1(x,t) - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\xi = 0, \quad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi).
\end{aligned} \tag{3.8}
$$

Put $w = 3\xi+\eta$. Then the above equation is equivalent to

$$
\begin{aligned}
&Lw + (\mu+2\nu)g(w) = 3f \quad \text{in} \quad \Omega, \\
&w = 0 \quad \text{on} \quad \partial\Omega.
\end{aligned} \tag{3.9}
$$

When f' is bounded, $f'(-\infty) < \lambda_1, \lambda_n < f'(+\infty) < \lambda_{n+1}$. By Theorem 2.1 there exists s_0 so that if $s \geq s_0$, equation (3.1) has at least two 2π-periodic solutions(say, w_{e1}, w_{e2}) if n is even, and at least three solutions(say, w_{o1}, w_{o2}, w_{o3})if n is odd.

When n is even, we get the solutions (ξ, η) of problem (3.1) from the following systems:

$$
\begin{aligned}
&\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\
&\xi + 2\eta = w_{e1}
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\
&\xi + 2\eta = w_{e2}
\end{aligned} \tag{3.11}
$$

Since $\frac{1}{3} + \frac{\mu}{\nu} \neq 0$, system (3.9) has a unique solution (ξ_1, η_1). From (3.10) we get the unique solution (ξ_2, η_2). Therefore system (3.1) has at least two solutions if n is even.

When n is odd, we get the solutions (ξ, η) of problem (3.1) from the following systems:

$$
\begin{aligned}
&\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\
&\xi + 2\eta = w_{o1}
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
&\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\
&\xi + 2\eta = w_{o2}
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
&\xi - \frac{\mu}{\nu}\eta = u_{1-\frac{\mu}{\nu}} \\
&\xi + 2\eta = w_{o3}
\end{aligned} \tag{3.14}
$$

Since $\frac{1}{3} + \frac{\mu}{\nu} \neq 0$, system (3.12) has a unique solution (ξ_{o1}, η_{o1}). From (3.13) we get the unique solution (ξ_{o2}, η_{o2}). From (3.14) we get the unique solution (ξ_{o3}, η_{o3}). Therefore system (3.1) has at least three solutions if n is odd.

$\square$

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