A BERBERIAN TYPE EXTENSION OF FUGLEDE-PUTNAM THEOREM FOR QUASI-CLASS A OPERATORS

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Abstract. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on a separable infinite dimensional complex Hilbert space $\mathcal{H}$. We say that $T \in \mathcal{L}(\mathcal{H})$ is a quasi-class $A$ operator if

$$T^*|T|^2T \geq T^*|T|^2T.$$

In this paper we prove that if $A$ and $B$ are quasi-class $A$ operators, and $B^*$ is invertible, then for a Hilbert-Schmidt operator $X$

$$AX = XB \implies A^*X = XB^*.$$

1. Introduction

Recall ([1], [5]) that $T \in \mathcal{L}(\mathcal{H})$ is called $p$-hyponormal if for $p \in (0, 1]$

$$(T^*T)^p \geq (TT^*)^p,$$

and $T$ is called paranormal if for all unit vector $x \in \mathcal{H}$

$$||T^2x|| \geq ||Tx||^2.$$

Following [5] and [6] we say that $T \in \mathcal{L}(\mathcal{H})$ belongs to class $A$ if

$$|T^2| \geq |T|^2.$$

Recall ([9]) that $T$ is called $p$-quasihyponormal if for $p \in (0, 1]$

$$T^*(T^*T)^pT \geq T^*(TT^*)^pT.$$

Received November 25, 2008. Revised December 2, 2008.
2000 Mathematics Subject Classification: 47B20.
Key words and phrases: quasi-class $A$ operator, Hilbert-Schmidt class, Fuglede-Putnam theorem.
This work was supported by the University of Incheon Research Grant in 2008.

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For brevity, we shall denote classes of $p$-hyponormal operators, $p$-quasihyponormal operators, paranormal operators, and class $A$ operators by $\mathcal{H}(p)$, $\mathcal{QH}(p)$, $\mathcal{PN}$, and $A$, respectively. It is well known that

\begin{equation}
\mathcal{H}(p) \subset A \subset \mathcal{PN} \text{ and } \mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{PN}.
\end{equation}

Recently, Jeon and Kim ([8]) considered an extension of class $A$ operators and $p$-quasihyponormal operators.

**Definition 1.1.** We say that $T \in \mathcal{L}(\mathcal{H})$ is quasi-class $A$ if

\[ T^*|T|^2 T \geq T^*|T|^2 T. \]

For brevity, we shall denote the set of quasi-class $A$ operators by $\mathcal{QA}$. As shown in [8], the class of quasi-class $A$ operators properly contains classes of class $A$ operators and $p$-quasihyponormal operators, i.e., the following inclusion holds;

\begin{equation}
\mathcal{H}(p) \subset \mathcal{QH}(p) \subset \mathcal{QA} \text{ and } \mathcal{H}(p) \subset A \subset \mathcal{QA}
\end{equation}

In view of inclusions (1.1) and (1.2), it seems reasonable to expect that operators in class $\mathcal{QA}$ are paranormal. But there exists an example which is not paranormal but quasi-class $A([8])$.

A familiar Fuglede-Putnam theorem is as follows.

**Proposition 1.2.** Let $A$, $B$, and $X$ be in $\mathcal{L}(\mathcal{H})$. If $A$ and $B$ are normal, then

\[ AX = XB \text{ implies } A^*X = XB^*. \]

In [2] S. K. Berberian relaxes the hypotheses on $A$ and $B$ in the above theorem at the cost of requiring $X$ to be of Hilbert-Schmidt class (denoted $X \in \mathcal{C}_2$, for definitions and details see [10]) as follows.

**Proposition 1.3.** Let $A$, $B \in \mathcal{L}(\mathcal{H})$ and $X \in \mathcal{C}_2$. Then

\[ AX = XB \text{ implies } A^*X = XB^* \]

under either of the following hypotheses:

(i) $A$ and $B^*$ are hyponormal,

(ii) $B$ is invertible and $|A| \cdot ||B^{-1}|| \leq 1$
In [4, Theorem 2] T. Furuta relaxed the hyponormality on $A$ and $B^*$ to $k$-quasihyponormality (to be defined below).

Recall [3] that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be $k$-quasihyponormal if $T^{*k}(T^*T - TT^*)T^k \geq 0$ for some non-negative integer $k$. It is well known that, for $k \geq 2$, the class of $k$-quasihyponormal operators has no inclusion relations with classes of the former mentioned operators.

In this paper, we prove an analogue result of T. Furuta as follows.

**Theorem 1.4.** Let $A \in \mathcal{QA}$ and $B^* \in \mathcal{QA}$ be invertible. Then for $X \in \mathcal{C}_2$

$$AX = XB \text{ implies } A^*X = XB^*.$$  

The following result immediately follows.

**Corollary 1.5.** Let $A \in \mathcal{A}$ (resp. $A \in \mathcal{QH}(p)$) and $B^* \in \mathcal{A}$ (resp. $B^* \in \mathcal{QH}(p)$) be invertible. Then for $X \in \mathcal{C}_2$

$$AX = XB \text{ implies } A^*X = XB^*.$$  

2. Proofs

In this section we give a proof of Theorem 1.4, modifying T. Furuta’s arguments in the proof of [4, Theorem 2]. We need some lemmas. Recall from [8] that

**Lemma 2.1.** Let $T \in \mathcal{QA}$ and $T$ not have a dense range. Then $T$ has the following matrix representation:

$$T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \text{ on } \overline{\text{ran}(T)} \oplus \ker(T^*),$$

where $A \in \mathcal{A}$. Furthermore, $\sigma(T) = \sigma(A) \cup \{0\}$.

From the above lemma we immediately have

**Corollary 2.2.** If $T \in \mathcal{QA}$ is invertible, then $T \in \mathcal{A}$.  

In [2] an operator $T$ on $C_2$ is defined by, for $A, B \in L(H)$,
$$TX = AXB.$$ 
Then, as in [2], simple calculations show that $T^*X = A^*XB^*$ and also 

\[(2.1) \quad A, B \geq 0 \text{ implies } T \geq 0.\]

**Lemma 2.3.** If $A, B^* \in QA$, then the operator $T$ belongs to $QA$.

**Proof.** From the hypotheses of $A$ and $B^*$, and (2.1) we have
\[
(T^*|T^2|T - T^*|T^2)X = (A^*|A^2|A - A^*|A^2A)XB|B^*|B^* + A^*|A^2A)(B|B^*|B^* - B|B^*|B^*) \geq 0,
\]
which shows that $T$ is a quasi-class $A$ operator on $C_2$. \qed

**Proof of Theorem 1.4.** If $S \in L(H)$ is invertible, let $T$ on $C_2$ be defined by
$$TY = AYS^{-1}.$$ 
Since $B^*$ is invertible quasi-class $A$, $B^*$ is just invertible class $A$ by Corollary 2.2, and $(B^*)^{-1} = (B^{-1})^*$ is also class $A$ by [11]. So it follows that from Lemma 2.3 that $T \in QA$. The hypotheses $AX = XB$ implies $TX = X$ and from the fact $T \in QA$ it follows (use the Hölder-McCarthy inequality[5])
\[
||T^*X||^2 = \langle T^*X, T^*X \rangle \\
= \langle T^*T^2T^*X, T^*T^2T^*X \rangle \\
\leq \langle T^*T^2T^*X, T^*T^2T^*X \rangle \\
= \langle (T^*T^2T^*)^{\frac{1}{2}}X, X \rangle \\
\leq \langle (T^*T^2T^*)X, X \rangle^{\frac{1}{2}} \cdot ||X|| \\
= ||X||^2,
\]
which implies
\[
||T^*X| - X||^2 \leq ||T^*X||^2 - 2||X||^2 + ||X||^2 \leq 0.
\]
Hence we have $T^*X = X$, i.e., $A^*X = XB^*$. \qed
A Berberian type extension of Fuglede-Putnam theorem

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