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NOTE ON TOTALLY DISCONNECTED AND CONNECTED SPACES

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Abstract. Every totally disconnected space is hereditarily disconnected. In this note, we provide an example of a hereditarily disconnected which is not a totally disconnected space. We further provide an example that not homogeneous space is the product of a totally disconnected and a connected.

1. Introduction

A space X is called *zero-dimensional* if it is nonempty and has a base consisting of clopen(both open and closed) sets, i.e., if for every point $x \in X$ and for every neighborhood N of x there exists a clopen subset $A \subseteq X$ such that $x \in A \subseteq N$. It is easy to see that a zero-dimensional space can be embedded in the real line $\mathbb{R}$, and that a nonempty subspace X of $\mathbb{R}$ is zero-dimensional if and only if it does not contain any non-degenerate interval. For example, the rational numbers $\mathbb{Q}$ is the only zero-dimensional countable space without isolated points, and the irrational numbers $\mathbb{Q}^c$ is the only zero-dimensional topologically complete space which is nowhere compact. The Cantor set is clearly closed in unit interval $[0, 1]$, hence is compact. It also has no isolated points and is zero-dimensional because it does not contain any nontrivial interval. That is, the Cantor set is the only zero-dimensional compact space without isolated points.

A subset A of a space X is called a $C - set$ in X if A can be written as an intersection of clopen subsets of X. It is well known that a space is zero-dimensional if and only if every cosed subset is a C-set. A space is called *almost zero-dimensional* if every point has a neighborhood basis consisting of C-sets. Note that almost zero-dimensionality is hereditary

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and productive. In [2] Oversteegen and Tymcnatyn proved that every almost zero-dimensional space is at most one-dimensional. In this note, all topological spaces are assumed to be separable and metrizable.

2. Main Results

Let $p \in (0, \infty)$ and consider the Banach space ℓ^p. This space consists of all sequences $x = (x_0, x_1, x_2, \cdots) \in \mathbb{R}^\infty$ such that $\sum_{n=0}^{\infty} |x_n|^p < \infty$. The topology on ℓ^p is generated by the norm $\|x\| = (\sum_{n=0}^{\infty} |x_n|^p)^{\frac{1}{p}}$. Recall that the norm topology on ℓ^p is generated by the product topology together with the sets $\{x \in \ell^p : \|x\| < k, k > 0\}$. The Erdös space $\mathcal{E}$ is the set of vectors in ℓ^2 the coordinates of which are all rational numbers, i.e.,

$$\mathcal{E} = \{(x_0, x_1, x_2, \cdots) \in \ell^2 : x_i \in \mathbb{Q}\}.$$

In [4], Erdös showed that $\mathcal{E}$ is one-dimensional by establishing that every nonempty clopen subset of $\mathcal{E}$ is unbounded. The complete Erdös space $\mathcal{E}^c$ is the set of vectors in ℓ^2 the coordinates of which are all irrational, i.e.,

$$\mathcal{E}^c = \{(x_0, x_1, x_2, \cdots) \in \ell^2 : x_i \in \mathbb{R}\backslash\mathbb{Q}\}.$$

It is easy to see that both $\mathcal{E}$ and $\mathcal{E}^c$ are almost zero-dimensional.

Theorem 2.1 ([1]). *The following statements are equivalent :*

(1) *X is almost zero-dimensional space*
(2) *X is imbeddable in Erdös space $\mathcal{E}$*
(3) *X is imbeddable in complete Erdös space $\mathcal{E}^c$.*

A space X is said to be *totally disconnected* if for any two distinct points $x, y \in X$ there is a clopen set $A \subseteq X$ such that $x \in A \subseteq X\backslash\{y\}$. It is clear that every zero-dimensional space is totally disconnected. It is know to see that both $\mathcal{E}$ and $\mathcal{E}^c$ are totally disconnected. Recall that a space X is called *hereditarily disconnected* if all components are singletons. It is known that every totally disconnected space is hereditarily disconnected. The Cantor set is a universal object for the class of all zero-dimensional spaces. And the Erdös space $\mathcal{E}$ is a universal object for the class of almost zero-dimensional spaces. But the class of totally disconnected spaces has no universal element [3].

In [4] Erdös proved that the empty set is the only bounded clopen subset of $\mathcal{E}^c$. This means that if we add a new point ∞ to $\mathcal{E}^c$ whose neighborhoods are the complements of bounded sets, then the resulting space $\mathcal{E}^c \cup \{\infty\}$ is a connected space. Let H be the convergent sequence $\{0\} \cup \{\frac{1}{n}\}$ for the natural number n. Consider the product space $(\mathcal{E}^c \cup \{\infty\} \times H$ and its subspace

$$\mathcal{T} = \left(\mathcal{E}^c \times \left\{\frac{1}{n}\right\}\right) \cup \{(\infty, 0)\}.$$

Since every $\mathcal{E}^c \times \{\frac{1}{n}\}$ is clopen in $\mathcal{T}$, we have that $\{(\infty, 0)\}$ is a C-set in $\mathcal{T}$, that $\mathcal{T} \backslash \{(\infty, 0)\}$ is almost zero-dimensional space, and that $\mathcal{T}$ is totally disconnected.

Let a be a fixed point in $\mathcal{E}^c$. Then $\{(a, \frac{1}{n})\}$ is a closed subset of $\mathcal{T}$. Since $\{(\infty, 0)\}$ cannot be separated from $\{(a, 0)\}$ by a clopen set, $\mathcal{T} \cup (a, 0)$ is not totally disconnected space. For if there is a clopen set F that contains $\{(\infty, 0)\}$ but not $\{(a, 0)\}$, then $\mathcal{T} \cap F$ is a C-set neighborhood of $\{(\infty, 0)\}$ in $\mathcal{T}$ such that $\{(a, \frac{1}{n})\} \cap F$ is finite. This is a contradiction. Note that these two points are the only points that cannot be separated. Thus $\mathcal{T} \cup (a, 0)$ is hereditarily disconnected.

Theorem 2.2. *There exists a complete space that is hereditarily disconnected but not a totally disconnected space.*

A space X is *homogeneous* if for every $x, y \in X$ there is a homeomorphism f of X such that $f(x) = y$. In [4] Erdös proved that $\mathcal{E}^c$ is one-dimensional homogeneous space. The *Lelek fan* L is a certain dendroid with the property that its set of endpoints G is a one-dimensional totally disconnected. It is known that G is homogeneous, and that G is homeomorphic to the complete Erdös space $\mathcal{E}^c$ [1].

Theorem 2.3 ([5])**.** *Let P be the pseudo-arc in the plane. Then there are a one-dimensional continuum $\tilde{L}$ and a continuous open surjection $\pi : \tilde{L} \to L$, such that*

(1) $\pi^{-1}(x) \simeq P$ *for all* $x \in L$
(2) *for a homeomorphism f of L, there is a homeomorphism $\tilde{f}$ of $\tilde{L}$ such that $f \circ \pi = \pi \circ \tilde{f}$*
(3) *if for some $x \in L$, $g : \pi^{-1}(x) \to \pi^{-1}(x)$ is a homeomorphism, then there is a homeomorphism $\tilde{g} : \tilde{L} \to \tilde{L}$ such that $\tilde{g}|_{\pi^{-1}(x)} = g$ and $\tilde{g}(\pi^{-1}(y)) = \pi^{-1}(y)$ for every $y \in L$.*

Recall that P is homogeneous. Thus Theorem 2.3 implies that $\pi^{-1}(G)$ is homogeneous.

Corollary 2.4. *If X is totally disconnected and Y is connected, then $\pi^{-1}(G)$ and $X \times Y$ are not homeomorphic.*

Proof. Consider the projection $p : X \times Y \to X$ and suppose that $h : \pi^{-1}(G) \to X \times Y$ is a homeomorphism. Since G is totally disconnected, $\{\pi^{-1}(a) : a \in G\}$ is the collection of components of $\pi^{-1}(G)$, and $\{\{x\} \times Y : x \in X\}$ is the collection of components of $X \times Y$. Since π is open, the map from G to X defined by

$$a \mapsto \pi^{-1}(a) \mapsto h(\pi^{-1}(a)) \mapsto \{p(h(\pi^{-1}(a)))\}$$

is continuous. Since it has a continuous inverse, $X \simeq G$ and $Y \simeq P$. Hence we have $\pi^{-1}(G) \simeq G \times P$. But since $dim(\pi^{-1}(G)) \leq 1$ being a subspace of the one-dimensional space $\tilde{L}$ and $dim(G \times P) = 2$, this is a contradiction. $\square$

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