SOME APPLICATIONS AND PROPERTIES OF
GENERALIZED FRACTIONAL CALCULUS
OPERATORS TO A SUBCLASS OF ANALYTIC AND
MULTIVALENT FUNCTIONS

S. K. LEE∗, S. M. KHAINAR AND MEENA MORE

Abstract. In this paper we introduce a new subclass $K_{\lambda, \phi, \eta}(n; p; \alpha)$ of analytic and multivalent functions with negative coefficients using fractional calculus operators. Connections to the well known and some new subclasses are discussed. A necessary and sufficient condition for a function to be in $K_{\lambda, \phi, \eta}(n; p; \alpha)$ is obtained. Several distortion inequalities involving fractional integral and fractional derivative operators are also presented. We also give results for radius of starlikeness, convexity and close-to-convexity and inclusion property for functions in the subclass. Modified Hadamard product, application of class preserving integral operator and other interesting properties are also discussed.

1. Introduction and definitions

Let $M(n, p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \ (a_k \geq 0; p, n \in \mathbb{N}) \quad (1.1)$$

which are analytic and multivalent in the unit open disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $U^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$. Consider the subclass $K_{\lambda, \phi, \eta}(n; p; \alpha)$ of functions $f(z) \in M(n, p)$ which also satisfy

2000 Mathematics Subject Classification: 30C45, 26A33.
Key words and phrases: Maximum modulus principle, analytic functions, univalent functions, simply connected region, generalized fractional integral operator, starlike functions.

∗Corresponding author.
the inequality:

\[
\left| zJ_{0,z}^{1+\lambda,1+\phi,1+\eta}f(z) + \mu z^2 J_{0,z}^{2+\lambda,2+\phi,2+\eta}f(z) \right| + \mu z J_{0,z}^{1+\lambda,1+\phi,1+\eta}f(z) < (p - \phi) + (p - \phi) < \alpha (1.2)
\]

\(z \in U; n \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \mu < 1; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda, \phi\} - (1 + p)\). Where \(J_{0,z}^{\lambda,\phi,\eta}f(z)\) denotes an operator of fractional calculus which is defined as follows:

**Definition 1.** The fractional integral of order \(\lambda\) of a function \(f(z)\) is defined by

\[
D_z^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt \quad (\lambda > 0) \quad (1.3)
\]

where \(f(z)\) is analytic in a simply-connected region of the \(z\)-plane containing the origin, and the multiplicity of \((z-t)^{\lambda-1}\) is removed by requiring \(\log(z-t)\) to be real when \((z-t) > 0\).

**Definition 2.** The fractional derivative of order \(\lambda\) of a function \(f(z)\) is defined by

\[
D_z^{\lambda}f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt \quad (0 \leq \lambda < 1) \quad (1.4)
\]

where \(f(z)\) is analytic in a simply-connected region of the \(z\)-plane containing the origin and the multiplicity of \((z-t)^{-\lambda}\) is removed as in Definition 1 above.

**Definition 3.** Let \(\lambda > 0 \) and \(\eta, \phi \in \mathbb{R}\). Then, in terms of the Gauss’s hypergeometric function \(\text{$_2F_1$}\), the generalized fractional integral operator \(I_{0,z}^{\lambda,\beta,\eta}\) of a function \(f(z)\) is defined by

\[
I_{0,z}^{\lambda,\beta,\eta}f(z) = z^{-\lambda-\beta} \frac{\Gamma(\lambda)}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} f(t) \cdot \text{$_2F_1$}(\lambda+\beta, -\eta; \lambda; 1-\frac{t}{z}) dt \quad (1.5)
\]

where the function \(f(z)\) is analytic in a simply-connected region of the \(z\)-plane containing the origin, with order

\[
f(z) = O(|z|^{\epsilon}), \quad z \to 0 \quad (1.6)
\]

for

\[
\epsilon > \max\{0, \beta - \eta\} - 1 \quad (1.7)
\]

and the multiplicity of \((z-t)^{\lambda-1}\) is removed by requiring \(\log(z-t)\) to be real when \((z-t) > 0\).
Definition 4. Let $0 \leq \lambda < 1$ and $\beta, \eta \in \mathbb{R}$. Then the generalized fractional derivative operator $J_{0,z}^{\lambda,\beta,\eta}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\lambda,\beta,\eta} \{ f(z) \} = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\beta} \int_0^z (z-t)^{-\lambda} f(t) \cdot \frac{2F_1(\beta-\lambda, 1-\eta; 1-\lambda; 1-t/z) dt}{z} \right\}$$

(1.8)

where the function $f(z)$ is analytic in a simply-connected region of the $z$-plane containing the origin, with the order as given in (1.6), and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Notice that for $f(z) \in M(n, p)$

$$J_{0,z}^{\lambda,\phi,\eta} \{ f(z) \} = \frac{\Gamma(1+p)\Gamma(1+p+\eta-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} z^{p-\phi} \mathcal{P}(z)$$

$$- \sum_{k=n+p}^{\infty} \frac{\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k z^{k-\phi}$$

(1.9)

For $(0 \leq \lambda < 1)$

$$J_{0,z}^{m+\lambda,m+\phi,m+\eta} \{ f(z) \} = d_m^{-\lambda} f(z) = \frac{d^m}{dz^m} D_z^{-\lambda} f(z).$$

(1.11)

Also

$$J_{0,z}^{\lambda,\phi,\eta} \{ f(z) \} = \frac{\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} z^{k-\phi}$$

if $0 \leq \lambda < 1$, $\phi, \eta \in \mathbb{R}$ and $k > \max\{0, \phi - \eta\} - 1$.

By comparing Definition 1 with 3 and Definition 2 with 4, we obtain the following relationships:

$$I_{0,z}^{\lambda,-\lambda,\eta} \{ f(z) \} = D_z^{-\lambda} f(z) \ (\lambda > 0)$$

(1.13)

and

$$J_{0,z}^{\lambda,\lambda,\eta} \{ f(z) \} = D_z^{\lambda} f(z) \ (0 \leq \lambda < 1).$$

(1.14)
From the general class $K^\lambda_{\mu}(n; p; \alpha)$ defined by (1.2) we take note of the following important subclasses:

$$Q^{\lambda,\phi,\eta}(n; p; \alpha) = K^\lambda_{0}(n; p; \alpha) \quad (n, p \in \mathbb{N}; 0 \leq \lambda < 1; 0 < \alpha \leq p)$$

(1.15)

$$R^{\lambda,\phi,\eta}(n; p; \alpha) = K^\lambda_{1}(n; p; \alpha) \quad (n, p \in \mathbb{N}; 0 \leq \lambda < 1; 0 < \alpha \leq p)$$

(1.16)

$$\Omega_{\lambda}(n; p; \alpha) = Q^{\lambda,\phi,\eta}(n; p; \alpha) \quad (n, p \in \mathbb{N}; 0 \leq \lambda < 1; 0 < \alpha \leq p)$$

(1.17)

$$\Delta_{\lambda}(n; p; \alpha) = R^{\lambda,\phi,\eta}(n; p; \alpha) \quad (n, p \in \mathbb{N}; 0 \leq \lambda < 1; 0 < \alpha \leq p)$$

(1.18)

$$S_{n}(p; \alpha) = \Omega_{n}(n; p; \alpha) \quad (n, p \in \mathbb{N}; 0 < \alpha \leq p)$$

(1.19)

$$C_{n}(p; \alpha) = \Delta_{n}(n; p; \alpha) \quad (n, p \in \mathbb{N}; 0 < \alpha \leq p)$$

(1.20)

$$S^{*}(p; \alpha) = S_{1}(p; \alpha) \quad (p \in \mathbb{N}; 0 < \alpha \leq p)$$

(1.21)

$$C^{*}(p; \alpha) = C_{1}(p; \alpha) \quad (p \in \mathbb{N}; 0 < \alpha \leq p).$$

(1.22)

The classes $S_{n}(p; \alpha)$ and $S^{*}(p; \alpha)$ consists of $p$-valently starlike functions of order $(p - \alpha), (0 < \alpha \leq p)$ and the classes $C_{n}(p; \alpha)$ and $C^{*}(p; \alpha)$ consists of $p$-valently convex functions of order $(p - \alpha), (0 < \alpha \leq p)$. For $p = 1$ we have $S^{*}(\alpha) = S^{*}(1; \alpha)$ the class of starlike functions of order $(1 - \alpha), (0 < \alpha \leq 1)$ and $C^{*}(\alpha) = C^{*}(1; \alpha)$ the class of convex functions of order $(1 - \alpha), (0 < \alpha \leq 1)$. The classes are popularly studied and of interest in Geometric Functions Theory (cf. [12]).

2. Coefficient bounds and distortion inequalities

We begin by stating a necessary and sufficient condition for a function $f(z) \in M(n, p)$ to be in the class $K^\lambda_{\mu}(n; p; \alpha)$.

**Theorem 1.** Let a function $f(z) \in M(n, p)$. Then the function $f(z)$ belongs to the class $K^\lambda_{\mu}(n; p; \alpha)$, if and only if

$$\sum_{k=n+p}^{\infty} \frac{(k - p + \alpha)[1 + \mu(k - \phi - 1)]\Gamma(1 + k)\Gamma(1 + k + \eta - \phi)}{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)} a_k \leq \frac{\alpha\Gamma(1 + p)\Gamma(1 + p + \eta - \phi)[1 + \mu(p - \phi - 1)]}{\Gamma(1 + p - \phi)\Gamma(1 + p + \eta - \lambda)} \max\{\lambda, \phi\} - (1 + p))$$

(2.1)

$n, p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > max\{\lambda, \phi\} - (1 + p))$.

The result is sharp for the function $f(z)$ given by
Some applications and properties of generalized fractional calculus 131

\[ f(z) = z^p \frac{\alpha \Gamma(1 + p) \Gamma(1 + p + \eta - \phi)[1 + \mu(p - \phi - 1)]}{\Gamma(1 + p - \phi)\Gamma(1 + p + \eta - \lambda)(k - p + \alpha)[1 + \mu(k - \phi - 1)]} \times \frac{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)}{\Gamma(1 + k)\Gamma(1 + k + \eta - \phi)} z^{n+p} (n \in \mathbb{N}) \]  

(2.2)

**Proof.** Assume that \( f(z) \) is defined by (1.1) and inequality (2.1) holds. Then

\[
\begin{align*}
|zJ_{0,\phi}^{\lambda,1+\phi,1+\eta}\{f(z)\} + \mu z^2 J_{0,\phi}^{2+\lambda,2+\phi,2+\eta}\{f(z)\}| \\
-(p - \phi)\{(1 - \mu)J_{0,\phi}^{\lambda,\phi,\eta}\{f(z)\} + \mu z J_{0,\phi}^{1+\lambda,1+\phi,1+\eta}\{f(z)\}\} | \\
-\alpha|(1 - \mu)J_{0,\phi}^{\lambda,\phi,\eta}\{f(z)\} + \mu z J_{0,\phi}^{1+\lambda,1+\phi,1+\eta}\{f(z)\} |
\end{align*}
\]

\[
= \left| \sum_{k=n+p}^{\infty} \frac{(k - p)[1 + \mu(k - \phi - 1)]\Gamma(1 + k)\Gamma(1 + k + \eta - \phi)}{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)} a_k z^{k-p} \right|
\]

\[
-\alpha \left| \frac{[1 + \mu(p - \phi - 1)]\Gamma(1 + p)\Gamma(1 + p + \eta - \phi)}{\Gamma(1 + p - \phi)\Gamma(1 + p + \eta - \lambda)} \right|
\]

\[
- \sum_{k=n+p}^{\infty} \frac{[1 + \mu(k - \phi - 1)]\Gamma(1 + k)\Gamma(1 + k + \eta - \phi)}{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)} a_k z^{k-p} \right| \\
\leq \sum_{k=n+p}^{\infty} \frac{(k - p + \alpha)[1 + \mu(k - \phi - 1)]\Gamma(1 + k)\Gamma(1 + k + \eta - \phi)}{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)} a_k \\
- \frac{\alpha \Gamma(1 + p)\Gamma(1 + p + \eta - \phi)[1 + \mu(p - \phi - 1)]}{\Gamma(1 + p - \phi)\Gamma(1 + p + \eta - \lambda)} \leq 0
\]

by hypothesis and maximum modulus principle \((n, p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda, \phi\} - (1 + p))\). Hence, \( f(z) \) defined by (1.1) belongs to the class \( K_{\mu}^{\lambda,\phi,\eta}(n; p; \alpha) \).
Conversely, assume \( f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n; p; \alpha) \). Then
\[
\left| z J_{0,z}^{1+\lambda,1+\phi,1+\eta} \{ f(z) \} + \mu z J_{0,z}^{2+\lambda,2+\phi,2+\eta} \{ f(z) \} - (p - \phi) \right| \\
\left| (1 - \mu) J_{0,z}^{\lambda,\phi,\eta} \{ f(z) \} + \mu z J_{0,z}^{1+\lambda,1+\phi,1+\eta} \{ f(z) \} \right|
\]

\[
= \left[ \sum_{k=n+p}^{\infty} \frac{(k-p)[1 + \mu(k - \phi - 1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k z^{k-p} \right] \]
\[
\left/ \left[ \frac{[1 + \mu(p - \phi - 1)]\Gamma(1+p)\Gamma(1+p+\eta-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} - \sum_{k=n+p}^{\infty} \frac{\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k z^{k-p} \right] \right| < \alpha \quad (2.3)
\]

Notice that \(|\text{Re}(z)| \leq |z|\) for any \( z \), and thus choosing \( z \) to be real and allowing \( z \to 1^- \) through real values, (2.3) yields
\[
\sum_{k=n+p}^{\infty} \frac{(k-p)[1 + \mu(k - \phi - 1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k \\
\leq \alpha \left\{ \frac{[1 + \mu(p - \phi - 1)]\Gamma(1+p)\Gamma(1+p+\eta-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} - \sum_{k=n+p}^{\infty} \frac{\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k \right\}
\]

which simplifies to (2.1). We also observe that \( f(z) \) given by (2.2) is an extremal function for the assertion (2.1).

**COROLLARY 1.** Let \( f(z) \in M(n,p) \). Then
\[
\sum_{k=n+p}^{\infty} a_k \leq \{ \alpha \Gamma(1+p)\Gamma(1+p+\eta-\phi)[1 + \mu(p - \phi - 1)] \\
\quad \Gamma(1+n+p-\phi)\Gamma(1+n+p+\eta-\lambda) / \{ \Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda) \} \times (n+\alpha)[1 + \mu(n + p - \phi - 1)]\Gamma(1+n+p)\Gamma(1+n+p+\eta-\phi) \}
\]

\((n, p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda, \phi\} - (1+p)) \) with equality for \( f(z) \) given by (2.2).
Corollary 2. Let a function $f(z) \in M(n, p)$. Then the function $f(z)$ belongs to the class $Q_{\lambda, \phi, \eta}^{\alpha}(n; p; \alpha)$, if and only if
\[
\sum_{k=n+p}^{\infty} \frac{(k - p + \alpha)k!\Gamma(1 + k + \eta - \phi)(k - \phi)}{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)} a_k \leq \frac{\alpha\Gamma(1 + p)\Gamma(1 + p + \eta - \phi)}{\Gamma(1 + p - \phi)\Gamma(1 + p + \eta - \lambda)}.
\]
\[
(n, p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda, \phi\} - (1 + p)).
\]

Corollary 3. Let a function $f(z) \in M(n, p)$. Then the function $f(z)$ belongs to the class $R_{\lambda, \phi, \eta}^{\alpha}(n; p; \alpha)$, if and only if
\[
\sum_{k=n+p}^{\infty} \frac{(k - p + \alpha)k!\Gamma(1 + k + \eta - \phi)(k - \phi)}{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)} a_k \leq \frac{\alpha\Gamma(1 + p)\Gamma(1 + p + \eta - \phi)}{\Gamma(1 + p - \phi)\Gamma(1 + p + \eta - \lambda)}.
\]
\[
(n, p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda, \phi\} - (1 + p)).
\]

Corollary 4. Let a function $f(z) \in M(n, p)$. Then the function $f(z)$ belongs to the class $\Omega_{\lambda}(n; p; \alpha)$, if and only if
\[
\sum_{k=n+p}^{\infty} \frac{(k - p + \alpha)k!\Gamma(1 + k - \lambda)}{\Gamma(1 + k - \lambda)} a_k \leq \frac{\alpha\Gamma(1 + p)}{\Gamma(1 + p - \lambda)}.
\]
\[
(n, p \in \mathbb{N}, 0 < \alpha \leq p; 0 \leq \lambda < 1).
\]

Corollary 5. Let a function $f(z) \in M(n, p)$. Then the function $f(z)$ belongs to the class $\Delta_{\lambda}(n; p; \alpha)$, if and only if
\[
\sum_{k=n+p}^{\infty} \frac{(k - p + \alpha)k!}{\Gamma(k - \lambda)} a_k \leq \frac{\alpha\Gamma(1 + p)}{\Gamma(p - \lambda)}.
\]
\[
(n, p \in \mathbb{N}, 0 < \alpha \leq p; 0 \leq \lambda < 1).
\]

Corollary 6. Let a function $f(z) \in M(n, p)$. Then the function $f(z)$ belongs to the class $S_{\alpha}(p; \alpha)$, if and only if
\[
\sum_{k=n+p}^{\infty} (k + p + \alpha) a_k \leq \alpha
\]
\[
(n, p \in \mathbb{N}; 0 < \alpha \leq p).
Corollary 7. Let a function \( f(z) \in M(n,p) \). Then the function \( f(z) \) belongs to the class \( C_{n}(p;\alpha) \), if and only if
\[
\sum_{k=n+p}^{\infty} k(k-p+\alpha) a_k \leq \alpha p
\]
\((n, p \in \mathbb{N}; 0 < \alpha \leq p)\).

Notice that substituting \( n = 1 \) and \( p = 1 \) in corollaries 6 and 7 above, we get the known results for starlike and convex functions.

Next, we prove the distortion inequalities involving the fractional operators \( I_{0,z}^{\lambda,\phi,\eta} \) and \( J_{0,z}^{\lambda,\phi,\eta} \).

Theorem 2. Let \( \beta \in \mathbb{R}^+ \) and \( \gamma, \eta \in \mathbb{R} \) such that \( \eta > \max\{-\beta, \gamma\} - (1 + p) \). If \( n \) is a positive integer satisfying
\[
n \geq \gamma \left( \frac{\beta + \eta}{\beta} \right) - (1 + p)
\]
and, if \( f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha) \), then
\[
\left| I_{0,z}^{\beta,\gamma,\eta} \{ f(z) \} \right| - \frac{\Gamma(1 + p)\Gamma(1 + p + \eta - \gamma)}{\Gamma(1 + p - \gamma)\Gamma(1 + p + \eta + \beta)} |z|^{p-\gamma} \leq \frac{\alpha\Gamma(1 + n + p - \gamma + \eta)\Gamma(1 + p + \eta - \phi)[1 + \mu(p - \phi - 1)]}{\Gamma(1 + n + p - \phi)\Gamma(1 + n + p + \eta - \lambda)|z|^{1+p-\gamma}} / \{\Gamma(1 + n + p - \gamma) \Gamma(1 + n + p + \eta + \beta) \Gamma(1 + n + p + \eta - \lambda)(n + \alpha) [1 + \mu(n + p - \phi - 1)]\Gamma(1 + n + p + \eta - \phi)\}
\]
for \( z \in U \) if \( \gamma \leq p \) and \( z \in U^* \) if \( \gamma > p \).

The result is sharp for the function \( f(z) \) given by
\[
f(z) = z^p - \{\alpha\Gamma(1 + p)\Gamma(1 + p + \eta - \phi)[1 + \mu(p - \phi - 1)] \\
\Gamma(1 + n + p - \phi)\Gamma(1 + n + p + \eta - \lambda)z^{p+1}\} / \{\Gamma(1 + p - \phi) \Gamma(1 + p + \eta - \lambda)(n + \alpha) [1 + \mu(n + p - \phi - 1)]\Gamma(1 + n + p) \\
\Gamma(1 + n + p + \eta - \phi)\}
\]
\((n, p \in \mathbb{N})\).
Proof. Let \( f(z) \in K_{\mu,\phi,\eta}^{n;p;\alpha} \). Then by Corollary 1 and the definition of \( I_{0,z}^{\beta,\gamma,\eta} \), we have

\[
I_{0,z}^{\beta,\gamma,\eta}\{f(z)\} = \frac{\Gamma(1 + p)\Gamma(1 + p + \eta - \gamma)}{\Gamma(1 + p - \gamma)\Gamma(1 + p + \eta + \beta)} z^{p-\gamma} - \sum_{k=n+p}^{\infty} \frac{\Gamma(1 + k)\Gamma(1 + k + \eta - \gamma)}{\Gamma(1 + k - \gamma)\Gamma(1 + k + \beta + \eta)} a_k z^{k-\gamma} \tag{2.13}
\]

Consider

\[
h(k) = \frac{\Gamma(1 + k)\Gamma(1 + k + \eta - \gamma)}{\Gamma(1 + k - \gamma)\Gamma(1 + k + \beta + \eta)} \quad (k \geq n + p; n, p \in \mathbb{N}).
\]

Notice that \( h(k) \) is a non-increasing function of \( k \) \((k \geq n + p; n, p \in \mathbb{N})\). By assuming the hypothesis of this theorem and the condition \((2.10)\), we have

\[
0 < h(k) \leq h(n+p) = \frac{\Gamma(1 + n + p)\Gamma(1 + n + p + \eta - \gamma)}{\Gamma(1 + n + p - \gamma)\Gamma(1 + n + p + \beta + \eta)} \quad (n, p \in \mathbb{N}).
\tag{2.14}
\]

Now the result in \((2.11)\) is an immediate consequence of \((2.13)\) and \((2.14)\).

**Theorem 3.** Let \( 0 \leq \beta < 1 \) and \( \gamma, \eta \in \mathbb{R} \) such that \( \gamma < 1 + p, \eta > \max\{\beta, \gamma\} - (1 + p) \). If \( n \) is a positive integer such that

\[
n \geq \frac{\gamma(\eta - \beta)}{\beta} - (1 + p) \tag{2.15}
\]

and if \( f(z) \in K_{\mu,\phi,\eta}^{n;p;\alpha} \), then

\[
\left| J_{0,z}^{\beta,\gamma,\eta} f(z) \right| \leq \frac{\alpha \Gamma(1 + p)\Gamma(1 + n + p + \eta - \gamma)\Gamma(1 + p + \eta - \phi)[1 + \mu(p - \phi - 1)]}{\Gamma(1 + n + p - \phi)\Gamma(1 + n + p + \eta - \lambda)[1 + \mu(n + p - \phi - 1)]}\Gamma(1 + n + p + \eta - \beta) \frac{1}{\Gamma(1 + n + p + \eta - \gamma)\Gamma(1 + n + p + \eta - \lambda)(n + \alpha)} \tag{2.16}
\]

for \( z \in U \) if \( \gamma \leq p \) and \( z \in U^* \) if \( \gamma > p \). The result is sharp for the function given by \((2.12)\).
Proof. Using the hypothesis of this theorem and function \( f(z) \) given by (1.1) we have

\[
J_{\beta, \gamma, \eta}^0 f(z) = \frac{\Gamma(1 + p)\Gamma(1 + p + \eta - \gamma)}{\Gamma(1 + p - \gamma)\Gamma(1 + p + \eta - \beta)} z^{p-\gamma} - \sum_{k=n+p}^{\infty} \frac{\Gamma(1 + k)\Gamma(1 + k + \eta - \gamma)}{\Gamma(1 + k - \gamma)\Gamma(1 + k + \eta - \beta)} a_k z^{k-\gamma}.
\]

(2.17)

Now following the arguments similar to those given in the proof of Theorem 2, result (2.16) is obtained.

The Theorems 2 and 3 can be used to derive a number distortion properties by suitable choice of the parameters \( \gamma, \phi, \mu, \alpha \) and \( n \) in equations (2.11) and (2.16). For \( \gamma = -\beta \) in Theorem 2 and \( \gamma = \beta \) in Theorem 3, we get the following distortion properties of the fractional integral and fractional derivative operator.

**Corollary 8.** If \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \), then

\[
\left| D_{\beta}^{-} f(z) \right| - \frac{\Gamma(1 + p)}{\Gamma(1 + p + \beta)} |z|^{p+\beta} \leq \left\{ \alpha \Gamma(1 + p) \right. \\
\Gamma(1 + p + \eta - \phi)(1 + \mu(p - \phi - 1))\Gamma(1 + n + p - \phi) \\
\Gamma(1 + n + p + \eta - \lambda) |z|^{1+p+\beta} / \{ \Gamma(1 + n + p + \beta)\Gamma(1 + p - \phi) \\
\times \Gamma(1 + p + \eta - \lambda)(n + \alpha)(1 + \mu(n + p - \phi - 1))\Gamma(1 + n + p + \eta - \phi) \} \\
\left. \right\} (2.18)
\]

for all \( \beta (\beta > 0), z \in U \) and \( n, p \in \mathbb{N} \).

**Corollary 9.** If \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \), then

\[
\left| D_{\beta}^{+} f(z) \right| - \frac{\Gamma(1 + p)}{\Gamma(1 + p - \beta)} |z|^{p-\beta} \leq \\
\left\{ \alpha \Gamma(1 + p)\Gamma(1 + p + \eta - \phi)(1 + \mu(p - \phi - 1))\Gamma(1 + n + p - \phi) \\
\Gamma(1 + n + p + \eta - \lambda) |z|^{1+p-\beta} / \{ \Gamma(1 + n + p - \beta)\Gamma(1 + p - \phi) \\
\times \Gamma(1 + p + \eta - \lambda)(n + \alpha)(1 + \mu(n + p - \phi - 1))\Gamma(1 + n + p + \eta - \phi) \} \\
\right\} (2.19)
\]

for all \( \beta (0 \leq \beta < 1), z \in U \) and \( n, p \in \mathbb{N} \).
Each of the results in (2.18) and (2.19) are sharp for the function \( f(z) \) given by
\[
f(z) = z^p - \{a \Gamma(1 + p) \Gamma(1 + p + \eta - \phi)[1 + \mu(p - \phi - 1)] \\
\Gamma(1 + n + p - \phi) \Gamma(1 + n + p + \eta - \lambda)z^{p+1}\}/\Gamma(1 + n + p) \\
\Gamma(1 + p + \eta - \lambda)(n + \alpha)[1 + \mu(n + p - \phi - 1)]\Gamma(1 + n + p + \eta - \phi)\}
\]
for \((n, p) \in \mathbb{N}\).

Next we state two corollaries for growth and distortion of the function \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \) using the fact that \( D_0^g f(z) = f(z) \) and \( D_1^f f(z) = f'(z) \). Thus we choose \( \beta = 0 \) and \( \beta = 1 \) in Corollary 9.

**Corollary 10.** If \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \), then
\[
|f(z)| - |z|^p \leq \{a \Gamma(1 + p) \Gamma(1 + p + \eta - \phi) \\
[1 + \mu(p - \phi - 1)]\Gamma(1 + n + p + \eta - \lambda)|z|^{p+1}\}/\Gamma(1 + n + p) \\
\Gamma(1 + n + p + \eta - \phi)\}
\]
\[
\Gamma(1 + n + p + \eta - \alpha)(n + \alpha)} \\
\Gamma(1 + n + p + \eta - \phi)\}
\]
(2.20)

**Corollary 11.** If \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \), then
\[
|f'(z)| - p|z|^{p-1} \leq \{a \Gamma(1 + p) \Gamma(1 + p + \eta - \phi) \\
[1 + \mu(p - \phi - 1)]\Gamma(1 + n + p + \eta - \lambda)|z|^p\}/\Gamma(1 + n + p + \eta - \phi)\}
\]
\[
\Gamma(1 + n + p + \eta - \alpha)(n + \alpha)} \\
\Gamma(1 + n + p + \eta - \phi)\}
\]
(2.21)

**3. Properties of the class \( K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \)**

Next, we investigate the radius of starlikeness, convexity and close-to-convexity for \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \).

**Theorem 4.** Let \( 0 \leq s < p \) and \( f(z) = K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \). Then \( f(z) \) is starlike of order \( s \) in \( |z| < r_1 \) where
\[
r_1 = \inf_k \left\{ \frac{(p - s)}{(k - s)} g(k, n, p, \mu, \alpha, \lambda, \phi, \eta) \right\} \frac{1}{k^p} \quad (3.1)
\]
for \((n, p) \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda, \phi\} - (1 + p); k \geq n + p)\).
Proof. Let \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \). \( f(z) \) is starlike of order \( s \), \( 0 \leq s < p \) in \( |z| < r_1 \) if \( \Re \left\{ z \frac{f''(z)}{f'(z)} \right\} > s \) which is equivalent to

\[
\left| \frac{z f'(z)}{f(z)} - p \right| < p - s. \tag{3.2}
\]

Simplifying by fairly straightforward calculations we obtain the required result, where

\[
g(k, n, p, \mu, \alpha, \lambda, \phi, \eta) = \frac{(k - p + \alpha)[1 + \mu(k - \phi - 1)]\Gamma(1 + k)}{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)\alpha} \times \frac{\Gamma(1 + k + \eta - \phi)\Gamma(1 + p - \phi)\Gamma(1 + p + n - \lambda)}{\Gamma(1 + p)\Gamma(1 + p + \eta - \phi)[1 + \mu(p - \phi - 1)]}. \tag{3.3}
\]

\[
\]

**Theorem 5.** Let \( 0 \leq c < p \) and \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \). Then \( f(z) \) is convex of order \( c \) in \( |z| < r_2 \), where

\[
r_2 = \inf_k \left\{ \frac{p(p - c)}{k(k - c)} g(k, n, p, \mu, \alpha, \lambda, \phi, \eta) \right\}^{\frac{1}{p - c}} \tag{3.4}
\]

\( (n, p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda, \phi\} - (1 + p); k \geq n + p) \).

**Proof.** Let \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \). \( f(z) \) is convex of order \( c \), \( 0 \leq c < p \) in \( |z| < r_2 \) if \( \Re \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} > c \) which is equivalent to

\[
\left| \frac{z f''(z)}{f'(z)} + 1 - p \right| < p - c, \tag{3.5}
\]

simplifying we get the required result for \( g(k, n, p, \mu, \alpha, \lambda, \phi, \eta) \) given in (3.3). \( \square \)

**Theorem 6.** Let \( 0 \leq d < p \) and \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \). Then \( f(z) \) is close-to-convex of order \( d \) in \( |z| < r_3 \) where

\[
r_3 = \inf_k \left\{ \frac{p - d}{k} g(k, n, p, \mu, \alpha, \lambda, \phi, \eta) \right\}^{\frac{1}{p - d}} \tag{3.6}
\]

\( (n, p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda, \phi\} - (1 + p); k \geq n + p) \).
Some applications and properties of generalized fractional calculus

Proof. Let \( f(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \). \( f(z) \) is close-to-convex of order \( d, 0 \leq d < p \) in \(|z| < r_3\) if \( \text{Re}\left\{ \frac{f(z)}{z^{p-1}} \right\} > d \) which is equivalent to

\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| < p - d,
\]

(3.7)
simplifying we get the required result for \( g(k, n, p, \mu, \alpha, \lambda, \phi, \eta) \) given in (3.3).

**Theorem 7.** Let \( f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \) \((a_{k,j} \geq 0, p, n \in \mathbb{N}, j = 1, 2, \cdots, \ell)\) be in the class \( K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \). Then the function \( h(z) \) defined by

\[
h(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z)
\]

(3.8)
also belongs to the class \( K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \).

Proof. By the definition of \( h(z) \) we have

\[
h(z) = z^p - \sum_{k=n+p}^{\infty} \left( \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j} \right) z^k
\]

(3.9)

since \( f_j(z) \in K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \) \((j = 1, 2, \cdots, \ell)\) by Theorem 1 we have inequality (2.1) with \( a_k \) replaced by \( a_{k,j} \). Consequently, \( h(z) \) can be easily shown to be in \( K_{\mu}^{\lambda, \phi, \eta}(n; p; \alpha) \).

4. Results on modified Hadamard product

**Theorem 8.** Let the function \( f(z) \) and \( g(z) \) be defined by

\[
f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k
\]

(4.1)

and

\[
g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k
\]

(4.2)
belong to $K^{\lambda,\phi,\eta}(n;p;\alpha)$ and $K^{\lambda,\phi,\eta}(n;p;\xi)$, respectively. Also assume that
\[ s(p + 1) = \frac{\Gamma(2 + p)\Gamma(2 + p + \eta - \gamma)[1 + (\mu(p - \phi))]}{\Gamma(2 + p - \gamma)\Gamma(2 + p + \beta + \eta)} \leq 1 \] (p \in \mathbb{N}).

Then $(f \ast g)(z) \in K^{\lambda,\phi,\eta}(n;p;\delta)$ where
\[ \delta = \frac{\alpha \xi s(p)(k - p)}{s(p + 1)(k - p + \alpha)(k - p + \xi) - \alpha \xi s(p)} \] (4.3)

The function $s(k)$ is given by
\[ s(k) = \frac{\Gamma(1 + k)\Gamma(1 + k + \eta - \phi)[1 + \mu(k - \phi - 1)]}{\Gamma(1 + k - \phi)\Gamma(1 + k + \eta - \lambda)} \] (4.4)

($k \geq n + p; n; p \in \mathbb{N}$).

The result is best possible for
\[ f(z) = z^p - \frac{\alpha s(p)}{(k - p + \alpha)s(p + 1)}z^{p+n} \]
\[ g(z) = z^p - \frac{\xi s(p)}{(k - p + \xi)s(p + 1)}z^{p+n}. \]

Proof. To prove the theorem it is sufficient to show that
\[ \sum_{k=p+n}^{\infty} \frac{(k - p + \delta)s(k)}{\delta s(p)}a_kb_k \leq 1 \] (4.5)

where $s(k)$ is defined by (4.4).

Now, $f(z) \in K^{\lambda,\phi,\eta}(n;p;\alpha)$ and $g(z) \in K^{\lambda,\phi,\eta}(n;p;\xi)$ and thus, we have
\[ \sum_{k=p+n}^{\infty} \frac{(k - p + \alpha)s(k)}{\alpha s(p)}a_k \leq 1 \] (4.6)
\[ \sum_{k=p+n}^{\infty} \frac{(k - p + \xi)s(k)}{\xi s(p)}b_k \leq 1. \] (4.7)

Applying Cauchy-Schwarz inequality to (4.6) and (4.7), we get
\[ \sum_{k=p+n}^{\infty} s(k)\sqrt{(k - p + \alpha)(k - p + \xi)} \frac{1}{s(p)\sqrt{\alpha \xi}} \sqrt{a_kb_k} \leq 1 \] (4.8)
In view of (4.5) it suffices to show that
\[ \sum_{k=p+n}^{\infty} \frac{(k-p+\delta)s(k)}{\delta s(p)} a_k b_k \leq \sum_{k=p+n}^{\infty} \frac{s(k)\sqrt{(k-p+\alpha)(k-p+\xi)}}{s(p)\sqrt{\alpha \xi}} \sqrt{a_k b_k} \]
or equivalently
\[ \sqrt{a_k b_k} \leq \frac{\delta \sqrt{(k-p+\alpha)(k-p+\xi)}}{(k-p+\delta)\sqrt{\alpha \xi}} \quad \text{for } k \geq p + 1 \quad (4.9) \]
In view of (4.8) and (4.9) it enough to show that
\[ \frac{s(p)\sqrt{\alpha \xi}}{s(k)\sqrt{(k-p+\alpha)(k-p+\xi)}} \leq \frac{\delta \sqrt{(k-p+\alpha)(k-p+\xi)}}{(k-p+\delta)\sqrt{\alpha \xi}} \]
which yields
\[ \delta \geq \frac{\alpha \xi s(p)(k-p)}{s(k)(k-p+\alpha)(k-p+\xi) - \alpha \xi s(p)} \quad (4.10) \]
s(k) is a decreasing function of k for \((k \geq (p + 1))\), we have
\[ \delta = \frac{\alpha \xi s(p)(k-p)}{s(p+1)(k-p+\alpha)(k-p+\xi) - \alpha \xi s(p)} \]
Hence the proof is complete.

**Corollary 12.** Let the function \( f(z) \) and \( g(z) \) be defined by (4.1) and (4.2), belong to the class \( K^{\lambda,\phi,\eta}_{\mu}(n; \alpha) \). Then \((f\ast g)(z) \in K^{\lambda,\phi,\eta}_{\mu}(n; \alpha; \delta)\) where
\[ \delta = \frac{\alpha^2 s(p)(k-p)}{s(p+1)(k-p+\alpha)^2 - \alpha^2 s(p)} \]

**Corollary 13.** Let the function \( f(z) \) define by (4.1) be in the class \( K^{\lambda,\phi,\eta}_{\mu}(n; p; \alpha) \). Also let
\[ g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \quad \text{for } (|b_k| \leq 1, n \in \mathbb{N}) \]
Then \((f \ast g)(z) \in K^{\lambda,\phi,\eta}_{\mu}(n; p; \alpha)\).

**Corollary 14.** Let the function \( f(z) \) define by (4.1) be in the class \( K^{\lambda,\phi,\eta}_{\mu}(n; p; \alpha) \). Also let
\[ g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \quad \text{for } (0 \leq b_k \leq 1, n \in \mathbb{N}) \]
142 S. K. Lee, S. M. Khairnar and Meena More

Then \((f \ast g)(z) \in K^\lambda,\phi,\eta_{\mu}(n;p;\alpha)\).

5. Integral transform of the class \(K^\lambda,\phi,\eta_{\mu}(n;p;\alpha)\)

The Komatu integral operator [7] of the function \(f(z)\) is defined by

\[
H(z) = P^d_{c,p} f(z) = \frac{(c + p)^d}{\Gamma(d)z^c} \int_0^z t^{c-1}(\log \frac{z}{t})^{d-1} f(t)dt \quad (5.1)
\]

\((d > 0, c > -p, z \in U)\)

**Theorem 9.** Let the function defined by (1.1) be in the class \(K^\lambda,\phi,\eta_{\mu}(n;p;\alpha)\). Also let \((d > 0, c > -p, z \in U)\), then the function \(H(z)\) defined by (5.1) is also in the class \(K^\lambda,\phi,\eta_{\mu}(n;p;\alpha)\).

**Proof.** From the definition of \(H(z)\) we notice that

\[
H(z) = z^p - \sum_{k=p+n}^{\infty} \left(\frac{c + p}{c + k}\right)^d a_k z^k. \quad (5.2)
\]

Since \((\frac{c + p}{c + k})^d \leq 1\), and in view of Theorem 1, the result follows.

Setting \(c = 1 - p, d = 1\) in (5.1) we obtain another integral operator

\[
F(z) = z^p - \int_0^z t^{-p} f(t)dt. \quad (5.3)
\]

Again if follows from Theorem 9 that \(F(z) \in K^\lambda,\phi,\eta_{\mu}(n;p;\alpha)\).

**Theorem 10.** Let \((d > 0, c > -p, z \in U)\). Also let \(H(z) \in K^\lambda,\phi,\eta_{\mu}(n;p;\alpha)\). Then \(H(z)\) given by (5.1) is \(p\)-valent in the disc \(|z| \leq r_4\), where

\[
r_4 = \inf_k \left\{ \frac{ph(k)(k - p + \alpha)}{kah(p)} \right\}^{\frac{1}{c-p}}. \quad (5.4)
\]

**Proof.** We have

\[
H(z) = z^p - \sum_{k=p+n}^{\infty} \left(\frac{c + p}{c + k}\right)^d a_k z^k \quad (5.5)
\]

We need to show that

\[
\left| \frac{H'(z)}{z^{p-1}} - p \right| \leq p \text{ in } |z| < r_4 \quad (5.6)
\]

where \(r_4\) is given by (5.4).
In view of (5.5), we have

\[ \left| \frac{H'(z)}{z^{p-1}} - p \right| = \left| - \sum_{k=p+n}^{\infty} k \left( \frac{c+p}{c+k} \right)^d a_k z^{k-p} \right| \leq \sum_{k=p+n}^{\infty} k \left( \frac{c+p}{c+k} \right)^d a_k |z|^{k-p}. \]

The last inequality is bounded above by \( p \) if

\[ \sum_{k=p+n}^{\infty} k \left( \frac{c+1}{c+k} \right)^d a_k |z|^{k-p} \leq 1. \]  

But \( H(z) \in K_{\mu}^{\lambda,\phi,\eta}(n; p; \alpha) \) and hence by Theorem 1, we have

\[ \sum_{k=p+n}^{\infty} (k - p + \alpha) s(k) \left( \frac{c+p}{c+k} \right)^d \frac{\alpha s(p)}{\alpha s(p)} a_k \leq 1. \]  

Thus (5.7) and hence (5.6) will hold if

\[ |z| \leq \frac{ps(k)(k-p+\alpha)}{k\alpha s(p)} \frac{1}{k^{1-p}} \text{ for } k \geq p + n, n \in \mathbb{N}. \]

This leads to precisely the main assertion of the theorem by setting \( |z| = r_4 \).

### 6. Extreme points of the class \( K_{\mu}^{\lambda,\phi,\eta}(n; p; \alpha) \)

**Theorem 11.** Let \( f_p(z) = z^p \) and

\[ f_k(z) = z^p - \frac{\alpha s(p)}{(k - p + \alpha)s(k)} z^k, \quad (k \geq p + 1). \]

Then \( f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n; p; \alpha) \), if and only if, \( f(z) \) can be expressed in the form

\[ f(z) = \sum_{k=p+n}^{\infty} \lambda_k f_k(z) \quad (n \in \mathbb{N}_0) \]  

where \( \lambda_k \geq 0 \) and \( \sum_{k=p+n}^{\infty} \lambda_k = 1. \)

**Proof.** Let \( f(z) \) be expressible in the form

\[ f(z) = \sum_{k=p+n}^{\infty} \lambda_k f_k(z) \quad (n \in \mathbb{N}_0) \]
\[ z^p - \sum_{k=p+1}^{\infty} \frac{\alpha s(p)}{(k-p+\alpha)s(k)} \lambda_k z^k \quad (n \in \mathbb{N}). \]

Now
\[ \sum_{k=p+1}^{\infty} \frac{\alpha s(p)}{(k-p+\alpha)s(k)} = \sum_{k=p+1}^{\infty} \lambda_k = 1 - \lambda_p \leq 1 \quad (n \in \mathbb{N}). \]

Therefore, \( f(z) \in K_{n; p}^{\lambda, \phi, \eta}(n; p; \alpha) \). Conversely, suppose that \( f(z) \in K_{n; p}^{\lambda, \phi, \eta}(n; p; \alpha) \). Then setting
\[ \lambda_k = \frac{\alpha s(p)}{(k-p+\alpha)s(k)} a_k \quad \text{and} \quad \lambda_p = 1 - \sum_{k=p+1}^{\infty} \lambda_k \quad (n \in \mathbb{N}). \]

Thus we notice that \( f(z) \) can be expressed in the form (6.1).

**Corollary 15.** The extreme points of the class \( K_{n; p}^{\lambda, \phi, \eta}(n; p; \alpha) \) are
\[ f_p(z) = z^p \quad \text{and} \quad f_k(z) = z^p - \frac{\alpha s(p)}{(k-p+\alpha)s(k)} z^k, \quad k \geq p + 1. \]

**References**


Some applications and properties of generalized fractional calculus


Department of Mathematics,
Gyeongsang National University,
Jinju 660-701, Korea

*E-mail*: sklee@gnu.ac.kr

Department of Mathematics,
Maharashtra Academy of Engineering,
Alandi, Pune - 412105,
Maharashtra, India

*E-mail*: smkhairnar2007@gmail.com

Department of Mathematics,
Maharashtra Academy of Engineering,
Alandi, Pune - 412105,
Maharashtra, India

*E-mail*: meenamores@gmail.com