Korean J. Math. **17** (2009), No. 2, pp. 127–145

SOME APPLICATIONS AND PROPERTIES OF GENERALIZED FRACTIONAL CALCULUS OPERATORS TO A SUBCLASS OF ANALYTIC AND MULTIVALENT FUNCTIONS

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Abstract. In this paper we introduce a new subclass $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$ of analytic and multivalent functions with negative coefficients using fractional calculus operators. Connections to the well known and some new subclasses are discussed. A necessary and sufficient condition for a function to be in $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$ is obtained. Several distortion inequalities involving fractional integral and fractional derivative operators are also presented. We also give results for radius of starlikeness, convexity and close-to-convexity and inclusion property for functions in the subclass. Modified Hadamard product, application of class preserving integral operator and other interesting properties are also discussed.

1. Introduction and definitions

Let $M(n,p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \ (a_k \geq 0; p, n \in I\!N) \tag{1.1}$$

which are analytic and multivalent in the unit open disc $U = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. Also let $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\}$. Consider the subclass $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$ of functions $f(z) \in M(n,p)$ which also satisfy

Received November 3, 2008. Revised June 5, 2009.
2000 Mathematics Subject Classification: 30C45, 26A33.
Key words and phrases: Maximum modulus principle, analytic functions, univalent functions, simply connected region, generalized fractional integral operator, starlike functions.
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the inequality:

$$\left| \frac{zJ_{0,z}^{1+\lambda,1+\phi,1+\eta}\{f(z)\} + \mu z^2 J_{0,z}^{2+\lambda,2+\phi,2+\eta}\{f(z)\}}{(1-\mu)J_{0,z}^{\lambda,\phi,\eta}\{f(z)\} + \mu z J_{0,z}^{1+\lambda,1+\phi,1+\eta}\{f(z)\}} - (p-\phi) \right| < \alpha \quad (1.2)$$

$(z \in U; n \in I\!N; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in I\!R; \phi < p; \eta > \max\{\lambda, \phi\} - (1+p))$. Where $J_{0,z}^{\lambda,\phi,\eta}$ denotes an operator of fractional calculus which is defined as follows:

Definition 1. *The fractional integral of order λ of a function $f(z)$ is defined by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt \ (\lambda > 0) \quad (1.3)$$

where $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 2. *The fractional derivative of order λ of a function $f(z)$ is defined by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt \ (0 \leq \lambda < 1) \quad (1.4)$$

where $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1 above.

Definition 3. *Let $\lambda > 0$ and $\eta, \phi \in I\!R$. Then, in terms of the Gauss's hypergeometric function ${}_2F_1$, the generalized fractional integral operator $I_{0,z}^{\lambda,\beta,\eta}$ of a function $f(z)$ is defined by*

$$I_{0,}^{\lambda,\beta,\eta}\{f(z)\} = \frac{z^{-\lambda-\beta}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} f(t) \cdot {}_2F_1(\lambda+\beta, -\eta; \lambda; 1-\frac{t}{z}) dt \quad (1.5)$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with order

$$f(z) = O(|z|^{\epsilon}), \ z \to 0 \quad (1.6)$$

for

$$\epsilon > \max\{0, \beta - \eta\} - 1 \quad (1.7)$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 4. *Let $0 \leq \lambda < 1$ and $\beta, \eta \in I\!R$. Then the generalized fractional derivative operator $J_{0,z}^{\lambda,\beta,\eta}$ of a function $f(z)$ is defined by*

$$J_{0,z}^{\lambda,\beta,\eta}\{f(z)\} = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz} \times$$

$$\left\{ z^{\lambda-\beta} \int_0^z (z-t)^{-\lambda} f(t) \cdot \ {}_2F_1(\beta-\lambda, 1-\eta; 1-\lambda; 1-\frac{t}{z})dt \right\} \tag{1.8}$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order as given in (1.6), and the multiplicity of $(z-t)^{-\lambda}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Notice that for $f(z) \in M(n,p)$

$$\begin{aligned} J_{0,z}^{\lambda,\phi,\eta}\{f(z)\} &= \frac{\Gamma(1+p)\Gamma(1+p+\eta-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} z^{p-\phi} \\ &- \sum_{k=n+p}^{\infty} \frac{\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k z^{k-\phi} \end{aligned} \tag{1.9}$$

$$D_z^\lambda f(z) = J_{0,z}^{\lambda,\lambda,\eta}\{f(z)\} \ (0 \leq \lambda < 1) \tag{1.10}$$

$$J_{0,z}^{m+\lambda,m+\phi,m+\eta}\{f(z)\} = \frac{d^m}{dz^m} J_{0,z}^{\lambda,\phi,\eta}\{f(z)\}, \ z \in U, m \in I\!N_0. \tag{1.11}$$

For $(0 \leq \lambda < 1)$

$$J_{0,z}^{m+\lambda,m+\lambda,m+\eta}\{f(z)\} = D_z^{m+\lambda} f(z) = \frac{d^m}{dz^m} D_z^\lambda f(z). \tag{1.12}$$

Also

$$J_{0,z}^{\lambda,\phi,\eta} z^k = \frac{\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} z^{k-\phi}$$

if $0 \leq \lambda < 1, \phi, \eta \in I\!R$ and $k > \max\{0, \phi-\eta\} - 1$.

By comparing Definition 1 with 3 and Definition 2 with 4, we obtain the following relationships:

$$I_{0,z}^{\lambda,-\lambda,\eta}\{f(z)\} = D_z^{-\lambda} f(z) \ (\lambda > 0) \tag{1.13}$$

and

$$J_{0,z}^{\lambda,\lambda,\eta}\{f(z)\} = D_z^\lambda f(z) \ (0 \leq \lambda < 1). \tag{1.14}$$

From the general class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$ defined by (1.2) we take note of the following important subclasses:

$$Q^{\lambda,\phi,\eta}(n;p;\alpha) = K_0^{\lambda,\phi,\eta}(n;p;\alpha) \ (n,p \in I\!N; 0 \le \lambda < 1; 0 < \alpha \le p) \tag{1.15}$$

$$R^{\lambda,\phi,\eta}(n;p;\alpha) = K_1^{\lambda,\phi,\eta}(n;p;\alpha) \ (n,p \in I\!N; 0 \le \lambda < 1; 0 < \alpha \le p) \tag{1.16}$$

$$\Omega_{\lambda}(n;p;\alpha) = Q^{\lambda,\lambda,\eta}(n;p;\alpha) \ (n,p \in I\!N; 0 \le \lambda < 1; 0 < \alpha \le p) \tag{1.17}$$

$$\Delta_{\lambda}(n;p;\alpha) = R^{\lambda,\lambda,\eta}(n;p;\alpha) \ (n,p \in I\!N; 0 \le \lambda < 1; 0 < \alpha \le p) \tag{1.18}$$

$$S_n(p;\alpha) = \Omega_0(n;p;\alpha) \ (n,p \in I\!N; 0 < \alpha \le p) \tag{1.19}$$

$$C_n(p;\alpha) = \Delta_0(n;p;\alpha) \ (n,p \in I\!N; 0 < \alpha \le p) \tag{1.20}$$

$$S^*(p;\alpha) = S_1(p;\alpha) \ (p \in I\!N; 0 < \alpha \le p) \tag{1.21}$$

$$C^*(p;\alpha) = C_1(p;\alpha) \ (p \in I\!N; 0 < \alpha \le p). \tag{1.22}$$

The classes $S_n(p;\alpha)$ and $S^*(p;\alpha)$ consists of p-valently starlike functions of order $(p-\alpha), (0 < \alpha \le p)$ and the classes $C_n(p;\alpha)$ and $C^*(p;\alpha)$ consists of p-valently convex functions of order $(p-\alpha), (0 < \alpha \le p)$. For $p = 1$ we have $S^*(\alpha) = S^*(1;\alpha)$ the class of starlike functions of order $(1-\alpha), (0 < \alpha \le 1)$ and $C^*(\alpha) = C^*(1;\alpha)$ the class of convex functions of order $(1-\alpha), (0 < \alpha \le 1)$. The classes are popularly studied and of interest in Geometric Functions Theory (cf. [12]).

2. Coefficient bounds and distortion inequalities

We begin by stating a necessary and sufficient condition for a function $f(z) \in M(n,p)$ to be in the class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$.

Theorem 1. *Let a function $f(z) \in M(n,p)$. Then the function $f(z)$ belongs to the class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$, if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-p+\alpha)[1+\mu(k-\phi-1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k \le \frac{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} \tag{2.1}$$

$(n,p \in I\!N; 0 < \alpha \le p; 0 \le \mu \le 1; 0 \le \lambda < 1; \phi,\eta \in I\!R; \phi < p; \eta > \max\{\lambda,\phi\} - (1+p))$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)(k-p+\alpha)[1+\mu(k-\phi-1)]} \times$$
$$\frac{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)}{\Gamma(1+k)\Gamma(1+k+\eta-\phi)} z^{n+p} (n \in I\!N) \qquad (2.2)$$

Proof. Assume that $f(z)$ is defined by (1.1) and inequality (2.1) holds. Then

$$\begin{aligned}
&|zJ_{0,z}^{1+\lambda,1+\phi,1+\eta}\{f(z)\} + \mu z^2 J_{0,z}^{2+\lambda,2+\phi,2+\eta}\{f(z)\} \\
&-(p-\phi)\{(1-\mu)J_{0,z}^{\lambda,\phi,\eta}\{f(z)\} + \mu z J_{0,z}^{1+\lambda,1+\phi,1+\eta}\{f(z)\}\}| \\
&-\alpha|(1-\mu)J_{0,z}^{\lambda,\phi,\eta}\{f(z)\} + \mu z J_{0,z}^{1+\lambda,1+\phi,1+\eta}\{f(z)\}| \\
&= \left|\sum_{k=n+p}^{\infty} \frac{(k-p)[1+\mu(k-\phi-1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k z^{k-p}\right| \\
&-\alpha\left|\frac{[1+\mu(p-\phi-1)]\Gamma(1+p)\Gamma(1+p+\eta-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)}\right. \\
&\left.-\sum_{k=n+p}^{\infty} \frac{[1+\mu(k-\phi-1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k z^{k-p}\right| \\
&\le \sum_{k=n+p}^{\infty} \frac{(k-p+\alpha)[1+\mu(k-\phi-1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k \\
&-\frac{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} \le 0
\end{aligned}$$

by hypothesis and maximum modulus principle ($n, p \in I\!N; 0 < \alpha \le p; 0 \le \mu \le 1; 0 \le \lambda < 1; \phi, \eta \in I\!R; \phi < p; \eta > \max\{\lambda, \phi\} - (1+p)$). Hence, $f(z)$ defined by (1.1) belongs to the class $K_{\mu}^{\lambda,\phi,\eta}(n; p; \alpha)$.

Conversely, assume $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. Then

$$\left|\frac{zJ_{0,z}^{1+\lambda,1+\phi,1+\eta}\{f(z)\}+\mu z^2 J_{0,z}^{2+\lambda,2+\phi,2+\eta}\{f(z)\}}{(1-\mu)J_{0,z}^{\lambda,\phi,\eta}\{f(z)\}+\mu z J_{0,z}^{1+\lambda,1+\phi,1+\eta}\{f(z)\}}-(p-\phi)\right|$$
$$= \Bigg| \left[\sum_{k=n+p}^{\infty} \frac{(k-p)[1+\mu(k-\phi-1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k z^{k-p}\right]$$
$$/ \Bigg[\frac{[1+\mu(p-\phi-1)]\Gamma(1+p)\Gamma(1+p+\eta-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} - \sum_{k=n+p}^{\infty}$$
$$\frac{[1+\mu(k-\phi-1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k z^{k-p}\Bigg]\Bigg| < \alpha \qquad (2.3)$$

Notice that $|Re(z)| \leq |z|$ for any z, and thus choosing z to be real and allowing $z \to 1^-$ through real values, (2.3) yields

$$\sum_{k=n+p}^{\infty} \frac{(k-p)[1+\mu(k-\phi-1)]\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k$$
$$\leq \alpha \left\{ \frac{[1+\mu(p-\phi-1)]\Gamma(1+p)\Gamma(1+p+\eta-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)} \right.$$
$$\left. - \sum_{k=n+p}^{\infty} \frac{\Gamma(1+k)\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k \right\}$$

which simplifies to (2.1). We also observe that $f(z)$ given by (2.2) is an extremal function for the assertion (2.1). □

Corollary 1. *Let $f(z) \in M(n,p)$. Then*

$$\sum_{k=n+p}^{\infty} a_k \leq \{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]$$
$$\Gamma(1+n+p-\phi)\Gamma(1+n+p+\eta-\lambda)\}/\{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)$$
$$\times(n+\alpha)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p)\Gamma(1+n+p+\eta-\phi)\}$$

$(n, p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda,\phi\} - (1+p))$ with equality for $f(z)$ given by (2.2).

Corollary 2. *Let a function $f(z) \in M(n,p)$. Then the function $f(z)$ belongs to the class $Q^{\lambda,\phi,\eta}(n;p;\alpha)$, if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-p+\alpha)k!\Gamma(1+k+\eta-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k \leq \frac{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)}. \tag{2.4}$$

$(n,p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \lambda < 1; \phi,\eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda,\phi\} - (1+p))$.

Corollary 3. *Let a function $f(z) \in M(n,p)$. Then the function $f(z)$ belongs to the class $R^{\lambda,\phi,\eta}(n;p;\alpha)$, if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-p+\alpha)k!\Gamma(1+k+\eta-\phi)(k-\phi)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} a_k \leq \frac{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)(p-\phi)}{\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)}. \tag{2.5}$$

$(n,p \in \mathbb{N}; 0 < \alpha \leq p; 0 \leq \lambda < 1; \phi,\eta \in \mathbb{R}; \phi < p; \eta > \max\{\lambda,\phi\} - (1+p))$.

Corollary 4. *Let a function $f(z) \in M(n,p)$. Then the function $f(z)$ belongs to the class $\Omega_\lambda(n;p;\alpha)$, if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-p+\alpha)k!}{\Gamma(1+k-\lambda)} a_k \leq \frac{\alpha\Gamma(1+p)}{\Gamma(1+p-\lambda)} \tag{2.6}$$

$(n,p \in \mathbb{N}, 0 < \alpha \leq p; 0 \leq \lambda < 1)$.

Corollary 5. *Let a function $f(z) \in M(n,p)$. Then the function $f(z)$ belongs to the class $\Delta_\lambda(n;p;\alpha)$, if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k-p+\alpha)k!}{\Gamma(k-\lambda)} a_k \leq \frac{\alpha\Gamma(1+p)}{\Gamma(p-\lambda)} \tag{2.7}$$

$(n,p \in \mathbb{N}, 0 < \alpha \leq p; 0 \leq \lambda < 1)$.

Corollary 6. *Let a function $f(z) \in M(n,p)$. Then the function $f(z)$ belongs to the class $S_n(p;\alpha)$, if and only if*

$$\sum_{k=n+p}^{\infty} (k+p+\alpha)\ a_k \leq \alpha \tag{2.8}$$

$(n,p \in \mathbb{N}; 0 < \alpha \leq p)$.

COROLLARY 7. *Let a function $f(z) \in M(n,p)$. Then the function $f(z)$ belongs to the class $C_n(p;\alpha)$, if and only if*

$$\sum_{k=n+p}^{\infty} k(k-p+\alpha)\; a_k \leq \alpha p \tag{2.9}$$

$(n,p \in I\!N; 0 < \alpha \leq p)$.

Notice that substituting $n = 1$ and $p = 1$ in corollaries 6 and 7 above, we get the known results for starlike and convex functions.

Next, we prove the distortion inequalities involving the fractional operators $I_{0,z}^{\lambda,\phi,\eta}$ and $J_{0,z}^{\lambda,\phi,\eta}$.

THEOREM 2. *Let $\beta \in I\!R^+$ and $\gamma, \eta \in I\!R$ such that $\eta > \max\{-\beta, \gamma\} - (1+p)$. If n is a positive integer satisfying*

$$n \geq \gamma\left(\frac{\beta+\eta}{\beta}\right) - (1+p) \tag{2.10}$$

and, if $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$, then

$$\begin{aligned}
&\left|\left|I_{0,z}^{\beta,\gamma,\eta}\{f(z)\}\right| - \frac{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)}{\Gamma(1+p-\gamma)\Gamma(1+p+\eta+\beta)}|z|^{p-\gamma}\right| \\
&\leq \{\alpha\Gamma(1+n+p-\gamma+\eta)\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)] \\
&\Gamma(1+n+p-\phi)\Gamma(1+n+p+\eta-\lambda)|z|^{1+p-\gamma}\}/\{\Gamma(1+n+p-\gamma) \\
&\Gamma(1+n+p+\eta+\beta)\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)(n+\alpha) \\
&[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+\eta-\phi)\}
\end{aligned} \tag{2.11}$$

for $z \in U$ if $\gamma \leq p$ and $z \in U^*$ if $\gamma > p$.

The result is sharp for the function $f(z)$ given by

$$\begin{aligned}
&f(z) = z^p - \{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)] \\
&\Gamma(1+n+p-\phi)\Gamma(1+n+p+\eta-\lambda)z^{p+1}\}/\{\Gamma(1+p-\phi) \\
&\Gamma(1+p+\eta-\lambda)(n+\alpha)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p) \\
&\Gamma(1+n+p+\eta-\phi)\}\ (2.12)
\end{aligned}$$

$(n,p \in I\!N)$.

Proof. Let $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. Then by Corollary 1 and the definition of $I_{0,z}^{\beta,\gamma,\eta}$, we have

$$I_{0,z}^{\beta,\gamma,\eta}\{f(z)\} = \frac{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)}{\Gamma(1+p-\gamma)\Gamma(1+p+\eta+\beta)} z^{p-\gamma} - \sum_{k=n+p}^{\infty} \frac{\Gamma(1+k)\Gamma(1+k+\eta-\gamma)}{\Gamma(1+k-\gamma)\Gamma(1+k+\beta+\eta)} a_k z^{k-\gamma} \quad (2.13)$$

Consider

$$h(k) = \frac{\Gamma(1+k)\Gamma(1+k+\eta-\gamma)}{\Gamma(1+k-\gamma)\Gamma(1+k+\beta+\eta)} \; (k \geq n+p; n; p \in \mathbb{N}).$$

Notice that $h(k)$ is a non-increasing function of k $(k \geq n+p; n,p \in \mathbb{N})$. By assuming the hypothesis of this theorem and the condition (2.10), we have

$$0 < h(k) \leq h(n+p) = \frac{\Gamma(1+n+p)\Gamma(1+n+p+\eta-\gamma)}{\Gamma(1+n+p-\gamma)\Gamma(1+n+p+\beta+\eta)} \; (n,p \in \mathbb{N}). \quad (2.14)$$

Now the result in (2.11) is an immediate consequence of (2.13) and (2.14). □

Theorem 3. *Let $0 \leq \beta < 1$ and $\gamma, \eta \in \mathbb{R}$ such that $\gamma < 1+p, \eta > \max\{\beta,\gamma\} - (1+p)$. If n is a positive integer such that*

$$n \geq \frac{\gamma(\eta-\beta)}{\beta} - (1+p) \quad (2.15)$$

and if $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$, then

$$\left| \left| J_{0,z}^{\beta,\gamma,\eta} f(z) \right| - \frac{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)}{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\beta)} |z|^{p-\gamma} \right| \leq$$
$$\{\alpha\Gamma(1+p)\Gamma(1+n+p+\eta-\gamma)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]$$
$$\Gamma(1+n+p-\phi)\Gamma(1+n+p+\eta-\lambda)|z|^{1+p-\gamma}\}/\{\Gamma(1+n+p-\gamma)$$
$$\Gamma(1+n+p+\eta-\beta)\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)(n+\alpha)$$
$$[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+\eta-\phi)\} \quad (2.16)$$

for $z \in U$ if $\gamma \leq p$ and $z \in U^*$ if $\gamma > p$. The result is sharp for the function given by (2.12).

Proof. Using the hypothesis of this theorem and function $f(z)$ given by (1.1) we have

$$J_{0,z}^{\beta,\gamma,\eta} f(z) = \frac{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)}{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\beta)} z^{p-\gamma} - \sum_{k=n+p}^{\infty} \frac{\Gamma(1+k)\Gamma(1+k+\eta-\gamma)}{\Gamma(1+k-\gamma)\Gamma(1+k+\eta-\beta)} a_k z^{k-\gamma}. \qquad (2.17)$$

Now following the arguments similar to those given in the proof of Theorem 2, result (2.16) is obtained. □

The Theorems 2 and 3 can be used to derive a number distortion properties by suitable choice of the parameters $\gamma, \phi, \mu, \alpha$ and n in equations (2.11) and (2.16). For $\gamma = -\beta$ in Theorem 2 and $\gamma = \beta$ in Theorem 3, we get the following distortion properties of the fractional integral and fractional derivative operator.

Corollary 8. *If $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$, then*

$$\left| \left| D_z^{-\beta} f(z) \right| - \frac{\Gamma(1+p)}{\Gamma(1+p+\beta)} |z|^{p+\beta} \right| \le \{\alpha\Gamma(1+p) \Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]\Gamma(1+n+p-\phi) \Gamma(1+n+p+\eta-\lambda)|z|^{1+p+\beta}\}/\{\Gamma(1+n+p+\beta)\Gamma(1+p-\phi) \times\Gamma(1+p+\eta-\lambda)(n+\alpha)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+\eta-\phi)\} \qquad (2.18)$$

for all β $(\beta > 0), z \in U$ and $n, p \in I\!N$.

Corollary 9. *If $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$, then*

$$\left| \left| D_z^{\beta} f(z) \right| - \frac{\Gamma(1+p)}{\Gamma(1+p-\beta)} |z|^{p-\beta} \right| \le \{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]\Gamma(1+n+p-\phi) \Gamma(1+n+p+\eta-\lambda)|z|^{1+p-\beta}\}/\{\Gamma(1+n+p-\beta)\Gamma(1+p-\phi) \times\Gamma(1+p+\eta-\lambda)(n+\alpha)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+\eta-\phi)\} \qquad (2.19)$$

for all β $(0 \le \beta < 1), z \in U$ and $n, p \in I\!N$.

Each of the results in (2.18) and (2.19) are sharp for the function $f(z)$ given by

$$f(z) = z^p - \{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]$$
$$\Gamma(1+n+p-\phi)\Gamma(1+n+p+\eta-\lambda)z^{p+1}\}/\{\Gamma(1+n+p)\Gamma(1+p-\phi)$$
$$\Gamma(1+p+\eta-\lambda)(n+\alpha)[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+\eta-\phi)\}$$
$$(n, p \in I\!N).$$

Next we state two corollaries for growth and distortion of the function $f(z) \in K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$ using the fact that $D_z^0 f(z) = f(z)$ and $D_z^1 f(z) = f'(z)$. Thus we choose $\beta = 0$ and $\beta = 1$ in Corollary 9.

Corollary 10. *If $f(z) \in K_\mu^{\lambda,\phi,\eta}(n;p.\alpha)$, then*

$$||f(z)| - |z|^p| \leq \{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)$$
$$[1+\mu(p-\phi-1)]\Gamma(1+n+p+\eta-\lambda)|z|^{p+1}\}/\{\Gamma(1+n+p)$$
$$\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)(n+\alpha)[1+\mu(n+p-\phi-1)]$$
$$\Gamma(1+n+p+\eta-\phi)\} \tag{2.20}$$

Corollary 11. *If $f(z) \in K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$, then*

$$\left||f'(z)| - p|z|^{p-1}\right| \leq \{\alpha\Gamma(1+p)\Gamma(1+p+\eta-\phi)$$
$$[1+\mu(p-\phi-1)]\Gamma(1+n+p-\phi)\Gamma(1+n+p+\eta-\lambda)|z|^p\}$$
$$/\{\Gamma(n+p)\Gamma(1+p-\phi)\Gamma(1+p+\eta-\lambda)(n+\alpha)$$
$$[1+\mu(n+p-\phi-1)]\Gamma(1+n+p+\eta-\phi)\} \tag{2.21}$$

3. Properties of the class $K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$

Next, we investigate the radius of starlikeness, convexity and close-to-convexity for $f(z) \in K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$.

Theorem 4. *Let $0 \leq s < p$ and $f(z) = K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$. Then $f(z)$ is starlike of order s in $|z| < r_1$ where*

$$r_1 = \inf_k \left\{ \frac{(p-s)}{(k-s)} g(k,n,p,\mu,\alpha,\lambda,\phi,\eta) \right\}^{\frac{1}{k-p}} \tag{3.1}$$

$(n, p \in I\!N; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi, \eta \in I\!R; \phi < p; \eta > \max\{\lambda, \phi\} - (1+p); k \geq n+p)$.

Proof. Let $f(z) \in K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$. $f(z)$ is starlike of order $s, 0 \leq s < p$ in $|z| < r_1$ if $Re\left\{z\frac{f'(z)}{f(z)}\right\} > s$ which is equivalent to

$$\left|z\frac{f'(z)}{f(z)} - p\right| < p - s. \tag{3.2}$$

Simplifying by fairly straightforward calculations we obtain the required result, where

$$g(k,n,p,\mu,\alpha,\lambda,\phi,\eta) = \frac{(k-p+\alpha)[1+\mu(k-\phi-1)]\Gamma(1+k)}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)\alpha} \times$$
$$\frac{\Gamma(1+k+\eta-\phi)\Gamma(1+p-\phi)\Gamma(1+p+n-\lambda)}{\Gamma(1+p)\Gamma(1+p+\eta-\phi)[1+\mu(p-\phi-1)]}. \tag{3.3}$$

□

Theorem 5. *Let $0 \leq c < p$ and $f(z) \in K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$. Then $f(z)$ is convex of order c in $|z| < r_2$, where*

$$r_2 = \inf_k \left\{\frac{p(p-c)}{k(k-c)} g(k,n,p,\mu,\alpha,\lambda,\phi,\eta)\right\}^{\frac{1}{k-p}} \tag{3.4}$$

$(n,p \in I\!N; 0 < \alpha \leq p; 0 \leq \mu \leq 1; 0 \leq \lambda < 1; \phi,\eta \in I\!R; \phi < p; \eta > \max\{\lambda,\phi\} - (1+p); k \geq n+p)$.

Proof. Let $f(z) \in K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$. $f(z)$ is convex of order $c, 0 \leq c < p$ in $|z| < r_2$ if $Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} > c$ which is equivalent to

$$\left|z\frac{f''(z)}{f'(z)} + 1 - p\right| < p - c, \tag{3.5}$$

simplifying we get the required result for $g(k,n,p,\mu,\alpha,\lambda,\phi,\eta)$ given in (3.3). □

Theorem 6. *Let $0 \leq d < p$ and $f(z) \in K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$. Then $f(z)$ is close-to-convex of order d in $|z| < r_3$ where*

$$r_3 = \inf_k \left\{\frac{p-d}{k} g(k,n,p,\mu,\alpha,\lambda,\phi,\eta)\right\}^{\frac{1}{k-p}} \tag{3.6}$$

Proof. Let $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. $f(z)$ is close-to-convex of order $d, 0 \leq d < p$ in $|z| < r_3$ if $Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > d$ which is equivalent to

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| < p - d, \tag{3.7}$$

simplifying we get the required result for $g(k,n,p,\mu,\alpha,\lambda,\phi,\eta)$ given in (3.3). □

THEOREM 7. *Let* $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k$ $(a_{k,j} \geq 0, p, n \in I\!N, j = 1, 2, \cdots, \ell)$ *be in the class* $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. *Then the function* $h(z)$ *defined by*

$$h(z) = \frac{1}{\ell}\sum_{j=1}^{\ell} f_j(z) \tag{3.8}$$

also belongs to the class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$.

Proof. By the definition of $h(z)$ we have

$$h(z) = z^p - \sum_{k=n+p}^{\infty}\left(\frac{1}{\ell}\sum_{j=1}^{\ell} a_{k,j}\right) z^k \tag{3.9}$$

since $f_j(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$ $(j = 1, 2, \cdots, \ell)$ by Theorem 1 we have inequality (2.1) with a_k replaced by $a_{k,j}$. Consequently, $h(z)$ can be easily shown to be in $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. □

4. Results on modified Hadamard product

THEOREM 8. *Let the function* $f(z)$ *and* $g(z)$ *be defined by*

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \tag{4.1}$$

and

$$g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \tag{4.2}$$

belong to $K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$ *and* $K_\mu^{\lambda,\phi,\eta}(n;p;\xi)$*, respectively. Also assume that*

$$s(p+1)=\frac{\Gamma(2+p)\Gamma(2+p+\eta-\gamma)[1+(\mu(p-\phi)]}{\Gamma(2+p-\gamma)\Gamma(2+p+\beta+\eta)}\le 1\ (p\in I\!N).$$

Then $(f*g)(z)\in K_\mu^{\lambda,\phi,\eta}(n;p;\delta)$ *where*

$$\delta=\frac{\alpha\xi s(p)(k-p)}{s(p+1)(k-p+\alpha)(k-p+\xi)-\alpha\xi s(p)} \tag{4.3}$$

The function s(k) is given by

$$s(k)=\frac{\Gamma(1+k)\Gamma(1+k+\eta-\phi)[1+\mu(k-\phi-1)]}{\Gamma(1+k-\phi)\Gamma(1+k+\eta-\lambda)} \tag{4.4}$$

$(k\ge n+p;n;p\in I\!N)$.

The result is best possible for

$$f(z)=z^p-\frac{\alpha s(p)}{(k-p+\alpha)s(p+1)}z^{p+n}$$

$$g(z)=z^p-\frac{\xi s(p)}{(k-p+\xi)s(p+1)}z^{p+n}.$$

Proof. To prove the theorem it is sufficient to show that

$$\sum_{k=p+n}^{\infty}\frac{(k-p+\delta)s(k)}{\delta s(p)}a_kb_k\le 1 \tag{4.5}$$

where $s(k)$ is defined by (4.4).

Now, $f(z)\in K_\mu^{\lambda,\phi,\eta}(n;p;\alpha)$ and $g(z)\in K_\mu^{\lambda,\phi,\eta}(n;p;\xi)$ and thus, we have

$$\sum_{k=p+n}^{\infty}\frac{(k-p+\alpha)s(k)}{\alpha s(p)}a_k\le 1 \tag{4.6}$$

$$\sum_{k=p+n}^{\infty}\frac{(k-p+\xi)s(k)}{\xi s(p))}b_k\le 1. \tag{4.7}$$

Applying Cauchy-Schwarz inequality to (4.6) and (4.7), we get

$$\sum_{k=p+n}^{\infty}\frac{s(k)\sqrt{(k-p+\alpha)(k-p+\xi)}}{s(p)\sqrt{\alpha\xi}}\sqrt{a_kb_k}\le 1 \tag{4.8}$$

In view of (4.5) it suffices to show that

$$\sum_{k=p+n}^{\infty} \frac{(k-p+\delta)s(k)}{\delta s(p)} a_k b_k \leq \sum_{k=p+n}^{\infty} \frac{s(k)\sqrt{(k-p+\alpha)(k-p+\xi)}}{s(p)\sqrt{\alpha\xi}} \sqrt{a_k b_k}$$

or equivalently

$$\sqrt{a_k b_k} \leq \frac{\delta\ \sqrt{(k-p+\alpha)(k-p+\xi)}}{(k-p+\delta)\sqrt{\alpha\xi}} \text{ for } k \geq p+1 \tag{4.9}$$

In view of (4.8)and (4.9) it enough to show that

$$\frac{s(p)\sqrt{\alpha\xi}}{s(k)\sqrt{(k-p+\alpha)(k-p+\xi)}} \leq \frac{\delta\ \sqrt{(k-p+\alpha)(k-p+\xi)}}{(k-p+\delta)\sqrt{\alpha\xi}}$$

which yields

$$\delta \geq \frac{\alpha\xi s(p)(k-p)}{s(k)(k-p+\alpha)(k-p+\xi) - \alpha\xi s(p)} \tag{4.10}$$

$s(k)$ is a decreasing function of k for $(k \geq (p+1))$, we have

$$\delta = \frac{\alpha\xi s(p)(k-p)}{s(p+1)(k-p+\alpha)(k-p+\xi) - \alpha\xi s(p)}$$

Hence the proof is complete. □

Corollary 12. *Let the function $f(z)$ and $g(z)$ be defined by (4.1) and(4.2), belong to the class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. Then $(f*g)(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\delta)$ where*

$$\delta = \frac{\alpha^2 s(p)(k-p)}{s(p+1)(k-p+\alpha)^2 - \alpha^2 s(p)}$$

Corollary 13. *Let the function $f(z)$ define by (4.1) be in the class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. Also let*

$$g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \text{ for } (|b_k| \leq 1, n \in I\!N)$$

Then $(f*g)(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$.

Corollary 14. *Let the function $f(z)$ define by (4.1) be in the class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. Also let*

$$g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \text{ for } (0 \leq b_k \leq 1, n \in I\!N)$$

Then $(f * g)(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$.

5. Integral transform of the class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$

The Komatu integral operator [7] of the function $f(z)$is defined by

$$H(z) = P_{c,p}^{d} f(z) = \frac{(c+p)^d}{\Gamma(d) z^c} \int_0^z t^{c-1} (\log \frac{z}{t})^{d-1} f(t) dt \tag{5.1}$$

$$(d > 0, c > -p, z \in U)$$

THEOREM 9. *Let the function defined by (1.1)be in the class* $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. *Also let* $(d > 0, c > -p, z \in U)$, *then the function* $H(z)$ *defined by (5.1) is also in the class* $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$.

Proof. From the definition of $H(z)$ we notice that

$$H(z) = z^p - \sum_{k=p+n}^{\infty} \left(\frac{c+p}{c+k} \right)^d a_k z^k. \tag{5.2}$$

Since $\left(\frac{c+p}{c+k}\right)^d \leq 1$, and in view of Theorem 1, the result follows.

Setting $c = 1 - p, d = 1$ in (5.1) we obtain another integral operator

$$F(z) = z^{p-1} \int_0^z t^{-p} f(t) dt. \tag{5.3}$$

Again if follows from Theorem 9 that $F(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. □

THEOREM 10. *Let* $(d > 0, c > -p, z \in U)$. *Also let* $H(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. *Then* $H(z)$ *given by (5.1) is p-valent in the disc* $|z| \leq r_4$, *where*

$$r_4 = \inf_k \left\{ \frac{ph(k)(k-p+\alpha)}{k\alpha h(p)} \right\}^{\frac{1}{k-p}}. \tag{5.4}$$

Proof. We have

$$H(z) = z^p - \sum_{k=p+n}^{\infty} \left(\frac{c+p}{c+k} \right)^d a_k z^k \tag{5.5}$$

We need to show that

$$\left| \frac{H'(z)}{z^{p-1}} - p \right| \leq p \text{ in } |z| < r_4 \tag{5.6}$$

where r_4 is given by (5.4).

In view of (5.5), we have

$$\left|\frac{H'(z)}{z^{p-1}} - p\right| = \left|-\sum_{k=p+n}^{\infty} k\left(\frac{c+p}{c+k}\right)^d a_k z^{k-p}\right| \leq \sum_{k=p+n}^{\infty} k\left(\frac{c+p}{c+k}\right)^d a_k |z|^{k-p}.$$

The last inequality is bounded above by p if

$$\sum_{k=p+n}^{\infty} \frac{k}{p}\left(\frac{c+1}{c+k}\right)^d a_k |z|^{k-p} \leq 1. \tag{5.7}$$

But $H(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$ and hence by Theorem 1, we have

$$\sum_{k=p+n}^{\infty} \frac{(k-p+\alpha)s(k)\left(\frac{c+p}{c+k}\right)^d}{\alpha s(p)} a_k \leq 1. \tag{5.8}$$

Thus (5.7) and hence (5.6) will hold if

$$|z| \leq \frac{ps(k)(k-p+\alpha)}{k\alpha s(p)}^{\frac{1}{k-p}} \quad \text{for } k \geq p+n, n \in I\!N.$$

This leads to precisely the main assertion of the theorem by setting $|z| = r_4$. □

6. Extreme points of the class $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$

THEOREM 11. *Let* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{\alpha s(p)}{(k-p+\alpha)s(k)} z^k, \ (k \geq p+1).$$

Then $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$ *, if and only if,* $f(z)$ *can be expressed in the form*

$$f(z) = \sum_{k=p+n}^{\infty} \lambda_k f_k(z) \ (n \in I\!N_0) \tag{6.1}$$

where $\lambda_k \geq 0$ *and* $\sum_{k=p+n}^{\infty} \lambda_k = 1$.

Proof. Let $f(z)$ be expressible in the form

$$f(z) = \sum_{k=p+n}^{\infty} \lambda_k f_k(z) \ (n \in I\!N_0)$$

$$= z^p - \sum_{k=p+n}^{\infty} \frac{\alpha s(p)}{(k-p+\alpha)s(k)} \lambda_k z^k \ (n \in I\!N).$$

Now

$$\sum_{k=p+n}^{\infty} \frac{\alpha s(p)}{(k-p+\alpha)s(k)} \frac{(k-p+\alpha)s(k)}{\alpha s(p)} = \sum_{k=p+n}^{\infty} \lambda_k = 1 - \lambda_p \leq 1 \ (n \in I\!N).$$

Therefore, $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. Conversely, suppose that $f(z) \in K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$. Then setting

$$\lambda_k = \frac{\alpha s(p)}{(k-p+\alpha)s(k)} a_k \text{ and } \lambda_p = 1 - \sum_{k=p+n}^{\infty} \lambda_k \ (n \in I\!N).$$

Thus we notice that $f(z)$ can be expressed in the form (6.1). □

COROLLARY 15. *The extreme points of the class* $K_{\mu}^{\lambda,\phi,\eta}(n;p;\alpha)$ *are* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{\alpha s(p)}{(k-p+\alpha)s(k)} z^k, \ k \geq p+1.$$

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