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LERAY-SCHAUDER DEGREE THEORY APPLIED TO THE PERTURBED PARABOLIC PROBLEM

Tacksun Jung and Q-Heung Choi*

Abstract. We show the existence of at least four solutions for the perturbed parabolic equation with Dirichlet boundary condition and periodic condition when the nonlinear part cross two eigenvalues of the eigenvalue problem of the Laplace operator with boundary condition. We obtain this result by using the Leray-Schauder degree theory, the finite dimensional reduction method and the geometry of the mapping. The main point is that we restrict ourselves to the real Hilbert space instead of the complex space.

1. Introduction

Let Ω be a bounded, connected open subset of R^n with smooth boundary $\partial\Omega$ and let Δ be the Laplace operator. In this paper we investigate the multiple solutions for the following perturbed parabolic equation with Dirichlet boundary condition and the periodic condition

$$D_t u = \Delta u + bu^+ - au^- + f_0(u) - s\phi_1 - h(x,t) \qquad \text{in } \Omega \times R, \quad (1.1)$$
$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in R,$$
$$u(x,t) = u(x,t+2\pi), \qquad \text{in } \Omega \times R,$$

where $\lim_{|\zeta|\to\infty} \frac{f_0(\zeta}{\zeta} = 0$ and $h(x,t)$ is a bounded function with the boundary condition and the periodic condition in (1.1). The physical model for this kind of the jumping nonlinearity problem can be furnished by travelling waves in suspension bridges. Jung and Choi [3] showed that

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*Corresponding author.

the problem (1.1) with $f(x,t) = 0$ and $h(x,t) = 0$ has at least four solutions by the finite dimensional reduction method and the geometry of the mapping from the finite dimensional subspace to the finite dimensional method. The nonlinear equations with jumping nonlinearity have been extensively studied by McKenna and Walter [8], Tarantello [14], Micheletti and Pistoia [10,11] and many the other authors. Tarantello, Micheletti and Pistoia dealt with the biharmonic equations with jumping nonlinearity and proved the existence of nontrivial solutions by degree theory and critical points theory. Lazer and McKenna [7] dealt with the one dimensional elliptic equation with jumping nonlinearity for the existence of nontrivial solutions by the global bifurcation method. For the multiplicity results of the solutions of the nonlinear parabolic problem we refer to [6, 9].

The steady-state case of (1.1) is the elliptic problem

$$\Delta w + bw^+ - aw^- + f_0(w) - s\phi_1 - h(x) = 0 \qquad \text{in } \Omega, \tag{1.2}$$
$$w = 0 \qquad \text{on } \partial\Omega.$$

For the multiplicity results for the solutions of (1.2) we refer to [9].

We observe that $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to \infty$ are the eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u|_{\partial\Omega} = 0$ and ϕ_k is the eigenfunction corresponding to the eigenvalue λ_k for each k. We note that the first eigenfunction $\phi_1(x) > 0$.

The purpose of this paper is to find the number of weak solutions of (1.1)

The main results are the following:

Theorem 1.1. *Assume that $a < \lambda_1 < \lambda_2 < b < \lambda_3$ and $s > 0$. Then there exists $s_0 > 0$ such that if $s \geq s_0$, (1.1) has at least four periodic solutions.*

Generally we have the following result:

Theorem 1.2. *Assume that $\lambda_n < a < \lambda_{n+1} < \lambda_{n+2} < b < \lambda_{n+3}$, $n \geq 1$, and $s > 0$. Then there exists $s_0 > 0$ such that if $s \geq s_0$, (1.1) has at least four periodic solutions.*

For the proof of Theorem 1.1 and Theorem 1.2 we use the Leray-Schauder degree theory, the finite dimensional reduction method and the geometry of the mapping from the finite dimensional subspace to the finite dimensional subspace. The organization of this paper is the following: In section 2 we introduce the Hilbert space H whose elements are expressed by the square integrable Fourier series expansions on $\Omega \times (0, 2\pi)$, consider the parabolic problem (1.2) on H and obtain some results on the operator $D_t - \Delta$. In section 3 we deal with the multiplicity of the solutions of the piecewise linear case of (1.1) by the degree theory and finite dimensional reduction method. In section 4 we obtain the multiplicity result of the nonlinear perturbed case of (1.1) from that of the piecewise linear case so that we prove Theorem 1.1 and Theorem 1.2.

2. Some results for the operator $D_t - \Delta$ on the Hilbert space H

Let Q be the space $\Omega \times (0, 2\pi)$. The space $L_2(\Omega \times (0, 2\pi))$ is a Hilbert space equipped with the usual inner product

$$< v, w >= \int_0^{2\pi} \int_\Omega v(x,t)\bar{w}(x,t)dxdt$$

and a norm

$$\|v\|_{L_2(Q)} = \sqrt{< v, v >}.$$

We shall work first in the complex space $L_2(\Omega \times (0, 2\pi))$ but shall later switch to the real space. The functions

$$\Phi_{jk}(x,t) = \phi_k \frac{e^{ijt}}{\sqrt{2\pi}}, \qquad j = 0, \pm 1, \pm 2, \ldots, \ k = 1, 2, 3, \ldots$$

form a complete orthonormal basis in $L_2(\Omega \times (0, 2\pi))$. Every elements $v \in L_2(\Omega \times (0, 2\pi))$ has a Fourier expansion

$$v = \sum_{jk} v_{jk}\Phi_{jk}$$

with $\sum |v_{jk}|^2 < \infty$ and $v_{jk} =< v, \Phi_{jk} >$. Let us define a subspace H of $L_2(\Omega \times (0, 2\pi))$ as

$$H = \{u \in L_2(\Omega \times (0, 2\pi)) | \sum_{jk} (j^2 + \lambda_k^2)^{\frac{1}{2}} u_{jk}^2 < \infty\}. \tag{2.1}$$

Then this is a complete normed space with a norm

$$\|u\| = [\sum_{jk} (j^2 + \lambda_k^2)^{\frac{1}{2}} u_{jk}^2]^{\frac{1}{2}}.$$

A weak solution of problem (1.1) is of the form $u = \sum u_{jk}\Phi_{jk}$ satisfying $\sum |u_{jk}|^2(j^2+\lambda_k^2)^{\frac{1}{2}} < \infty$, which implies $u \in H$. Thus we have that if u is a weak solution of (1.1), then $u_t = D_t u = \sum_{j\ k} iju_{jk}\Phi_{jk}$ belong to H and $-\Delta u = \sum \lambda_k u_{jk}\Phi_{jk}$ belong to H.

We have some properties on $\|\cdot\|$ and $D_t - \Delta$. Since $|ij+\lambda_k| \geq 1$ for all j, k, we have that:

LEMMA 2.1. *(i)* $\|u\| \geq \|u(x,0)\| \geq \|u(x,0)\|_{L_2(\Omega)}$.
(ii) $\|u\|_{L_2(Q)} = 0$ *if and only if* $\|u\| = 0$.
(iii) $u_t - \Delta u \in H$ *implies* $u \in H$.

Proof. (i) Let $u = \sum_{j\ k} u_{jk}\Phi_{jk}$. Then

$$\begin{aligned}\|u\|^2 = \sum (j^2+\lambda_k^2)^{\frac{1}{2}} u_{jk}^2 &\geq \sum \lambda_k^2 u_{jk}^2(x.0) = \|u(x.0)\|^2 \\ &\geq \sum u_{jk}^2(x,0) = \|u(x,0)\|^2_{L_2(\Omega)}.\end{aligned}$$

(ii) Let $u = \sum_{j\ k} u_{jk}\Phi_{jk}$.

$$\|u\| = 0 \Leftrightarrow \sum_{j\ k}(j^2+\lambda_k^2)^{\frac{1}{2}} u_{jk}^2 = 0 \Leftrightarrow \sum_{j\ k} u_{jk}^2 = 0 \Leftrightarrow \|u\|_{L_2(Q)} = 0.$$

(iii) Let $u_t - \Delta u = f \in H$. Then f can be expressed by

$$f = \sum f_{jk}\Phi_{jk}, \qquad \sum_{j\ k}(j^2+\lambda_k^2)^{\frac{1}{2}} f_{jk}^2 < \infty.$$

Then we have

$$\|(D_t-\Delta)^{-1}f\|^2 = \sum_{j\ k}\frac{(j^2+\lambda_k^2)^{\frac{1}{2}}}{j^2+\lambda_k^2} f_{jk}^2 < C\sum_{j\ k} f_{jk}^2 < \infty$$

for some $C > 0$. □

LEMMA 2.2. *For any real* $\alpha \neq \lambda_k$, *the operator* $(D_t - \Delta - \alpha)^{-1}$ *is linear, self-adjoint, and a compact operator from* $L_2(\Omega \times (0,2\pi))$ *to* H *with the operator norm* $\frac{1}{|\alpha-\lambda_k|}$, *where* λ_k *is an eigenvalue of* $-\Delta$ *closest to* α.

Proof. Suppose that $\alpha \neq \lambda_k$. Since $\lambda_k \to +\infty$, the number of elements in the set $\{\lambda_k | \ \lambda_k < \alpha\}$ is finite, where λ_k is an eigenvalue of $-\Delta$. Let $h = \sum_{j\ k} h_{jk}\Phi_{jk}$, where $\Phi_{jk} = \phi_k \frac{e^{ijt}}{\sqrt{2\pi}}$. Then

$$(D_t - \Delta - \alpha)^{-1}h = \sum_{j\ k}\frac{1}{im+\lambda_n-\alpha} h_{jk}\Phi_{jk}.$$

Hence

$$\|(D_t-\Delta-\alpha)^{-1}h\|^2 = \sum_{j\ k} \frac{1}{j^2+(\lambda_k-\alpha)^2}(j^2+(\lambda_k-\alpha)^2)^{\frac{1}{2}}h_{jk}^2 \leq \sum_{j\ k} Ch_{jk}^2 < \infty$$

for some $C > 0$. Thus $(D_t - \Delta - \alpha)^{-1}$ is a bounded operator from $L_2(\Omega\times(0,2\pi))$ to H and also send bounded subset of $L_2(\Omega\times(0,2\pi))$ to a compact subset of H, hence $(D_t-\Delta-\alpha)^{-1}$ is a compact operator. □

From Lemma 2.2 we obtain the following lemma:

LEMMA 2.3. *Let $F(x,t,u) \in L_2(\Omega\times(0,2\pi))$. Then all the solutions of*

$$u_t - \Delta u = F(x,t,u) \qquad \text{in } L_2(\Omega\times(0,2\pi))$$

belong to H.

With the aid of Lemma 2.3 it is enough to investigate the existence of solutions of (1.1) in the subspace H of $L_2(\Omega\times(0,2\pi))$, namely

$$D_t u = \Delta u + bu^+ - au^- + f_0(u) - s\phi_1 - h(x,t) \qquad \text{in } H. \tag{2.2}$$

From now on we restrict ourselves to the real L_2-space and observe that this is an invariant space for R. So $L_2(\Omega\times(0,2\pi))$ denotes the real square-integrable functions on $\Omega\times(0,2\pi)$ and H the subspace of $L_2(\Omega\times(0,2\pi))$ satisfying (2.1).

3. The piecewise linear case

Assume that $a < \lambda_1 < \lambda_2 < b < \lambda_3$ and $s > 0$. In this section we first investigate the multiplicity of the solutions of the piecewise linear case of (1.1)

$$D_t u = \Delta u + bu^+ - au^- - s\phi_1 \qquad \text{in } H. \tag{3.1}$$

We shall use the contraction mapping theorem to reduce the problem from an infinite dimensional one in $L_2(Q)$ to a finite dimensional one.

Let V be the two dimensional subspace of H spanned by $\Phi_{01}(x)$ and $\Phi_{02}(x)$ and W the subspace spanned by Φ_{0n}, $n \geq 3$ and Φ_{mn}^c, Φ_{mn}^s, $m \geq 1$. Then W is the orthogonal complement of V in H.

From now on we restrict ourselves to the real L_2-space and observe that this is an invariant space for R. So $L_2(\Omega\times(0,2\pi))$ denotes the real square-integrable functions on $\Omega\times(0,2\pi)$ and H the subspace of $L_2(\Omega\times(0,2\pi))$ satisfying (2.1). Let P be an orthogonal projection from H onto V. Then for all $u \in H$, $u = v + w$, where $v = Pu$, $w = (I-P)u$.

Therefore (3.1) is equivalent to

$$\begin{aligned} &(a) \qquad w = (D_t - \Delta)^{-1}(I-P)(b(v+w)^+ - a(v+w)^{-1}), \\ &(b) \qquad D_t v = \Delta v + P(b(v+w)^+ - a(v+w)^- - s\phi_1), \end{aligned} \tag{3.2}$$

where $D_t = \frac{\partial}{\partial t}$.

Let us show that for fixed v, (3.2.a) has a unique solution $w = \theta(v)$ and that $\theta(v)$ is Lipschitz continuous in terms of v. Let σ be the spectrum of $D_t - \Delta$. Then $\sigma = \{\lambda_n \pm im | \ n \geq 1, \ m \geq 0\}$. Let $\alpha = \frac{1}{2}(\lambda_1 + \lambda_2)$. (3.2.a) can be rewritten as

$$(D_t - \Delta - \alpha)w = (I-P)(b(v+w)^+ - a(v+w)^{-1} - \alpha(v+w))$$

or

$$w = (D_t - \Delta - \alpha)^{-1}(I-P)g_v(w) \tag{3.3}$$

where

$$g_v(w) = b(v+w)^+ - a(v+w)^{-1} - \alpha(v+w).$$

Since

$$|g_v(w_1) - g_v(w_2)| \leq \max\{|b-\alpha|, |a-\alpha|\}|w_2 - w_1|,$$
$$\|g_v(w_1) - g_v(w_2)\| \leq \max\{|b-\alpha|, |a-\alpha|\}\||w_2 - w_1\||,$$

where $\|\cdot\|$ is the norm in H. Since the operator $(D_t - \alpha)^{-1}(I-P)$ is a self-adjoint, compact linear map from $(I-P)H$ onto itself, it follows that

$$\|(D_t - \Delta - \alpha I)^{-1}(I-P)\| = \mathrm{dist}(\alpha, \{(\lambda_n \pm im - \alpha)^{-1} | \ m \geq 0, \ n \geq 2\}).$$

Therefore for fixed $v \in V$, the right hand side of (3.3) defines a Lipschitz mapping $(I-P)H$ into itself with Lipschitz constant $\gamma < 1$. Therefore by the contraction mapping principle, for given $v \in V$, there exists a unique $w = \theta(v) \in W$ which satisfies (3.3). it follows that, by the standard argument principle, $\theta(v)$ is Lipschitz continuous in terms of v.

Thus we have a reduced equation (3.1) to the equivalent equation

$$D_t v = \Delta v + P(b(v+\theta(v))^+ - a(v+\theta(v))^- - s\phi_1) \tag{3.4}$$

defined on the two dimensional subspace PH spanned by $\{\Phi_{01}(x), \Phi_{02}(x)\}$.

We note that if $v \geq 0$ or $v \leq 0$, then $\theta(v) = 0$. If we put $v \geq 0$ ($v \leq 0$) and $\theta(v) = 0$ in (3.2.a), equation (3.2.a) is satisfied, respectively. Since $v = c_1\Phi_{01} + c_2\Phi_{02}$, there exists a cone C_1 defined by $c_1 \geq 0$, $|c_2| \leq \epsilon_0 c_1$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_3, $c \leq 0$, $|c_2| \leq \epsilon_0|c_1|$ so that $v \leq 0$ for all $v \in C_2$. We know that $w = \theta(v) = 0$ for $v \in C_1 \cup C_3$, but we do not know $\theta(v)$ for all $v \in PH$. Let C_2 be a cone defined by $c_2 < 0$, $|c_1| \leq \epsilon_0 c_2$ and a cone C_4 defined by $c_2 \leq 0$, $|c_1| \leq \epsilon_0|c_2|$.

We consider the map T from V to V by

$$v \mapsto T(v) = -D_t v + \Delta v + P(b(v+\theta(v))^+ - a(v+\theta(v))^-).$$

First we consider the image of the cone C_1. If $v = c_1\Phi_{01} + c_2\Phi_{02}$, we have that

$$\begin{aligned} T(v) &= -\lambda_1 c_1\Phi_{01} - \lambda_2 C_2\Phi_{02} + b(c_1\Phi_{01} + c_2\Phi_{02}) \\ &= (\lambda_1 - b)c_1\Phi_{01} + (\lambda_2 - b)c_2\Phi_{02}. \end{aligned}$$

Then there exists $d > 0$ such that

$$(T(c_1\Phi_{01} + c_2\Phi_{02}), \Phi_{01}) \geq d|c_2|$$

(see [3]). Hence the map $T : V \to V$ takes the value Φ_{01}, once in each of four different regions C_i $i \leq i \leq 4$ of the plane which was proved in the paper written by Jung and Choi [3]. Let us define a map $F : R^2 \to R^2$

$$F(t_1, t_2) = (s_1, s_2) \qquad \text{if } \; v = t_1\Phi_{01} + t_2\Phi_{02} \qquad \text{and}$$

$$T(v) = s_1\Phi_{01} + s_2\Phi_{02}.$$

Let us set

$$\begin{aligned} A_1 &= \{(t_1, t_2)|\ 0 < t_1 < k,\ |t_2| < t_1\}, \\ A_2 &= \{(t_1, t_2)|\ |t_1| \leq k,\ |t_1| < t_2 < k\}, \\ A_3 &= \{(t_1, t_2)|\ -k < s_1 < 0,\ |s_2| < |s_1|\}, \\ A_4 &= \{(t_1, t_2)|\ |s_1| \leq k,\ -k < s_2 < -|s_1|\}. \end{aligned}$$

Now we calculate the degree of T in the regions A_i $(1 \leq i \leq 4)$.

LEMMA 3.1. *Let $p = (0, 1)$. Let k be so large that $k > 1$, $k(b - \lambda_1) > 1$ and $kd > 1$. If deg(F, A_i, p) denotes the Brouwer degree of F with respect to A_i and p for $1 \leq i \leq 4$, then deg(F, A_i, p) is defined for $1 \leq i \leq 4$ and*

$$\text{deg}(F, A_i, p) = (-1)^{i+1}.$$

Proof. First we calculate the Brouwer degree of F with respect to A_1. If $(t_1, t_2) \in \bar{A}_1$ and $v = t_1\Phi_{01} + t_2\Phi_{02}$, then $\theta(v) = 0$. Since $v \geq 0$ in A_1, we have

$$\begin{aligned} T(v) &= -D_t v + \Delta v + P(b(v+\theta(v))^+ - a(v+\theta(v))^-) \\ &= -(t_1\Phi_{01} + t_2\Phi_{20})_t + \Delta(t_1\Phi_{01} + t_2\Phi_{02}) + P(b(t_1\Phi_{01} + t_2\Phi_{02})) \\ &= -\lambda_1 t_1\Phi_{01} - \lambda_2 t_2\Phi_{02} + b(t_1\Phi_{01} + t_2\Phi_{02}) \\ &= (b - \lambda_1)t_1\Phi_{01} + (b - \lambda 2)t_2\Phi_{02}. \end{aligned}$$

Thus we have that for $(t_1, t_2) \in \bar{A}_1$,

$$F(t_1, t_2) = ((b - \lambda_1)s_1, (b - \lambda_2)s_2).$$

Since $k(b - \lambda_1) > 1$, the equation $F(t_1, t_2) = p$ has a unique solution $(t_1, t_2) = (\frac{1}{b-\lambda_1}, 0)$. Since the determinant of the diagonal map is positive,

$$\deg(F, A_1, p) = 1.$$

Similarly in the case of $(t_1, t_2) \in \bar{A}_3$, we have

$$T(v) = (a - \lambda_1)t_1\Phi_{01} + (a - \lambda 2)t_2\Phi_{02}.$$

Thus we have

$$F(t_1, t_2) = ((a - \lambda_1)s_1, (a - \lambda_2)s_2).$$

Since the determinant of the diagonal map is positive,

$$\deg(F, A_3, p) = 1.$$

Now we calculate the Brouwer degree of F with respect to A_2. We claim that $\deg(F, A_2, p) = -1$. We note that the boundary of A_2 consists of three line segments:
(i) a ray I in the first quadrant A_1, $t_1 > 0$ and $t_2 = t_1$,
(ii) a ray II in the third quadrant A_3, $t_1 < 0$ and $t_2 = -t_1$,
(iii) a line segment L of $t_2 = k$, paralleled to the t_1 axis.
The image of I under F is a straight line segment in the first quadrant, the image of II under F is a straight line segment in the fourth quadrant and the image of L is to the right of the line $t_1 = 1$ by the condition $kd > 1$. We consider the map $u \mapsto Gu$, where G is defined by

$$G = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The image of I under F will be a straight line in the first quadrant. So if $0 \le \tau \le 1$, we have

$$\tau Gt + (1 - \tau)F(t) \neq p, \qquad t = (t_1, t_2) \in I.$$

The image of II under G is in the fourth quadrant and we have, $0 \le \tau \le 1$,

$$\tau Gt + (1 - \tau)F(t) \neq p, \qquad t = (t_1, t_2) \in II.$$

If $t \in L$, then $t_2 = k > 1$ and $Gt \in \{(t_1, t_2) | \ t_1 > 1\}$. Thus we have

$$\tau Gt + (1 - \tau)F(t) \neq p \qquad \text{for} \ \ t \in L.$$

By the homotopy argument we have

$$\deg(F, A_2, p) = \deg(G, A_2, p).$$

We note that $Gt - p$ has exactly one zero in A_2 and the sign of the determinant of G is -1. Thus we have

$$\deg(F, A_2, p) = -1.$$

Similarly we have

$$\deg(F, A_4, p) = -1.$$

Thus we prove the lemma. $\square$

From Lemma 3.1 we obtain the degree of a mapping on the finite dimensional subspace V.
Let us set

$$E_i = \{v \in V|\ v = t_1\Phi_{01} + t_2\Phi_{02}, \qquad (t_1, t_2) \in A_i \qquad \text{for}\ \ 1 \le i \le 4.\}$$

Let us define a map $\Gamma : V \to V$ by

$$\Gamma v = PL^{-1}(b(v + \theta(v))^+ - a(v + \theta(v))^-),$$

where $Lu = -u_t + \Delta u$.

From Lemma 3.1 we obtain the following lemma.

Lemma 3.2. *For $1 \le i \le 4$,*

$$deg(I + \Gamma, E_i, \frac{\Phi_{01}}{\lambda_1}) = (-1)^{i+1}.$$

From now on we restrict ourselves to the real L_2-space and $L_2(\Omega \times (0, 2\pi))$ denotes the real square-integrable functions on $\Omega \times (0, 2\pi)$ and H the subspace of $L_2(\Omega \times (0, 2\pi))$ satisfying (2.1). Now we shall obtain a result on the degree of a mapping on the infinite dimensional space H from the degree of the mapping on the two dimensional subspace V. Let us define the mapping $N : H \to H$ by

$$Nu = L^{-1}(bu^+ - au^-).$$

We note that N is a compact operator from H to H. Let us set

$$X_i = \{u \in H)|\ Pu \in E_i,\ \|(I-P)u\| < M_1\} \qquad \text{for large number } M_1 > 0.$$

Then the Leray-Schauder degree $\deg(I + N, X_i, \frac{\Phi_{01}}{\lambda_1})$ is well defined.

Lemma 3.3. *Let $M_1 > 0$ be a large number. Then we have*

$$d(I + N, X_i, \frac{\Phi_{01}}{\lambda_1}) = d(I + \Gamma, E_i, \frac{\Phi_{01}}{\lambda_1}) = (-1)^{i+1}.$$

Proof. Let $v \in \bar{E}_i$ and $w = (1-t)(I-P)N(v+w)$, $1 \le i \le 4$. Since $w \mapsto (1-t)(I-P)N(v+w)$, $0 \le t \le 1$ is a contraction mapping, there exists $M_1 > 0$ such that $\|w\| \le M_1$. Let us choose $M_2 > M_1$ and define $\Psi_1 : X_i \times [0,1] \to L^2$ by

$$\Psi_1(u,t) = (I-P)N(v+w) + PN(v+w+t(\theta(v)-w)),$$

where $v = Pu$ and $w = (I-P)u$. Then we have

$$u + \Psi_1(u,t) \ne \frac{\Phi_{01}}{\lambda_1} \qquad \text{for } (u,t) \in \partial X_i \times [0,1].$$

By the homotopy invariance property of degree

$$d(I+N, X_i, \frac{\Phi_{01}}{\lambda_1}) = d(I+\Psi_1(\cdot,1), X_i, \frac{\Phi_{01}}{\lambda_1}).$$

Let $\Psi_2|X_i \times [0,1] \to L^2(Q)$ be defined by

$$\Psi_2(u,t) = (1-t)(I-P)N(u) + PN(v+\theta(v)), \qquad v = Pu.$$

We claim that

$$u + \Psi_2(u,t) \ne \frac{\Phi_{01}}{\lambda_1} \qquad \text{for } (u,t) \in \partial X_i \times [0,1].$$

In fact, if $v \in \partial E_i$, $w = (I-P)H$, $\|w\| = M_2$, $0 \le t \le 1$, $u = v+w$ and $u + \Psi_2(u,t) = \frac{\Phi_{01}}{\lambda_1}$, then

$$0 = (I-P)(u+\Psi_2(u,t)) = w + (1-t)(I-P)N(v+w),$$

which implies that $\|w\| \le M_2$ which is a contradiction. We note that $\Psi_1(u,1) = \Psi_2(u,0)$. By the homotopy invariance property of degree we have

$$d(I+N, X_i, \frac{\Phi_{01}}{\lambda_1}) = d(I+\Psi_2(\cdot,1), X_i, \frac{\Phi_{01}}{\lambda_1}).$$

Let B be the open ball of radius M_2 in $(I-P)H$. If $u \in \bar{X}_i$, $v = Pu$, $w = (I-P)u$, then

$$u + \Psi_2(u,1) = v + PN(v+\theta(v)) + w.$$

The map $w \mapsto w + \Psi_2(u,1)$ is uncoupled an $PH \oplus (I-P)H$ and is the identity on $(I-P)H$. Therefore by the product property of degree we have

$$d(I+N, X_i, \frac{\Phi_{01}}{\lambda_1}) = d(I+\Gamma, X_i, \frac{\Phi_{01}}{\lambda_1}).$$

Thus we prove the lemma. □

4. The proof of Theorem 1.1 and Theorem 1.2

Now we consider the multiplicity of the solutions of the nonlinear perturbed problem

$$D_t u = \Delta u + bu^+ - au^- + f_0(u) - s\phi_1 - h(x,t) \qquad \text{in } \ H. \tag{4.1}$$

We shall obtain the multiplicity result for the nonlinear perturbed problem from the piecewise linear one. Let

$$f_1(u) = bu^+ - au^-.$$

Then (4.1) can be rewritten as

$$-D_t z + \Delta z + f_1(z) + \frac{f_0(sz)}{s} = \Phi_{01} + \frac{h(x,t)}{s}, \tag{4.2}$$

where $z = \frac{u}{s}$. Let

$$N_s(z) = L^{-1}(f_1(z) + \frac{f_0(sz)}{s} - \frac{h(x,t)}{s}).$$

We note that

$$\lim_{s\to\infty} \|N(z) - N_s(z)\| = 0$$

uniformly for z in bounded subsets of $L_2(Q)$.

In section 3 we show that

$$z + N(z) \neq \frac{\Phi_{01}}{\lambda_1} \qquad \text{for all } \ z \in \partial X_i, \ 1 \le i \le 4.$$

Since ∂X_i is closed and bounded and N is continuous and compact, there exists $\eta > 0$ such that

$$\|z + N(z) - \frac{\Phi_{01}}{\lambda_1}\| \ge \eta \qquad z \in \partial X_i.$$

Now we choose s_0 so that

$$\|N_s(z) - N(z)\| < \frac{\eta}{2} \qquad \text{for all } \ z \in \partial X_i, \ 1 \le i \le 4.$$

Then

$$\|z + N(z) + (1-\tau)(N_s(z) - N(z) - \frac{\Phi_{01}}{\lambda_1})\| \ge \frac{\eta}{2}$$

for $0 \le \tau \le 1$. Thus we have

$$d(I + N_s, X_i, \frac{\Phi_{01}}{\lambda_1}) = d(I + N, X_i, \frac{\Phi_{01}}{\lambda_1}) = (-1)^{i+1}, \ \ 1 \le i \le 4.$$

Thus we prove Theorem 1.1. For the proof we set V the two dimensional subspace spanned by $\Phi_{0\ n+1}(x)$ and $\Phi_{0\ n+2}(x)$ and W the complement of V in H. The other parts of the proof of Theorem 1.2 are similar to that of Theorem 1.1.

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Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail: tsjung@kunsan.ac.kr

Department of Mathematics Education
Inha University
Incheon 402-751, Korea
E-mail: qheung@inha.ac.kr