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ON REFLEXIVE PRINCIPALLY QUASI-BAER RINGS

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Abstract. We investigate in this paper some equivalent conditions for right principally quasi-Baer rings to be reflexive. Using these results we are able to prove that if R is a reflexive right principally quasi-Baer ring then R is a left principally quasi-Baer ring. In addition, for an idempotent reflexive principally quasi-Baer ring R we show that R is prime if and only if R is torsion free.

Throughout this paper, R denotes an associative ring with identity. According to Clark [3], a ring R is called quasi-Baer if the right annihilator of each right ideal is generated (as a right ideal) by an idempotent. He considered quasi-Baer rings for characterizing a finite dimensional algebra over an algebraically closed field to be a twisted semigroup algebra of a matrix units semigroup. Recently principally quasi-Baer ring were defined in [2] to provide a lattice connection between the right principally quasi-Baer and quasi-Baer conditions. They also provide numerous examples and develop several basic results. For example it was proved that if R is semiprime then R is right principally quasi-Baer if and only if R is left principally quasi-Baer. We will show in this paper that there are several equivalent conditions which are sufficient to yield the above result. Actually we prove that if R is reflexive then R is right principally quasi-Baer if and only if R is left principally quasi-Baer. Also we show that if R is an idempotent reflexive principally quasi-Baer ring then R is prime if and only if R is torsion free.

For a nonempty subset A of R, $r(a)$ and $l(a)$ denote the right and left annihilators of A, respectively.

Recall that an idempotent in a ring R is called *left (resp. right) semicnetral* if $xe = exe$ *(resp.* $ex = exe$*)* for all $x \in R$. It can be easily

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checked that an idempotent e is left (resp. right) semicental if and only if eR (resp. Re) is an ideal of R. For a ring R, $S_l(R)$ (resp. $S_r(R)$) denotes the set of all left (resp. right) semicentral idempotents of R. We say that an idempotent $e \in R$ is called *semicentral reduced* if $S_l(eRe) = \{0, e\}$. Note that e is semicentral reduced if and only if $S_r(eRe) = \{0, e\}$. If 1 is semicentral reduced, then we say that R is semicentral reduced.

Definition 1. *A ring R is right (resp. left) principally quasi-Baer (or simply right (resp. left)* pq*-Baer) [2] if the right (resp. left) annihilator of a principally right (resp/ left) ideal is generated (as a right (resp. left) ideal) by an idempotent.*

Note that biregular rings and quasi-Baer rings are *pq*-Baer.

Definition 2. *A left ideal I is said to be reflexive [5] if $aRb \subseteq I$ implies $bRa \subseteq I$ for $a, b \in R$. A ring R is called reflexive if 0 is a reflexive ideal.*

Recall that a ring R is said to be *idempotent reflexive* [4] if $aRe = 0$ implies $eRa = 0$ for $a, e = e^2 \in R$.

Remark 3. *Note that a ring R is reflexive if and only if $r(aR) = l(Ra)$ for all $a \in R$. Similarly we can observe that a ring R is idempotent reflexive if and only if $r(eR) = l(Re)$ for all $e = e^2 \in R$.*

The next proposition contains [2, Proposition 1.17]

Proposition 4. *Let R be a right* pq*-Baer ring. Then the following conditions are equivalent:*

(1) *R is semiprime.*
(2) *R is reflexive.*
(3) *$S_l(R) = S_r(R) = B(R)$, where $B(R)$ is the set of all central idempotents of R.*
(4) *R is idempotent reflexive.*

Proof. (1) $\Rightarrow$ (2) : Let $aRb = 0$ for $a, b \in R$. Then $(bRaR)(bRaR) = bR(aRb)RaR = 0$. Since R is semiprime, $bRaR = 0$. Hence $bRa = 0$.
(2) $\Rightarrow$ (3) : Let $e \in S_l(R)$. Then $(1 - e)Re = 0$, so $eR(1 - e) = 0$ since R is reflexive. Hence $ex = exe = xe$ for all $x \in R$. Therefore $S_l(R) = B(R)$. Similarly we have $S_r(R) = B(R)$.
(3) $\Rightarrow$ (4) : Let $e = e^2 \in R$. We will show that $r(eR) = l(Re)$. Let $x \in r(eR) = fR$ where $f \in S_l(R) = B(R)$. Thus $eRf = 0$, so $fRe = 0$. Hence $xRe = 0$, so $x \in l(Re)$. Similarly we have $l(Re) \subseteq r(eR)$. Hence

$r(eR) = l(Re)$.
(4) $\Rightarrow$ (1) : Suppose that $aRa = 0$ for $a \in R$. Then $a \in r(aR) = eR$ where $e \in S_l(R)$. Thus $aRe = 0$, so $eRa = 0$ since R is idempotent reflexive. ?Hence $a \in r(eR) = fR$, where $f \in S_l(R)$. Now $a = eb$, $a = fc$ and $eRf = 0$ for some $b, c \in R$. Therefore $a = ea = efc = 0$. □

Note that if R is idempotent reflexive, then $S_l(R) = S_r(R) = B(R)$. Obviously any Abelian ring is idempotent reflexive but the converse is not true. However we have the next proposition.

PROPOSITION 5. *The following conditions are equivalent :*
(1) *R is Abelian.*
(2) *$r(e) = r(eR)$ for every $e = e^2 \in R$.*

Recall from [1] that R is said to satisfy the IFP (insertion of factors property) if $r(a)$ is an ideal for all $a \in R$. Observe that if R has the IFP then $r(a) = r(aR)$ for all $a \in R$. Now we will give a nice characterization of reduced rings.

PROPOSITION 6. *The following are equivalent :*
(1) *R is a reduced ring.*
(2) *R has the IFP and every nonzero principal right ideal is not nil.*

Proof. (1) $\Rightarrow$ (2) : Clear.
(2) $\Rightarrow$ (1) : Suppose that $a^2 = 0$ and $a \neq 0$. Then aR is not nil. Thus there is $c \in aR$ such that c is not nilpotent. Now $c = ab$ for some $b \in R$, hence $ac = a^2b = 0$. Since $r(a)$ is two-sided ideal of R and $c \in r(a)$, $bc \in r(a)$. Hence $abc = c^2 = 0$. Therefore c is nilpotent, it is absurd. □

Using Proposition 4, we can give another proof of [2, Corollary 1.11] as follows.

PROPOSITION 7. *Let R be a reflexive ring. Then R is right* pq-*Baer if and only if R is left* pq-*Baer.*

Proof. Let R be a right *pq*-Baer and $a \in R$. Then we have $r(aR) = eR$ where $e \in B(R)$. By Remark 3, $l(Ra) = r(aR) = eR = Re$. Hence R is left *pq*-Baer. Similarly we can get the converse. □

Recall [2 , Lemma 1.2] that R is a semicentral reduced *pq*-Baer ring if and only if R is a prime ring. For a prime ideal P, $O(P) = \{a \in R \mid aRb = 0 \text{ for some } b \in R \setminus P\}$. According to [6], a ring R is called *torsion free* if there exists a prime ideal P of R such that $O(P) = 0$.

Proposition 8. *Let R be an idempotent reflexive* pq-*Baer ring. Then following conditions are equivalent:*

(1) *R is prime.*
(2) *R is torsion free.*

Proof. (1) $\Rightarrow$ (2) : Since R is prime, we have $O(0) = \{a \in R \mid aRs = 0$ for some $s \in R \setminus \{0\}\} = \{0\}$. Hence R is torsion free.
(2) $\Rightarrow$ (1) : Let P be a prime ideal of R such that $O(P) = 0$. By [2, Lemma 1.2] it is enough to show that $S_l(R) = \{0, 1\}$. Let $e \in S_l(R)$. Then $(1-e)Re = 0$. Since R is idempotent reflexive, we have $eR(1-e) = 0$. Hence $e \in B(R)$. If $e \notin P$, then $1 - e \in O(P) = \{0\}$. Thus $e = 1$. If $e \in P$, then $1 - e \notin P$. Thus $e \in O(P) = \{0\}$, so $e = 0$. Therefore R is semicentral reduced. □

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