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PROJECTIVE AND INJECTIVE PROPERTIES OF REPRESENTATIONS OF A QUIVER $Q = \bullet \to \bullet \to \bullet$

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Abstract. We define injective and projective representations of a quiver $Q = \bullet \to \bullet \to \bullet$. Then we show that a representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is projective if and only if each M_1, M_2, M_3 is projective left R-module and $f_1(M_1)$ is a summand of M_2 and $f_2(M_2)$ is a summand of M_3. And we show that a representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is injective if and only if each M_1, M_2, M_3 is injective left R-module and $ker(f_1)$ is a summand of M_1 and $ker(f_2)$ is a summand of M_2.

1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) ([1]). We may consider many different types of quivers. We allow multiple edges and multiple arrows, and edges going from a vertex back to the same vertex. Originally a representation of quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex to the one assigned to the terminal vertex. For example, a representation of the quiver $Q = \bullet \to \bullet$ is $V_1 \xrightarrow{f} V_2$, V_1 and V_2 are vector spaces and f is a linear map (morphism). Then we can define a morphism of two representations of the same quiver i.e., given a quiver $Q = \bullet \to \bullet$, we can define two representations $V_1 \xrightarrow{f} V_2$ and $W_1 \xrightarrow{g} W_2$.

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Now we can define a morphism between these two representations. A morphism of $V_1 \xrightarrow{f} V_2$ to $W_1 \xrightarrow{g} W_2$ is given by a commutative diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{f} & V_2 \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle s_2} \\ W_1 & \xrightarrow[g]{} & W_2 \end{array}$$

with s_1, s_2 linear maps.

In ([3]) a homotopy of quiver was developed and in ([2]) cyclic quiver ring was studied. The theory of projective representations was developed in ([4]) and the theory of injective representation was studied in ([5]). Recently, in ([7]) injective covers and envelopes of representations of linear quivers was studied, and in ([6]) properties of multiple edges of quivers was studied.

2. Projective representation of a quiver $Q = \bullet \to \bullet \to \bullet$

Definition 2.1. A representation $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is called a projective representation if every diagram of representations

$$\begin{array}{ccccccccccccccccc} & & & & & & (P_1 & \xrightarrow{f_1} & P_2 & \xrightarrow{f_2} & P_3) & & & & & & \\ & & & & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & & & & & \\ (M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3) & \longrightarrow & (N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow & (0 & \longrightarrow & 0 & \longrightarrow & 0) \end{array}$$

can be completed to a commutative diagram as follows :

$$\begin{tikzcd} & & & & & & (P_1 \arrow[r, "f_1"] \arrow[d, "\alpha"] \arrow[llllll, dotted, "s"'] & P_2 \arrow[r, "f_2"] \arrow[d, "\beta"] \arrow[lllll, dotted, "t"'] & P_3) \arrow[d, "\gamma"] \arrow[llll, dotted, "u"'] & & & & \\ (M_1 \arrow[r, "g_1"] & M_2 \arrow[r, "g_2"] & M_3) \arrow[r] & (N_1 \arrow[r, "h_1"] & N_2 \arrow[r, "h_2"] & N_3) \arrow[r] & (0 \arrow[r] & 0 \arrow[r] & 0) \end{tikzcd}$$

Theorem 2.2. *If* $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$ *is a projective representation of a quiver* $Q = \bullet \to \bullet \to \bullet$, *then* P_1, P_2, *and* P_3 *are projective left* R*-modules.*

Proof. Let M, N be left R-modules and $\alpha : P_1 \to N$ be an R-linear map and $k : M \to N$ be an onto R-linear map. Then since $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$ is a projective representation we can complete the following diagram

$$
\begin{array}{ccccc}
 & & (P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) & & \\
 & & \downarrow{\scriptstyle \alpha} \quad \downarrow{\scriptstyle 0} \quad \downarrow{\scriptstyle 0} & & \\
(M \longrightarrow 0 \longrightarrow 0) & \longrightarrow & (N \longrightarrow 0 \longrightarrow 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0)
\end{array}
$$

as a commutative diagram. Thus P_1 is a projective left R-module.

Let $\beta : P_2 \to N$ be a R-linear map and $k : M \to N$ be a onto R-linear map. Then since $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$ is a projective representation we can complete the following diagram

$$
\begin{array}{ccccc}
 & & (P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) & & \\
 & & \downarrow{\scriptstyle \beta f_1} \quad \downarrow{\scriptstyle \beta} \quad \downarrow{\scriptstyle 0} & & \\
(M \xrightarrow{id} M \longrightarrow 0) & \longrightarrow & (N \xrightarrow{id} N \longrightarrow 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0)
\end{array}
$$

as a commutative diagram. Thus P_2 is a projective left R-module.

Let $\gamma : P_3 \to N$ be an R-linear map and $k : M \to N$ be an onto R-linear map. Then since $P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3$ is a projective representation we can complete the following diagram

$$
\begin{array}{ccccc}
 & & (P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) & & \\
 & & \downarrow{\scriptstyle \gamma f_2 f_1} \quad \downarrow{\scriptstyle \gamma f_2} \quad \downarrow{\scriptstyle \gamma} & & \\
(M \xrightarrow{id} M \xrightarrow{id} M) & \longrightarrow & (N \xrightarrow{id} N \xrightarrow{id} N) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0)
\end{array}
$$

as a commutative diagram. Thus P_3 is a projective left R-module. □

Lemma 2.3. *If P is a projective left R-module, then a representation $0 \longrightarrow 0 \longrightarrow P$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is a projective representation.*

Proof. The lemma follows by completing the diagram

$$\begin{array}{ccccccccccccccc}
 & & & & & & (0 & \longrightarrow & 0 & \longrightarrow & P) & & & & \\
 & & & & & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma} & & & & \\
(M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3) & \longrightarrow & (N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) & &
\end{array}$$

as a commutative diagram. □

Lemma 2.4. *If P is a projective left R-module, then a representation $0 \longrightarrow P \xrightarrow{id} P$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is a projective representation.*

Proof. Let $\beta : P \longrightarrow N_2$ be an R-linear map and $k_2 : M_2 \longrightarrow N_2$ be an onto R-linear map and choose $\beta h_2 : P \longrightarrow N_3$ as an R-linear map. Then since P is a projective left R-module, there exist $t : P \longrightarrow M_2$ such that $k_2 t = \beta$. Now choose $g_2 t : P \longrightarrow M_3$ as an R-linear map. Then t and $g_2\alpha$ complete the following diagram

$$\begin{array}{ccccccccccccccc}
 & & & & & & (0 & \longrightarrow & P & \xrightarrow{id} & P) & & & & \\
 & & & & & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle h_2\beta} & & & & \\
(M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3) & \longrightarrow & (N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) & &
\end{array}$$

as a commutative diagram. Therefore, $0 \longrightarrow P \xrightarrow{id} P$ is a projective representation. □

Lemma 2.5. *If P is a projective left R-module, then a representation $P \xrightarrow{id} P \xrightarrow{id} P$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is a projective representation.*

Proof. Let $\alpha : P \longrightarrow N_1$ be an R-linear map and $k_1 : M_1 \longrightarrow N_1$ be an onto R-linear map and choose $h_1\alpha : P \longrightarrow N_2$ as an R-linear map, and choose $h_2h_1\alpha : P \longrightarrow N_3$ as an R-linear map. Then since P is a projective left R-module, there exist $S : P \longrightarrow M_1$ such that $k_1 S = \alpha$.

Now choose $g_1\alpha : P \longrightarrow M_2$ and $g_2 g_1 \alpha : P \longrightarrow M_3$ as an R-linear map. Then $g_1\alpha$ and $g_2g_1\alpha$ complete the following diagram

$$\begin{array}{ccccccc} & & (P & \xrightarrow{id} & P & \xrightarrow{id} & P) & & \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle h_1\alpha} & & \downarrow{\scriptstyle h_2h_1\alpha} & & \\ (M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) & \longrightarrow & (N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) \end{array}$$

as a commutative diagram. Therefore, $P \xrightarrow{id} P \xrightarrow{id} P$ is a projective representation. □

Remark 1. A representation $P \longrightarrow 0 \longrightarrow 0$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is not a projective representation if $P \neq 0$. Because we can not complete the following diagram

$$\begin{array}{ccccccc} & & (P & \longrightarrow & 0 & \longrightarrow & 0) & & \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\ (P \xrightarrow{id} P \longrightarrow 0) & \longrightarrow & (P & \longrightarrow & 0 & \longrightarrow & 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) \end{array}$$

as a commutative diagram.

Remark 2. A representation $P \xrightarrow{id} P \longrightarrow 0$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is not a projective representation if $P \neq 0$. Because we can not complete the following diagram

$$\begin{array}{ccccccc} & & (P & \xrightarrow{id} & P & \longrightarrow & 0) & & \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle 0} & & \\ (P \xrightarrow{id} P \xrightarrow{id} P) & \longrightarrow & (P & \longrightarrow & P & \longrightarrow & 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) \end{array}$$

as a commutative diagram.

Theorem 2.6. *A representation* $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ *of a quiver* $Q = \bullet \to \bullet \to \bullet$ *is projective if and only if each* M_1, M_2, M_3 *is projective left* R*-module and* $f_1(M_1)$ *is a summand of* M_2 *and* $f_2(M_2)$ *is a summand* *of M_3. That is,*

$$(M_1 \longrightarrow M_2 \longrightarrow M_3) \cong$$

$$(P_1 \xrightarrow{id} P_1 \xrightarrow{id} P_1) \oplus (0 \longrightarrow P_2 \xrightarrow{id} P_2) \oplus (0 \longrightarrow 0 \longrightarrow P_3),$$

where P_1, P_2, and P_3 are projective left R-modules.

Proof. The diagram

$$\begin{array}{ccc} & & (M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3) & & \\ & & \downarrow id \qquad \downarrow \qquad \downarrow & & \\ (M_1 \xrightarrow{id} M_1 \longrightarrow 0) & \longrightarrow & (M_1 \longrightarrow 0 \longrightarrow 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) \end{array}$$

can be completed to a commutative diagram by $id : M_1 \longrightarrow M_1$, $t : M_2 \longrightarrow M_1$, $0 : M_3 \longrightarrow 0$. Then we can get $tf_1 = id_{M_1}$ so that $M_2 \cong M_1 \oplus Ker(t)$. Now the following diagram

$$\begin{array}{ccc} & & (M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3) & & \\ & & \downarrow f_1 \qquad \downarrow \qquad \downarrow & & \\ (M_2 \xrightarrow{id} M_2 \xrightarrow{id} M_2) & \longrightarrow & (M_2 \xrightarrow{id} M_2 \longrightarrow 0) & \longrightarrow & (0 \longrightarrow 0 \longrightarrow 0) \end{array}$$

can be completed to a commutative diagram by $f_1 : M_1 \longrightarrow M_2$, $id : M_2 \longrightarrow M_2$, $u : M_3 \longrightarrow M_2$. Then we can get $uf_2 = id_{M_2}$ so that $M_3 \cong M_2 \oplus Ker(u)$. Therefore,

$$M_3 \cong M_2 \oplus Ker(u) \cong M_1 \oplus Ker(t) \oplus Ker(u).$$

This completes the proof. □

3. Injective representation of a quiver $Q = \bullet \to \bullet \to \bullet$

DEFINITION 3.1. A representation $E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is called an injective representation if every diagram of representations

$$\begin{array}{ccc} (0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (S_1 \xrightarrow{{}_{S_2}|g_1|_{S_1}} S_2 \xrightarrow{{}_{S_3}|g_2|_{S_2}} S_3) & \longrightarrow (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3) \\ & \downarrow \alpha \qquad \downarrow \beta \qquad \downarrow \gamma & \\ & (E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3) & \end{array}$$

can be completed to a commutative diagram as follows :

$$\begin{array}{ccc} (0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (S_1 \xrightarrow{{}_{S_2}|g_1|_{S_1}} S_2 \xrightarrow{{}_{S_3}|g_2|_{S_2}} S_3) & \longrightarrow (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3) \\ & \downarrow \alpha \qquad \downarrow \beta \qquad \downarrow \gamma & s: N_1 \dashrightarrow E_1, \ t: N_2 \dashrightarrow E_2, \ u: N_3 \dashrightarrow E_3 \\ & (E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3) & \end{array}$$

THEOREM 3.2. *If* $E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3$ *is a injective representation of a quiver* $Q = \bullet \to \bullet \to \bullet$, *then* E_1, E_2, *and* E_3 *are injective left* R*-modules.*

Proof. Let N be a left R-module, S be a submodule of N and $\gamma : S \longrightarrow E_3$ be an R-linear map. The since $E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3$ is an injective representation we can complete the following diagram

$$\begin{array}{ccc} (0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (0 \longrightarrow 0 \longrightarrow S) & \longrightarrow (0 \longrightarrow 0 \longrightarrow N) \\ & \downarrow \qquad \downarrow \qquad \downarrow \gamma & \\ & (E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3) & \end{array}$$

as a commutative diagram. Thus E_3 is an injective left R-module.

Let N be a left R-module, S be a submodule of N and $\beta : S \longrightarrow E_2$ be an R-linear map. The since $E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3$ is an injective representation we can complete the following diagram

$$\begin{array}{ccc} (0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (0 \longrightarrow S \xrightarrow{id} S) & \longrightarrow (0 \longrightarrow N \xrightarrow{id} N) \\ & \downarrow \qquad \downarrow \beta \qquad \downarrow f_2\beta & \\ & (E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3) & \end{array}$$

as a commutative diagram. Thus E_2 is an injective left R-module.

Let N be a left R-module, S be a submodule of N and $\alpha : S \longrightarrow E_1$ be an R-linear map. The since $E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3$ is an injective representation we can complete the following diagram

$$\begin{array}{ccccccc}
(0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (S & \xrightarrow{id} & S & \xrightarrow{id} & S) & \longrightarrow (N \xrightarrow{id} N \xrightarrow{id} N) \\
& \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle f_1\alpha} & & \downarrow{\scriptstyle f_2 f_1 \alpha} & \\
& (E_1 & \xrightarrow{f_1} & E_2 & \xrightarrow{f_2} & E_3) &
\end{array}$$

as a commutative diagram. Thus E_1 is an injective left R-module. □

Lemma 3.3. *If E is an injective left R-module, then a representation $E \longrightarrow 0 \longrightarrow 0$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is an injective representation.*

Proof. The lemma follows by completing the diagram

$$\begin{array}{ccccccc}
(0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (S_1 & \xrightarrow{S_2|g_1|S_1} & S_2 & \xrightarrow{S_3|g_2|S_2} & S_3) & \longrightarrow (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3) \\
& \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow & \\
& (E & \longrightarrow & 0 & \longrightarrow & 0) &
\end{array}$$

as a commutative diagram □

Lemma 3.4. *If E is an injective left R-module, then a representation $E \xrightarrow{id} E \longrightarrow 0$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is an injective representation.*

Proof. Let $\beta : S_2 \longrightarrow E$ be an R-linear map and choose $\beta g_1 : S_1 \longrightarrow E$ as an R-linear map. Then since E is a injective left R-module, there exist $t : N_2 \longrightarrow E$ such that $g_1 t = \beta$. Now choose $tg_1 : N_1 \longrightarrow E$ as an R-linear map. Then t and tg_1 complete the following diagram

$$\begin{array}{ccccccc}
(0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (S_1 & \xrightarrow{S_2|g_1|S_1} & S_2 & \xrightarrow{S_3|g_2|S_2} & S_3) & \longrightarrow (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3) \\
& \downarrow{\scriptstyle \beta g_1} & & \downarrow{\scriptstyle \beta} & & \downarrow & \\
& (E & \xrightarrow{id} & E & \longrightarrow & 0) &
\end{array}$$

as a commutative diagram. Therefore, $E \xrightarrow{id} E \longrightarrow 0$ is an injective representation. $\square$

LEMMA 3.5. *If E is a injective left R-module, then a representation $E \xrightarrow{id} E \xrightarrow{id} E$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is an injective representation.*

Proof. Let $\gamma : S_3 \longrightarrow E$ be an R-linear map and choose $\gamma g_2 : S_2 \longrightarrow E$ and $\gamma g_2 g_1 : S_1 \longrightarrow E$ as R-linear maps. Then since E is an injective left R-module, there exist $u : N_3 \longrightarrow E$ such that $ug_2 = \gamma$. Now choose $ug_2 : N_2 \longrightarrow E$ and $ug_2g_1 : N_1 \longrightarrow E$ as R-linear maps. Then u and ug_2, and ug_2g_1 complete the following diagram

$$\begin{array}{ccccccccccc}
(0 \longrightarrow 0 \longrightarrow 0) & \longrightarrow & (S_1 & \xrightarrow{S_2|g_1|S_1} & S_2 & \xrightarrow{S_3|g_2|S_2} & S_3) & \longrightarrow & (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3) \\
 & & \downarrow{\scriptstyle \gamma g_2 g_1} & & \downarrow{\scriptstyle \gamma g_2} & & \downarrow{\scriptstyle \gamma} & & \\
 & & (E & \xrightarrow{id} & E & \longrightarrow & 0) & &
\end{array}$$

as a commutative diagram. Therefore, $E \xrightarrow{id} E \xrightarrow{id} E$ is an injective representation. $\square$

REMARK 3. A representation $0 \longrightarrow 0 \longrightarrow E$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is not a injective representation if $E \neq 0$. Because we can not complete the following diagram

$$\begin{array}{ccccccccccc}
(0 \longrightarrow 0 \longrightarrow 0) & \longrightarrow & (0 & \longrightarrow & 0 & \longrightarrow & E) & \longrightarrow & (0 \longrightarrow E \xrightarrow{id} E) \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & & \\
 & & (0 & \longrightarrow & 0 & \longrightarrow & E) & &
\end{array}$$

as a commutative diagram.

REMARK 4. A representation $0 \longrightarrow E \xrightarrow{id} E$ of a quiver $Q = \bullet \to \bullet \to \bullet$ is not an injective representation if $E \neq 0$. Because we can not complete the following diagram

$$\begin{array}{ccccccc}
(0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (0 & \longrightarrow & E & \xrightarrow{id} & E) & \longrightarrow (E \xrightarrow{id} E \xrightarrow{id} E) \\
& \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & \\
& (0 & \longrightarrow & E & \longrightarrow & E) &
\end{array}$$

as a commutative diagram.

Theorem 3.6. *A representation* $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ *of a quiver* $Q = \bullet \to \bullet \to \bullet$ *is injective if and only if each* M_1, M_2, M_3 *is injective left* R*-module and* $ker(f_1)$ *is a summand of* M_1 *and* $ker(f_2)$ *is a summand of* M_2*. That is*

$$(M_1 \longrightarrow M_2 \longrightarrow M_3) \;\cong$$

$$(E_1 \xrightarrow{id} E_1 \xrightarrow{id} E_1) \oplus (E_2 \xrightarrow{id} E_2 \longrightarrow 0) \oplus (E_3 \longrightarrow 0 \longrightarrow 0)\ ,$$

where E_1, E_2*, and* E_3 *are injective left* R*-modules.*

Proof. The diagram

$$\begin{array}{ccccccc}
(0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (0 & \longrightarrow & M_2 & \xrightarrow{id} & M_2) & \longrightarrow (M_2 \xrightarrow{id} M_2 \xrightarrow{id} M_2) \\
& \downarrow & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle f_2} & \\
& (M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3) &
\end{array}$$

can be completed to a commutative diagram by $s : M_2 \to M_1$, $id : M_2 \to M_2$ and $f_2 : M_2 \to M_3$. Then we can get $f_1 s = id_{M_2}$ so that $M_1 \cong M_2 \oplus ker(f_1)$. Now the diagram

$$\begin{array}{ccccccc}
(0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (0 & \longrightarrow & 0 & \longrightarrow & M_3) & \longrightarrow (0 \longrightarrow M_3 \xrightarrow{id} M_3) \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle id} & \\
& (M_1 & \xrightarrow{f_1} & M_2 & \xrightarrow{f_2} & M_3) &
\end{array}$$

can be completed to a commutative diagram by $0 : 0 \to M_1$, $t : M_3 \to M_2$, $id : M_3 \to M_3$. Then, we can get $f_2 t = id_{M_3}$ so that $M_2 \cong M_3 \oplus ker(f_2)$. Therefore, $M_1 \cong M_3 \oplus ker(f_2) \oplus ker(f_1)$. This completes the proof.

□

References

[1] R. Diestel, *Graph Theory*, G.T.M. No. 88, Springer-Verlag, New York (1997).
[2] E. Enochs, I. Herzog, S. Park, *Cyclic quiver rings and polycyclic-by-finite group rings*, Houston J. Math. (1), **25** (1999) 1–13.
[3] E. Enochs, I. Herzog, *A homotopy of quiver morphism with applications to representations*, Canad J. Math. (2), **51** (1999), 294–308.
[4] S. Park, *Projective representations of quivers*, IJMMS(2), **31** (2002), 97–101.
[5] S. Park, D. Shin, *Injective representation of quiver*, Commun. Korean Math. Soc. (1), **21** (2006), 37–43.
[6] S. Park, *Injective and projective properties of representation of quivers with n edges*, Korean J. Math. (3), **16** (2008), 323–334.
[7] S. Park, E. Enochs, H. Kim *Injective covers and envelopes of representation of linear quiver*, Commun. Algebra (2), **37** (2009), 515–524.
[8] R. S. Pierce, *Associative Algebras*, G.T.M. No. 173, Springer-Verlag, New York (1982).

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