PROJECTIVE AND INJECTIVE PROPERTIES OF REPRESENTATIONS OF A QUIVER $Q = \bullet \rightarrow \bullet \rightarrow \bullet$

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Abstract. We define injective and projective representations of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$. Then we show that a representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is projective if and only if each $M_1, M_2, M_3$ is projective left $R$-module and $f_1(M_1)$ is a summand of $M_2$ and $f_2(M_2)$ is a summand of $M_3$. And we show that a representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is injective if and only if each $M_1, M_2, M_3$ is injective left $R$-module and $\ker(f_1)$ is a summand of $M_1$ and $\ker(f_2)$ is a summand of $M_2$.

1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) ([1]). We may consider many different types of quivers. We allow multiple edges and multiple arrows, and edges going from a vertex back to the same vertex. Originally a representation of quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex to the one assigned to the terminal vertex. For example, a representation of the quiver $Q = \bullet \rightarrow \bullet$ is $V_1 \xrightarrow{f} V_2$, $V_1$ and $V_2$ are vector spaces and $f$ is a linear map (morphism). Then we can define a morphism of two representations of the same quiver i.e., given a quiver $Q = \bullet \rightarrow \bullet$, we can define two representations $V_1 \xrightarrow{f} V_2$ and $W_1 \xrightarrow{g} W_2$. 

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Now we can define a morphism between these two representations. A morphism of \( V_1 \xrightarrow{f} V_2 \) to \( W_1 \xrightarrow{g} W_2 \) is given by a commutative diagram

\[
\begin{align*}
V_1 & \xrightarrow{f} V_2 \\
\downarrow{s_1} & \downarrow{s_2} \\
W_1 & \xrightarrow{g} W_2
\end{align*}
\]

with \( s_1, s_2 \) linear maps.

In ([3]) a homotopy of quiver was developed and in ([2]) cyclic quiver ring was studied. The theory of projective representations was developed in ([4]) and the theory of injective representation was studied in ([5]). Recently, in ([7]) injective covers and envelopes of representations of linear quivers was studied, and in ([6]) properties of multiple edges of quivers was studied.

2. Projective representation of a quiver \( Q = \bullet \to \bullet \to \bullet \)

**Definition 2.1.** A representation \( P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \) of a quiver \( Q = \bullet \to \bullet \to \bullet \) is called a projective representation if every diagram of representations

\[
\begin{align*}
(P_1 & \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) \\
\downarrow{\alpha} & \downarrow{\beta} \downarrow{\gamma} \\
(M_1 & \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) \to (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) \to (0 \to 0 \to 0)
\end{align*}
\]

can be completed to a commutative diagram as follows:

\[
\begin{align*}
(P_1 & \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) \\
\downarrow{\alpha} & \downarrow{\beta} \downarrow{\gamma} \\
(M_1 & \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) \to (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) \to (0 \to 0 \to 0)
\end{align*}
\]
Theorem 2.2. If \( P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \) is a projective representation of a quiver \( Q = \bullet \rightarrow \bullet \rightarrow \bullet \), then \( P_1, P_2, \) and \( P_3 \) are projective left \( R \)-modules.

Proof. Let \( M, N \) be left \( R \)-modules and \( \alpha : P_1 \to N \) be an \( R \)-linear map and \( k : M \to N \) be an onto \( R \)-linear map. Then since \( P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \) is a projective representation we can complete the following diagram

\[
\begin{array}{c}
(P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) \\
\downarrow{\alpha} \quad \downarrow{0} \quad \downarrow{0}
\end{array}
\]

\[
(M \xrightarrow{\text{id}} M \xrightarrow{0} 0) \quad (N \xrightarrow{\text{id}} N \xrightarrow{0} 0) \quad (0 \xrightarrow{0} 0 \xrightarrow{0})
\]

as a commutative diagram. Thus \( P_1 \) is a projective left \( R \)-module.

Let \( \beta : P_2 \to N \) be a \( R \)-linear map and \( k : M \to N \) be a onto \( R \)-linear map. Then since \( P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \) is a projective representation we can complete the following diagram

\[
\begin{array}{c}
(P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) \\
\downarrow{\beta f_1} \quad \downarrow{\beta} \quad \downarrow{0}
\end{array}
\]

\[
(M \xrightarrow{id} M \xrightarrow{\text{id}} M \xrightarrow{0} 0) \quad (N \xrightarrow{id} N \xrightarrow{\text{id}} N) \quad (0 \xrightarrow{0} 0 \xrightarrow{0})
\]

as a commutative diagram. Thus \( P_2 \) is a projective left \( R \)-module.

Let \( \gamma : P_3 \to N \) be an \( R \)-linear map and \( k : M \to N \) be an onto \( R \)-linear map. Then since \( P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3 \) is a projective representation we can complete the following diagram

\[
\begin{array}{c}
(P_1 \xrightarrow{f_1} P_2 \xrightarrow{f_2} P_3) \\
\downarrow{\gamma f_2 f_1} \quad \downarrow{\gamma f_2} \quad \downarrow{\gamma}
\end{array}
\]

\[
(M \xrightarrow{id} M \rightarrow M) \quad (N \xrightarrow{id} N \rightarrow N) \quad (0 \xrightarrow{0} 0 \xrightarrow{0})
\]

as a commutative diagram. Thus \( P_3 \) is a projective left \( R \)-module. \( \Box \)
Lemma 2.3. If \( P \) is a projective left \( R \)-module, then a representation \( 0 \to 0 \to P \) of a quiver \( Q = \bullet \to \bullet \to \bullet \) is a projective representation.

Proof. The lemma follows by completing the diagram

\[
\begin{array}{c}
0 \to 0 \to P \\
\downarrow \downarrow \downarrow \\
(M_1 \overset{g_1}{\to} M_2 \overset{g_2}{\to} M_3) \to (N_1 \overset{h_1}{\to} N_2 \overset{h_2}{\to} N_3) \to (0 \to 0 \to 0)
\end{array}
\]

as a commutative diagram.

\[
\text{Lemma 2.4. If } P \text{ is a projective left } R \text{-module, then a representation } 0 \to P \xrightarrow{id} P \text{ of a quiver } Q = \bullet \to \bullet \to \bullet \text{ is a projective representation.}
\]

Proof. Let \( \beta : P \to N_2 \) be an \( R \)-linear map and \( k_2 : M_2 \to N_2 \) be an onto \( R \)-linear map and choose \( \beta h_2 : P \to N_3 \) as an \( R \)-linear map. Then since \( P \) is a projective left \( R \)-module, there exist \( t : P \to M_2 \) such that \( k_2 t = \beta \). Now choose \( g_2 t : P \to M_3 \) as an \( R \)-linear map. Then \( t \) and \( g_2 \alpha \) complete the following diagram

\[
\begin{array}{c}
0 \to P \xrightarrow{id} P \\
\downarrow \downarrow \downarrow \\
(M_1 \overset{g_1}{\to} M_2 \overset{g_2}{\to} M_3) \to (N_1 \overset{h_1}{\to} N_2 \overset{h_2}{\to} N_3) \to (0 \to 0 \to 0)
\end{array}
\]

as a commutative diagram. Therefore, \( 0 \to P \xrightarrow{id} P \) is a projective representation.

\[
\text{Lemma 2.5. If } P \text{ is a projective left } R \text{-module, then a representation } P \xrightarrow{id} P \xrightarrow{id} P \text{ of a quiver } Q = \bullet \to \bullet \to \bullet \text{ is a projective representation.}
\]

Proof. Let \( \alpha : P \to N_1 \) be an \( R \)-linear map and \( k_1 : M_1 \to N_1 \) be an onto \( R \)-linear map and choose \( h_1 \alpha : P \to N_2 \) as an \( R \)-linear map, and choose \( h_2 h_1 \alpha : P \to N_3 \) as an \( R \)-linear map. Then since \( P \) is a projective left \( R \)-module, there exist \( S : P \to M_1 \) such that \( k_1 S = \alpha \).
Now choose $g_1 \alpha : P \rightarrow M_2$ and $g_2 g_1 \alpha : P \rightarrow M_3$ as an $R$-linear map. Then $g_1 \alpha$ and $g_2 g_1 \alpha$ complete the following diagram

\[
\begin{array}{ccc}
(P \xrightarrow{id} P \xrightarrow{id} P) \\
\downarrow \alpha & \downarrow h_1 \alpha & \downarrow h_2 h_1 \alpha \\
(M_1 \xrightarrow{g_1} M_2 \xrightarrow{g_2} M_3) \rightarrow (N_1 \xrightarrow{h_1} N_2 \xrightarrow{h_2} N_3) \rightarrow (0 \rightarrow 0 \rightarrow 0)
\end{array}
\]
as a commutative diagram. Therefore, $P \xrightarrow{id} P \xrightarrow{id} P$ is a projective representation.

Remark 1. A representation $P \rightarrow 0 \rightarrow 0$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is not a projective representation if $P \neq 0$. Because we can not complete the following diagram

\[
\begin{array}{ccc}
(P \xrightarrow{id} 0 \xrightarrow{id} 0) \\
\downarrow {\text{id}} & \downarrow 0 & \downarrow 0 \\
(P \xrightarrow{id} P \xrightarrow{id} 0) \rightarrow (P \xrightarrow{id} 0 \xrightarrow{id} 0) \rightarrow (0 \rightarrow 0 \rightarrow 0)
\end{array}
\]
as a commutative diagram.

Remark 2. A representation $P \xrightarrow{id} P \xrightarrow{id} 0$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is not a projective representation if $P \neq 0$. Because we can not complete the following diagram

\[
\begin{array}{ccc}
(P \xrightarrow{id} P \xrightarrow{id} 0) \\
\downarrow {\text{id}} & \downarrow {\text{id}} & \downarrow 0 \\
(P \xrightarrow{id} P \xrightarrow{id} P) \rightarrow (P \xrightarrow{id} P \xrightarrow{id} 0) \rightarrow (0 \rightarrow 0 \rightarrow 0)
\end{array}
\]
as a commutative diagram.

Theorem 2.6. A representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is projective if and only if each $M_1, M_2, M_3$ is projective left $R$-module and $f_1(M_1)$ is a summand of $M_2$ and $f_2(M_2)$ is a summand.
of $M_3$. That is,

$$(M_1 \longrightarrow M_2 \longrightarrow M_3) \cong (P_1 \longrightarrow P_1 \longrightarrow P_1) \oplus (0 \longrightarrow P_2 \longrightarrow P_2) \oplus (0 \longrightarrow 0 \longrightarrow P_3),$$

where $P_1$, $P_2$, and $P_3$ are projective left $R$-modules.

**Proof.** The diagram

$$
\begin{array}{ccc}
(M_1 \overset{f_1}{\longrightarrow} M_2 \overset{f_2}{\longrightarrow} M_3) & \overset{id}{\downarrow} & \\
(M_1 \overset{id}{\longrightarrow} M_1 \longrightarrow 0) & \longrightarrow & (M_1 \overset{0}{\longrightarrow} M_1 \longrightarrow 0) \longrightarrow (0 \longrightarrow 0 \longrightarrow 0)
\end{array}
$$

can be completed to a commutative diagram by $id : M_1 \longrightarrow M_1$, $t : M_2 \longrightarrow M_1$, $0 : M_3 \longrightarrow 0$. Then we can get $tf_1 = id_{M_1}$ so that $M_2 \cong M_1 \oplus Ker(t)$. Now the following diagram

$$
\begin{array}{ccc}
(M_1 \overset{f_1}{\longrightarrow} M_2 \overset{f_2}{\longrightarrow} M_3) & \overset{id}{\downarrow} & \\
(M_2 \overset{id}{\longrightarrow} M_2 \longrightarrow M_2) & \longrightarrow & (M_2 \overset{id}{\longrightarrow} M_2 \longrightarrow 0) \longrightarrow (0 \longrightarrow 0 \longrightarrow 0)
\end{array}
$$

can be completed to a commutative diagram by $f_1 : M_1 \longrightarrow M_2$, $id : M_2 \longrightarrow M_2$, $u : M_3 \longrightarrow M_2$. Then we can get $uf_2 = id_{M_2}$ so that $M_3 \cong M_2 \oplus Ker(u)$. Therefore,

$$M_3 \cong M_2 \oplus Ker(u) \cong M_1 \oplus Ker(t) \oplus Ker(u).$$

This completes the proof. 

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### 3. Injective representation of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$

**Definition 3.1.** A representation $E_1 \overset{f_1}{\longrightarrow} E_2 \overset{f_2}{\longrightarrow} E_3$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is called an injective representation if every diagram of representations

Theorem 3.2. If \( E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3 \) is a injective representation of a quiver \( Q = \bullet \rightarrow \bullet \rightarrow \bullet \), then \( E_1, E_2, \) and \( E_3 \) are injective left \( R \)-modules.

Proof. Let \( N \) be a left \( R \)-module, \( S \) be a submodule of \( N \) and \( \gamma : S \rightarrow E_3 \) be an \( R \)-linear map. The since \( E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3 \) is an injective representation we can complete the following diagram

\[
\begin{array}{c}
(0 \rightarrow 0 \rightarrow 0) \\
\downarrow \\
(0 \rightarrow 0 \rightarrow S) \\
\downarrow \\
(0 \rightarrow 0 \rightarrow N)
\end{array}
\]

as a commutative diagram. Thus \( E_3 \) is an injective left \( R \)-module.

Let \( N \) be a left \( R \)-module, \( S \) be a submodule of \( N \) and \( \beta : S \rightarrow E_2 \) be an \( R \)-linear map. The since \( E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3 \) is an injective representation we can complete the following diagram

\[
\begin{array}{c}
(0 \rightarrow 0 \rightarrow 0) \\
\downarrow \\
(0 \rightarrow S) \\
\downarrow \\
(0 \rightarrow N)
\end{array}
\]

as a commutative diagram. Thus \( E_2 \) is an injective left \( R \)-module.
Let $N$ be a left $R$-module, $S$ be a submodule of $N$ and $\alpha : S \rightarrow E_1$ be an $R$-linear map. The since $E_1 \xrightarrow{f_1} E_2 \xrightarrow{f_2} E_3$ is an injective representation we can complete the following diagram

\[
\begin{array}{c}
(0 \rightarrow 0 \rightarrow 0) \rightarrow (S \xrightarrow{id} S \xrightarrow{id} S) \rightarrow (N \xrightarrow{id} N \xrightarrow{id} N)
\end{array}
\]

as a commutative diagram. Thus $E_1$ is an injective left $R$-module.

**Lemma 3.3.** If $E$ is an injective left $R$-module, then a representation $E \rightarrow 0 \rightarrow 0$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is an injective representation.

**Proof.** The lemma follows by completing the diagram

\[
\begin{array}{c}
(0 \rightarrow 0 \rightarrow 0) \rightarrow (S_1 \xrightarrow{s_2|s_1} S_2 \xrightarrow{s_3|s_2} S_3) \rightarrow (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3)
\end{array}
\]

as a commutative diagram

**Lemma 3.4.** If $E$ is an injective left $R$-module, then a representation $E \xrightarrow{id} E \rightarrow 0$ of a quiver $Q = \bullet \rightarrow \bullet \rightarrow \bullet$ is an injective representation.

**Proof.** Let $\beta : S_2 \rightarrow E$ be an $R$-linear map and choose $\beta g_1 : S_1 \rightarrow E$ as an $R$-linear map. Then since $E$ is a injective left $R$-module, there exist $t : N_2 \rightarrow E$ such that $g_1 t = \beta$. Now choose $tg_1 : N_1 \rightarrow E$ as an $R$-linear map. Then $t$ and $tg_1$ complete the following diagram

\[
\begin{array}{c}
(0 \rightarrow 0 \rightarrow 0) \rightarrow (S_1 \xrightarrow{s_2|s_1} S_2 \xrightarrow{s_3|s_2} S_3) \rightarrow (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3)
\end{array}
\]
as a commutative diagram. Therefore, \( E \xrightarrow{id} E \xrightarrow{id} 0 \) is an injective representation.

\[ \text{Lemma 3.5. If } E \text{ is a injective left } R \text{-module, then a representation } E \xrightarrow{id} E \xrightarrow{id} E \text{ of a quiver } Q = \bullet \rightarrow \bullet \rightarrow \bullet \text{ is an injective representation.} \]

\[ \text{Proof. Let } \gamma : S_3 \rightarrow E \text{ be an } R \text{-linear map and choose } \gamma g_2 : S_2 \rightarrow E \text{ and } \gamma g_2 g_1 : S_1 \rightarrow E \text{ as } R \text{-linear maps. Then since } E \text{ is an injective left } R \text{-module, there exist } u : N_3 \rightarrow E \text{ such that } ug_2 = \gamma. \text{ Now choose } ug_2 : N_2 \rightarrow E \text{ and } ug_2 g_1 : N_1 \rightarrow E \text{ as } R \text{-linear maps. Then } u \text{ and } ug_2 \text{, and } ug_2 g_1 \text{ complete the following diagram} \]

\[
\begin{array}{c}
(0 \rightarrow 0 \rightarrow 0) \xrightarrow{(S_1 \xrightarrow{g_1} S_2 \xrightarrow{g_2} S_3)} (N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3) \\
\downarrow \gamma g_2 g_1 \downarrow \gamma g_2 \downarrow \gamma \\
(0 \rightarrow E \xrightarrow{id} E \rightarrow 0)
\end{array}
\]

as a commutative diagram. Therefore, \( E \xrightarrow{id} E \xrightarrow{id} E \) is an injective representation. \[ \square \]

\[ \text{Remark 3. A representation } 0 \rightarrow 0 \rightarrow 0 \rightarrow E \text{ of a quiver } Q = \bullet \rightarrow \bullet \rightarrow \bullet \text{ is not a injective representation if } E \neq 0. \text{ Because we can not complete the following diagram} \]

\[
\begin{array}{c}
(0 \rightarrow 0 \rightarrow 0) \xrightarrow{(0 \rightarrow 0 \rightarrow E)} (0 \rightarrow E \xrightarrow{id} E) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
(0 \rightarrow 0 \rightarrow E)
\end{array}
\]

as a commutative diagram.

\[ \text{Remark 4. A representation } 0 \rightarrow E \xrightarrow{id} E \text{ of a quiver } Q = \bullet \rightarrow \bullet \rightarrow \bullet \text{ is not an injective representation if } E \neq 0. \text{ Because we can not complete the following diagram} \]
A representation \( M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \) of a quiver \( Q = \bullet \rightarrow \bullet \rightarrow \bullet \) is injective if and only if each \( M_1, M_2, M_3 \) is injective left \( R \)-module and \( \ker(f_1) \) is a summand of \( M_1 \) and \( \ker(f_2) \) is a summand of \( M_2 \). That is
\[
(M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3) \cong (E_1 \xrightarrow{id} E_1 \xrightarrow{id} E_1) \oplus (E_2 \xrightarrow{id} E_2 \rightarrow 0) \oplus (E_3 \rightarrow 0 \rightarrow 0),
\]
where \( E_1, E_2, \) and \( E_3 \) are injective left \( R \)-modules.

**Proof.** The diagram
\[
(0 \rightarrow 0 \rightarrow 0) \xrightarrow{f_1} (0 \rightarrow M_2 \xrightarrow{id} M_2) \xrightarrow{f_2} (M_2 \xrightarrow{id} M_2 \xrightarrow{id} M_2)
\]
can be completed to a commutative diagram by \( s : M_2 \rightarrow M_1, \ id : M_2 \rightarrow M_2 \) and \( f_2 : M_2 \rightarrow M_3 \). Then we can get \( f_1 s = id_{M_2} \) so that \( M_1 \cong M_2 \oplus \ker(f_1) \). Now the diagram
\[
(0 \rightarrow 0 \rightarrow 0) \xrightarrow{f_1} (0 \rightarrow 0 \rightarrow M_3) \xrightarrow{f_2} (0 \rightarrow M_3 \xrightarrow{id} M_3)
\]
can be completed to a commutative diagram by \( 0 : 0 \rightarrow M_1, \ t : M_3 \rightarrow M_2, \ id : M_3 \rightarrow M_3 \). Then, we can get \( f_2 t = id_{M_3} \) so that \( M_2 \cong M_3 \oplus \ker(f_2) \). Therefore, \( M_1 \cong M_3 \oplus \ker(f_2) \oplus \ker(f_1) \). This completes the proof.
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