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IMPROVED MULTIPLICITY RESULTS FOR FULLY NONLINEAR PARABOLIC SYSTEMS

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Abstract. We investigate the existence of multiple solutions (ξ, η) for perturbations of the parabolic system with Dirichlet boundary condition

$$(0.1)\qquad \begin{aligned} \xi_t &= -L\xi + g_1(3\xi + \eta) - s\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ \eta_t &= -L\eta + g_2(3\xi + \eta) - s\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0, 2\pi). \end{aligned}$$

We show the existence of multiple solutions (ξ, η) for perturbations of the parabolic system when the nonlinearity g_1', g_2' are bounded and $3g_1'(-\infty) + g_2'(-\infty) < \lambda_1$, $\lambda_n < 3g_1'(+\infty) + g_2'(+\infty) < \lambda_{n+1}$.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$ and let L denote the elliptic differential operator. In [2, 4, 5, 7, 8] the authors investigate multiplicity of solutions of the nonlinear elliptic equation with Dirichlet boundary condition

$$(1.1)\qquad \begin{aligned} &Lu + g(u) = f(x) \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where g is the semilinear term and L is a second order linear elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated according to its multiplicity,

$$0 < \lambda_1 < \lambda_2 < \lambda_3 \le \cdots \le \lambda_i \le \cdots \to \infty.$$

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Equation (1.1) and the following type nonlinear equation with Dirichlet boundary condition was studied by many authors:

$$
\begin{aligned}
Lu &= bu^+ - au^- + f \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial\Omega.
\end{aligned} \tag{1.2}
$$

In [9] Lazer and McKenna point out that this kind of nonlinearity $bu^+ - au^-$ can furnish a model to study traveling waves in suspension bridges. So the nonlinear equation with jumping nonlinearity have been extensively studied by many authors. For fourth elliptic equation Tarantello [15] , Micheletti and Pistoia [12][13] proved the existence of nontrivial solutions used degree theory and critical points theory separately. For one-dimensional case Lazer and McKenna [10] proved the existence of nontrivial solution by the global bifurcation method. For this jumping nonlinearity we are interest in the multiple nontrivial solutions of the equation. Here we used variational reduction method to find the nontrivial solutions of problem (1.2).

In [6, 11] the authors investigate multiplicity of solutions of the nonlinear parabolic equation with Dirichlet boundary condition

$$
\begin{aligned}
u_t &= -Lu + f(u) - s\phi_1 - h(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\
u &= 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi).
\end{aligned} \tag{1.3}
$$

In [5] the authors investigate the existence of solutions (ξ, η) for perturbations of the parabolic system with Dirichlet boundary condition

$$
\begin{aligned}
\xi_t &= -L\xi + \mu g(3\xi + \eta) - s_1\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\
\eta_t &= -L\eta + \nu g(3\xi + \eta) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\
\xi &= 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi),
\end{aligned} \tag{1.4}
$$

where we assume that $h_1, h_2 \in H^*$ and g' is bounded, $(3\mu + \nu)g'(-\infty) < \lambda_1, \lambda_n < (3\mu + \nu)g'(+\infty) < \lambda_{n+1}$. Here they assume that the nonlinear term $\mu g(3\xi + \eta)$ is a multiple of the other nonlinear term $\nu g(3\xi + \eta)$.

In this paper we investigate the existence of solutions (ξ, η) for perturbations of the parabolic system with Dirichlet boundary condition

$$
\begin{aligned}
\xi_t &= -L\xi + g_1(3\xi + \eta) - s_1\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\
\eta_t &= -L\eta + g_2(3\xi + \eta) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\
\xi &= 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi),
\end{aligned} \tag{1.5}
$$

where we assume that $h_1, h_2 \in H^*$ and g_1', g_2' are bounded and $3g_1'(-\infty)+g_2'(-\infty) < \lambda_1$, $\lambda_n < 3g_1'(+\infty) + g_2'(+\infty) < \lambda_{n+1}$. We improve the result of [5]. Here we do not assume that the nonlinear term $g_1(3\xi + \eta)$ is a multiple of the other nonlinear term $g_2(3\xi + \eta)$.

In section 2, we state the result for the parabolic equation with Dirichlet boundary condition when the nonlinearity crosses eigenvalues. We investigate the multiplicity of solutions for the single nonlinear parabolic equation. In section 3, we investigate the uniqueness when the nonlinearity does not cross eigenvalues. We also investigate multiple solutions $(\xi(x,t), \eta(x,t))$ for perturbations of the parabolic system with Dirichlet boundary condition when the nonlinearities g_1', g_2' are bounded and $3g_1'(-\infty)+g_2'(-\infty) < \lambda_1$, $\lambda_n < 3g_1'(+\infty)+g_2'(+\infty) < \lambda_{n+1}$. Here we do not assume that the nonlinear term $g_1(3\xi+\eta)$ is a multiple of the other nonlinear term $g_2(3\xi + \eta)$.

2. Parabolic equations with source terms

In this section we state the result for the parabolic equation with Dirichlet boundary condition when the nonlinearity crosses eigenvalues.

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$ and let L denote the elliptic differential operator. We look for weak solutions of the parabolic equation with Dirichlet boundary condition

$$\begin{aligned} u_t &= -Lu + f(u) - s\phi_1 - h(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi). \end{aligned} \tag{2.1}$$

We assume that the eigenfunctions ϕ_i of L are an orthonormal basis for $L^2(\Omega)$ with eigenfunctions $-\lambda_i, \lambda_1 > 0, \lambda_i \to +\infty$, and that $\phi_1(x) > 0, x \in \Omega$. These are the assumptions of this section. For the more results for the parabolic equation we refer to [6, 11].

We shall work with the complex Hilbert space $H_T^* = L^2(\Omega \times (0,T))$, equipped with the usual inner product

$$\langle v, \omega\rangle^* = \int_0^{2\pi}\int_\Omega v(x,t)\overline{\omega}(x,t)dxdt$$

and norm $\|v\| = \langle v, v\rangle^{*\frac{1}{2}}$. Later we shall switch to the real subspace H_T. The functions $\phi_{mn} = \frac{\phi_n(x)e^{imt}}{\sqrt{2\pi}}, n \geq 1, m = 0, \pm 1, \pm 2, \ldots$ are a complete orthornormal basis for H^*. Let Σ^* denote sums over the indices m, n. Every $v \in H^*$ has a Fourier expansion

$$v = \Sigma^* v_{mn} \phi_{mn},$$

with $\Sigma |v_{mn}|^2 = \|v\|^2, v_{mn} = \langle v, \phi^*_{mn} \rangle$. A weak solution to the boundary value problem (2.1) is, by definition, a function $u \in H$ satisfying $Lu \in H$, i.e. $\Sigma^* |u_{mn}|^2 (m^2 + \lambda_n^2) < \infty$ satisfying (2.1) in H.

For real $\alpha \neq \lambda_n$, the operator $R = (L + \alpha - D_t)^{-1}$ denoted by

$$u = Rh \leftrightarrow u_{mn} = \frac{h_{mn}}{-\lambda_n + \alpha + im}$$

is a compact linear operator on H^* and the operator norm of R, $\|R\| = \frac{1}{|\alpha - \lambda_n|}$, where λ_n is an eigenvalue of $-L$ closest to α.

From now on, we restrict ourselves to the real subspace H and observe that it is invariant under R.

Our first theorem is a non-self-adjoint problem.

THEOREM 2.1. *Assume that f' is bounded, that $f'(+\infty) = \alpha$ satisfies $\lambda_n < \alpha < \lambda_{n+1}$ and that $h \in H$. Then there exists $s_0 > 0, \epsilon > 0$ such that the Leray-Schauder degree*

$$\deg(u - (-L + D_t)^{-1}(f(u) - s\phi_1 - h), B^*_{s\epsilon}(s\theta), 0) = (-1)^n \qquad (2.2)$$

for $s \geq s_0$. Here B^*_r denotes a ball of radius r in H and

$$\theta = -(-L - \alpha + D_t)^{-1} \phi_1 = \frac{\phi_1}{\alpha - \lambda_1}.$$

PROPOSITION 1. *If f' is bounded, and $\overline{\alpha} = f'(-\infty) < \lambda_1$, then there exist positive constants s_0, ϵ such that*

$$\deg(u - (D_t - L)^{-1}(f_(u) - s\phi_1 - h), B^*_{s\epsilon}(s\overline{\theta}), 0) = 1$$

for $s \geq s_0$, where $\overline{\theta} = \frac{\phi_1}{\overline{\alpha} - \lambda_1} < 0$.

LEMMA 2.1. *Assume that $|f(u)| \leq a + c|u|, f'(-\infty), f'(+\infty)$ exist, that $f(u) - \lambda_1 u \geq \epsilon |u| - b$, and that $h \in H$ satisfies $\|h\| \leq r$, where a, b, c, r, ϵ are positive constants. Then there exists C depending only on a, b, c, r, ϵ such that*

$$D_t u = Lu + f(u) - s\phi_1 - h$$

$$u(x, t + 2\pi) = u(x, t)$$

satisfies $\|u\| \leq C$.

LEMMA 2.2. *Let $s_1 \in R$ under the assumptions of the preceding lemma, there exists $C_1 > 0$, depending on s_1 and the constants of Lemma 1, such that*

$$\deg(u - (D_t - Lu)^{-1}(f(u) - (h + s\phi_1)), B^*_\beta(0), 0) = 0$$

for $s \le s_1$ and $\beta > C_1$.

The proof of Lemma 2.2 is the same as those for the self-adjoint case, as done in Chapter I(cf. [11]). There is no solution on the boundary of the ball for $s \le s_1$, by the previous lemma. Therefore, by homotopy, the degree is the same for all $s \le s_1$, and since it must be zero for large negative s, it must be zero for all $s \le s_1$.

We have now assembled all the ingredients for our first existence theorem.

THEOREM 2.2. *Let $h \in H^*$. Assume f' is bounded, $f'(-\infty) < \lambda_1, \lambda_n < f'(+\infty) < \lambda_{n+1}$. Then there exists s_0 so that if $s \ge s_0$, equation (2.1) has at least two 2π-periodic solutions if n is even, and at least three if n is odd.*

The proof is by now obvious. The degree on a large ball is zero. By Theorem 2.1, we can find a ball near $\overline{\theta}$, on which the degree of the map

$$u - (D_t - L)^{-1}(f(u) - (s\phi_1 + h(x)))$$

is 1, and a ball on which the degree is zero, we have two solutions if n is odd, and three if n is even. This concludes the proof.

3. Periodic solutions of the parabolic system

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$ and let L denote the elliptic differential operator. In this section we investigate the existence of multiple solutions (ξ, η) for perturbations of the parabolic system with Dirichlet boundary condition

$$\begin{aligned} &\xi_t = -L\xi + g_1(3\xi + \eta) - s_1\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ (3.1) \qquad &\eta_t = -L\eta + g_2(3\xi + \eta) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0, 2\pi), \\ &\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi), \end{aligned}$$

where we assume that $h_1, h_2 \in H^*$ and g_1', g_2' are bounded and $3g_1'(-\infty) + g_2'(-\infty) < \lambda_1$, $\lambda_n < 3g_1'(+\infty) + g_2'(+\infty) < \lambda_{n+1}$. We also assume that $s_1, s_2 > 0$.

THEOREM 3.1. *Let $s_1, s_2 > 0$. Assume that $3A+B < \lambda_1$ and $h \in H^*$. Then the parabolic system with Dirichlet boundary condition*

$$\begin{aligned} &\xi_t = -L\xi + A(3\xi+\eta)^+ - s_1\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0,2\pi),\\ &\eta_t = -L\eta + B(3\xi+\eta)^+ - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0,2\pi),\\ &\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0,2\pi), \end{aligned} \tag{3.2}$$

has a unique solution (ξ, η).

Proof. From problem (3.2) we get the equation

$$\begin{aligned} &(3\xi+\eta)_t = -L(3\xi+\eta) + (3A+B)(3\xi+\eta)^+\\ &\quad - (3s_1+s_2)\phi_1 - 3h_1(x,t) - h_2(x,t) \quad \text{in } \Omega \times (0,2\pi),\\ &\xi = 0, \quad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0,2\pi). \end{aligned} \tag{3.3}$$

Put $w = 3\xi + \eta$. Then the above equation is equivalent to

$$\begin{aligned} &w_t + Lw + (3A+B)w^+ =\\ &\quad - (3s_1+s_2)\phi_1 - 3h_1 - h_2 \quad \text{in } \Omega \times (0,2\pi),\\ &w = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.4}$$

When $3A+B < \lambda_1$, by the contraction mapping principle, the above equation has a unique solution, say w_1. For any $F \in H_0$ the linear problem

$$\begin{aligned} &u_t + Lu = F \quad \text{in} \quad \Omega \times (0,2\pi),\\ &u = 0 \quad \text{on} \quad \partial\Omega \times (0,2\pi) \end{aligned} \tag{3.5}$$

has a unique solution.

Hence we get the unique solution (ξ, η) of problem (3.2) from the following system

$$\begin{aligned} &\xi_t = -L\xi + Aw_1^+ - s_1\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0,2\pi),\\ &\eta_t = -L\eta + Bw_1^+ - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0,2\pi),\\ &\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0,2\pi). \end{aligned} \tag{3.6}$$

□

THEOREM 3.2. *Let $s_1 = s_2 = s > 0$. Assume that $h_1, h_2 \in H^*$ and g_1', g_2' are bounded and $3g_1'(-\infty) + g_2'(-\infty) < \lambda_1$, $\lambda_n < 3g_1'(+\infty) + g_2'(+\infty) < \lambda_{n+1}$. Then there exists s_0 so that if $s \geq s_0$, equation (3.1) has at least two 2π-periodic solutions if n is even, and at least three if n is odd.*

Proof. From problem (3.2) we get the equation

$$
\begin{aligned}
&(3\xi+\eta)_t = -L(3\xi+\eta) + 3g_1(3\xi+\eta) + g_2(3\xi+\eta) \\
&- 4s_1\phi_1 - 3h_1(x,t) - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\xi = 0, \quad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi).
\end{aligned} \tag{3.7}
$$

Put $w = 3\xi+\eta$. Then the above equation is equivalent to

$$
\begin{aligned}
&w_t + Lw + 3g_1(w) + g_2(w) = \\
&- 4s_1\phi_1 - 3h_1(x,t) - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&w = 0 \quad \text{on} \quad \partial\Omega.
\end{aligned} \tag{3.8}
$$

Since g_1', g_2' are bounded and $3g_1'(-\infty)+g_2'(-\infty) < \lambda_1$, $\lambda_n < 3g_1'(+\infty)+g_2'(+\infty) < \lambda_{n+1}$, by Theorem 2.1 there exists s_0 so that if $s \geq s_0$, equation (3.1) has at least two 2π-periodic solutions(say, w_{e1}, w_{e2}) if n is even, and at least three solutions(say, w_{o1}, w_{o2}, w_{o3}) if n is odd.

When n is even, we get the solutions (ξ,η) of problem (3.1) from the following systems:

$$
\begin{aligned}
&\xi_t = -L\xi + g_1(w_{e1}) - s\phi_1 - h_1(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\eta_t = -L\eta + +g_2(w_{e1}) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi).
\end{aligned} \tag{3.9}
$$

$$
\begin{aligned}
&\xi_t = -L\xi + g_1(w_{e2}) - s\phi_1 - h_1(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\eta_t = -L\eta + +g_2(w_{e2}) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi).
\end{aligned} \tag{3.10}
$$

Therefore system (3.1) has at least two solutions if n is even.

When n is odd, we get the solutions (ξ,η) of problem (3.1) from the following systems:

$$
\begin{aligned}
&\xi_t = -L\xi + g_1(w_{o1}) - s\phi_1 - h_1(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\eta_t = -L\eta + +g_2(w_{o1}) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi).
\end{aligned} \tag{3.11}
$$

$$
\begin{aligned}
&\xi_t = -L\xi + g_1(w_{o2}) - s\phi_1 - h_1(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\eta_t = -L\eta + +g_2(w_{o2}) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega\times(0,2\pi), \\
&\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega\times(0,2\pi).
\end{aligned} \tag{3.12}
$$

$$\begin{aligned}
&\xi_t = -L\xi + g_1(w_{o3}) - s\phi_1 - h_1(x,t) \quad \text{in } \Omega \times (0, 2\pi),\\
(3.13) \qquad &\eta_t = -L\eta + +g_2(w_{o3}) - s_2\phi_1 - h_2(x,t) \quad \text{in } \Omega \times (0, 2\pi),\\
&\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega \times (0, 2\pi).
\end{aligned}$$

Therefore system (3.1) has at least three solutions if n is odd.

□

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