Abstract. In this paper we make a connection between the specific class of weighted shifts and general study of $k$-hyponormality. We show how the $k$-hyponormality of an arbitrary operator can be ascertained by examining the $k$-hyponormality of an associated family of weighted shifts.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces, let $\mathcal{L}(\mathcal{H}, \mathcal{K})$ be the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ and write $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be normal if $T^*T = TT^*$, hyponormal if $T^*T \geq TT^*$, and subnormal if $T = N|_\mathcal{H}$, where $N$ is normal on some Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. If $T$ is subnormal then $T$ is also hyponormal. The Bram-Halmos criterion for subnormality states that an operator $T$ is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0$$

for all finite collections $x_0, x_1, \cdots, x_k \in \mathcal{H}$ ([2], [3, II.1.9]). It is easy to see that this is equivalent to the following positivity test:

$$(0.1) \begin{pmatrix} I & T^* & \cdots & T^*k \\ T & T^*T & \cdots & T^*kT \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \cdots & T^{*k}T^k \end{pmatrix} \geq 0 \quad \text{(all } k \geq 1).$$

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Condition (0.1) provides a measure of the gap between hyponormality and subnormality. In fact, the positivity condition (0.1) for $k = 1$ is equivalent to the hyponormality of $T$, while subnormality requires the validity of (0.1) for all $k$. Let $[A, B] := AB - BA$ denote the commutator of two operators $A$ and $B$, and define $T$ to be $k$-hyponormal whenever the $k \times k$ operator matrix

\begin{equation}
M_k(T) := ([T^{*j}, T^i])_{i,j=1}^k
\end{equation}

is positive. An application of the Choleski algorithm for operator matrices shows that the positivity of (0.2) is equivalent to the positivity of the $(k+1) \times (k+1)$ operator matrix in (0.1); the Bram-Halmos criterion can then be rephrased as saying that $T$ is subnormal if and only if $T$ is $k$-hyponormal for every $k \geq 1$ ([11]).

Recall ([1],[4],[11]) that $T \in L(H)$ is said to be weakly $k$-hyponormal if

\begin{equation}
LS(T, T^2, \cdots, T^k) := \left\{ \sum_{j=1}^k \alpha_j T^j : \alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{C}^k \right\}
\end{equation}

consists entirely of hyponormal operators, or equivalently, $M_k(T)$ is weakly positive, i.e., ([11])

\begin{equation}
\left\langle M_k(T) \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix}, \begin{pmatrix} \lambda_1 x \\ \vdots \\ \lambda_k x \end{pmatrix} \right\rangle \geq 0 \quad \text{for all } x \in H \text{ and } \lambda_1, \cdots, \lambda_k \in \mathbb{C}.
\end{equation}

If $k = 2$ then $T$ is said to be quadratically hyponormal and if $k = 3$ then $T$ is said to be cubically hyponormal. Similarly, $T \in L(H)$ is said to be polynomially hyponormal if $p(T)$ is hyponormal for every polynomial $p \in \mathbb{C}[z]$. It is known that $k$-hyponormal $\Rightarrow$ weakly $k$-hyponormal, but the converse is not true in general. Recall that given a bounded sequence of positive numbers $\alpha : \alpha_0, \alpha_1, \cdots$ (called weights), the (unilateral) weighted shift $W_\alpha$ associated with $\alpha$ is the operator on $\ell^2(\mathbb{Z}_+)$ defined by $W_\alpha e_n := \alpha_n e_{n+1}$ for all $n \geq 0$, where $\{e_n\}_{n=0}^\infty$ is the canonical orthonormal basis for $\ell^2$. It is straightforward to check that $W_\alpha$ can never be normal, and that $W_\alpha$ is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \geq 0$. The moments of $\alpha$ are given as

\begin{equation}
\gamma_k \equiv \gamma_k(\alpha) := \begin{cases} 1 & \text{if } k = 0 \\ \alpha_0^2 \cdots \alpha_{k-1}^2 & \text{if } k > 0 \end{cases}.
\end{equation}
We now recall a well known characterization of subnormality for single variable weighted shifts, due to C. Berger (cf. [3, III.8.16]): $W_{\alpha}$ is subnormal if and only if there exists a probability measure $\xi$ supported in $[0, \|W_{\alpha}\|^2]$ such that $\gamma_k(\alpha) := \alpha_0^2 \cdots \alpha_{k-1}^2 = \int t^k \, d\xi(t)$ ($k \geq 1$). If $W_{\alpha}$ is subnormal, and if for $h \geq 1$ we let $M_h := \bigvee \{e_n : n \geq h\}$ denote the invariant subspace obtained by removing the first $h$ vectors in the canonical orthonormal basis of $l^2(\mathbb{Z}_+)$, then the Berger measure of $W_{\alpha}|_{M_h}$ is $\frac{1}{\gamma_h} t^h \, d\xi(t)$.

The classes of (weakly) $k$-hyponormal operators have been studied in an attempt to bridge the gap between subnormality and hyponormality ([5], [6], [7], [8], [9], [10], [11], [12], [13], [15]).

2. Main Results

Let $T \in \mathcal{L}(\mathcal{H})$ with $\|T\| = 1$. For each nonzero vector $x$ in $\mathcal{H}$, define $W_x$ to the weighted shift with weighted sequence $\{ \|T^{n+1}x\| / \|T^n x\| \}_{n=0}^\infty$. A. Lambert([14]) showed the following result.

**Theorem 1.** ([14]) $T$ is subnormal if and only if $W_x$ is subnormal for each $x \in \mathcal{H}$.

More generally, T. Trent gave([16]) a criterion for subnormality which involves looking at only one vector of $\mathcal{H}$ at a time. For $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, denote the orbit of $T$ i.e., the closed linear span of $\{ T^n x : n = 0, 1, \cdots \}$ by $\mathcal{H}_x(T)$.

**Theorem 2.** ([16]) $T$ is subnormal if and only if the restriction of $T$ to $\mathcal{H}_x(T)$, $T|_{\mathcal{H}_x(T)}$ is subnormal for each $x \in \mathcal{H}$.

In this paper we discuss about these analogy for $k$-hyponormal operator. For weighted shifts, there is no gap between hyponormality and paranormality (i.e., $\|T^2 x\| \geq \|T x\|^2$ for all unit vector $x \in \mathcal{H}$). So the analogy of Theorem 1 does not work for hyponormality (see Example 4). However, we have:

**Theorem 3.** If $T$ is $k$-hyponormal for $k \geq 1$, then $W_x$ is $k$-hyponormal for each $x \in \mathcal{H}$.
Proof. Note that \[
\frac{||T^{n+1}x||}{||T^nx||} = \frac{||T^{n+1}\frac{ix}{||T^nx||}||}{||T^n\frac{ix}{||T^nx||}||}
\] for each \(n \geq 0\). It thus suffices to show that \(W_x\) is \(k\)-hyponormal for each \(x \in \mathcal{H}\) with \(||x|| = 1\). Recall that ([6, Theorem 4(d)] \(W_x\) is \(k\)-hyponormal if and only if
\[
A(n; k) := \begin{pmatrix}
\gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\
\gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k}
\end{pmatrix} \geq 0 \text{ for all } n \geq 0,
\]
where \(\gamma_i\) is the moments of \(\alpha\). Note that \(\gamma_n = ||T^nx||^2\) for \(W_x\) with \(||x|| = 1\). If \(T\) is \(k\)-hyponormal, then
\[
B(n; k) := D(n)^* \begin{pmatrix}
I & T^* & \cdots & T^{*k} \\
T & T^*T & \cdots & T^{*k}T \\
\vdots & \vdots & \ddots & \vdots \\
T^k & T^{*k} & \cdots & T^{*k}T
\end{pmatrix} D(n) \geq 0 \text{ for all } n \geq 0,
\]
where
\[
D(n) := \begin{pmatrix}
T^n & 0 & \cdots & 0 \\
0 & T^{n+1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & T^{n+k}
\end{pmatrix}.
\]
In particular, (1) implies \(B(n; k)\) is weakly positive, i.e.,
\[
\left\langle B(n; k) \begin{pmatrix}
\lambda_1 x \\
\vdots \\
\lambda_k x
\end{pmatrix}, \begin{pmatrix}
\lambda_1 x \\
\vdots \\
\lambda_k x
\end{pmatrix} \right\rangle \geq 0 \text{ for all } x \in \mathcal{H} \text{ and } \lambda_1, \cdots, \lambda_k \in \mathbb{C}.
\]
But, (2) is equivalent to
\[
\left\langle \begin{pmatrix}
\langle T^{*n}T^nx, x \rangle \\
\langle T^{*(n+1)}T^{n+1}x, x \rangle \\
\vdots \\
\langle T^{*(n+k)}T^{n+k}x, x \rangle
\end{pmatrix}, \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_k
\end{pmatrix} \right\rangle \geq 0 \text{ for all } x \in \mathcal{H} \text{ and } \lambda_1, \cdots, \lambda_k \in \mathbb{C}.
\]
is positive for all \(x \in \mathcal{H}\) and \(\lambda_1, \cdots, \lambda_k \in \mathbb{C}\). Thus \(A(n; k) \geq 0\) for all \(n \geq 0\). Therefore \(W_x\) is \(k\)-hyponormal for each \(x \in \mathcal{H}\) with \(||x|| = 1\). \(\square\)

The converse is not true in general.
Example 4. Let $T \in \mathcal{L}(\mathcal{H})$ be a paranormal nonhyponormal operator. Then $W_x$ is hyponormal for each $x \in \mathcal{H}$ but $T$ is not hyponormal.

Proof. Recall that $T$ is paranormal if and only if $\|T^2x\| \geq \|Tx\|^2$ for all unit vector $x \in \mathcal{H}$. Hence replacing $x$ by $\frac{x}{\|x\|}$, we have that if $T$ is paranormal then $\frac{\|T^2x\|}{\|x\|} \geq \frac{\|Tx\|}{\|x\|}$ for every $x \in \mathcal{H}$. Substituting $Tx$ for $x$ and repeating this process, we get $\frac{\|T^{n+2}x\|}{\|T^{n+1}x\|} \geq \frac{\|T^{n+1}x\|}{\|T^nx\|}$ for every $x \in \mathcal{H}$. Thus, $W_x$ is hyponormal for each $x \in \mathcal{H}$. \hfill \Box

References

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