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FUZZY PARTIAL ORDER RELATIONS AND FUZZY LATTICES

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Abstract. We characterize a fuzzy partial order relation using its level set, find sufficient conditions for the image of a fuzzy partial order relation to be a fuzzy partial order relation, and find sufficient conditions for the inverse image of a fuzzy partial order relation to be a fuzzy partial order relation. Also we define a fuzzy lattice as fuzzy relations, characterize a fuzzy lattice using its level set, show that a fuzzy totally ordered set is a distributive fuzzy lattice, and show that the direct product of two fuzzy lattices is a fuzzy lattice.

1. Introduction

The concept of a fuzzy set was first introduced by Zadeh ([5]) and this concept was adapted by Goguen ([2]) and Sanchez ([3]) to define and study fuzzy relations. Yuan and Wu ([4]) introduced the concepts of fuzzy sublattices and fuzzy ideals of a lattice. Ajmal and Thomas ([1]) defined a fuzzy lattice as a fuzzy algebra and characterized fuzzy sublattices. As a continuation of these studies, we define a fuzzy lattice as a fuzzy relation and work on fuzzy posets and fuzzy lattices in this note.

In section 2, we characterize a fuzzy partial order relation using its level set, find sufficient conditions for the image of a fuzzy partial order relation to be a fuzzy partial order relation, and find sufficient conditions for the inverse image of a fuzzy partial order relation to be a fuzzy partial order relation. In section 3, we define a fuzzy lattice as a fuzzy relation, develop some basic properties of fuzzy lattices, characterize a fuzzy lattice using its level set, show that a fuzzy totally

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ordered set is a distributive fuzzy lattice, and show that the direct product of two fuzzy lattices is a fuzzy lattice.

2. Fuzzy partial order relations

In this section we give some definitions and develop some properties of fuzzy partial order relations.

DEFINITION 2.1. Let X be a set. A function $A : X \times X \to [0,1]$ is called a *fuzzy relation* in X. The fuzzy relation A in X is *reflexive* iff $A(x,x) = 1$ for all $x \in X$, A is *transitive* iff $A(x,z) \geq \sup_{y \in X} \min(A(x,y),\ A(y,z))$, and A is *antisymmetric* iff $A(x,y) > 0$ and $A(y,x) > 0$ implies $x = y$. A fuzzy relation A is a *fuzzy partial order relation* if A is reflexive, antisymmetric, and transitive. A fuzzy partial order relation A is a *fuzzy total order relation* iff $A(x,y) > 0$ or $A(y,x) > 0$ for all $x, y \in X$. If A is a fuzzy partial order relation in a set X, then (X, A) is called a *fuzzy partially ordered set* or a *fuzzy poset*. If B is a fuzzy total order relation in a set X, then (X, B) is called a *fuzzy totally ordered set* or a *fuzzy chain*.

PROPOSITION 2.2. *Let (X, A) be a fuzzy poset (or chain) and $Y \subseteq X$. If $B = A|_{Y \times Y}$, then (Y, B) is a fuzzy poset (or chain), where $B = A_{Y \times Y}$.*

Proof. Straightforward. □

If A is a fuzzy relation on a set X, then the fuzzy relation $A^{-1} : X \times X \to [0,1]$ defined by $A^{-1}(x,y) = A(y,x)$ is called a *converse* of A. Note that the converse of any fuzzy partial order relation is itself a fuzzy partial order relation.

PROPOSITION 2.3. *Let $\{A_i : i \in I\}$ be a collection of fuzzy partial order relations in a set X. Then $(X, \bigcap_{i \in I} A_i)$ is a fuzzy poset.*

Proof. It is obvious that $\bigcap_{i \in I} A_i$ is reflexive and antisymmetric.

$$\begin{aligned}\bigcap_{i\in I} A_i(x,z) &= \min_{i\in I} A_i(x,z) \geq \min_{i\in I} \sup_{y\in X} \min[A_i(x,y), A_i(y,z)] \\ &\geq \sup_{y\in X} \min \min_{i\in I}[A_i(x,y), A_i(y,z)] \\ &= \sup_{y\in X} \min[\min_{i\in I} A_i(x,y), \min_{i\in I} A_i(y,z)] \\ &= \sup_{y\in X} \min[(\bigcap_{i\in I} A_i)(x,y), (\bigcap_{i\in I} A_i)(y,z)].\end{aligned}$$

Thus $(X, \bigcap_{i=1}^{n} A_i)$ is a fuzzy poset in X. □

However, it is easy to see that for fuzzy partial order relations A and B in a set X, $(X, A \cup B)$ is not necessarily a fuzzy poset.

We define the level set $B_p = \{(x,y) \in X \times X : B(x,y) \geq p\}$ of a fuzzy relation B in a set X and characterize a relationship between a fuzzy partial order relation and its level set.

Proposition 2.4. *Let $B : X \times X \to [0,1]$ be a fuzzy relation and let $B_p = \{(x,y) \in X \times X : B(x,y) \geq p\}$. Then B is a fuzzy partial order relation iff the level set B_p is a partial order relation in $X \times X$ for all p such that $0 < p \leq 1$.*

Proof. (⇒) Let B be a fuzzy partial order relation. Since $B(x,x) = 1$ for all $x \in X$, $(x,x) \in B_p$ for all p such that $0 < p \leq 1$. Suppose $(x,y) \in B_p$ and $(y,x) \in B_P$. Then $B(x,y) \geq p > 0$ and $B(y,x) \geq p > 0$, and hence $x = y$ for all p such that $0 < p \leq 1$. Suppose $(x,y) \in B_p$ and $(y,z) \in B_P$. Then $B(x,y) \geq p$ and $B(y,z) \geq p$. Since $B(x,z) \geq \sup_{r\in X} \min\,[B(x,r), B(r,z)]$, $B(x,z) \geq \min(B(x,y), B(y,z)) \geq p$, that is, $(x,z) \in B_p$ for all p such that $0 < p \leq 1$.
(⇐) Let B_p be a partial order relation for all p such that $0 < p \leq 1$. Then $(x,x) \in B_p$ for all p such that $0 < p \leq 1$. Thus $(x,x) \in B_1$, that is, $B(x,x) = 1$. Suppose $B(x,y) > 0$ and $B(y,x) > 0$. Then $B(x,y) > v > 0$ for some $v \in \mathbb{R}$ and $B(y,x) > w > 0$ for some $w \in \mathbb{R}$. Let $u = \min(v,w)$. Then $B(x,y) > u > 0$ and $B(y,x) > u > 0$. Thus $(x,y), (y,x) \in B_u$. Since B_u is antisymmetric, $x = y$. Suppose

$(x,y),(y,z) \in B_p$. Since B_p is transitive, $(x,z) \in B_p$. That is, if $B(x,y) \geq p$ and $B(y,z) \geq p$, then $B(x,z) \geq p$. Thus

$$B(x,z) \geq \sup_{r \in X} \min(B(x,r), B(r,z)).$$

□

We find sufficient conditions for the image of a fuzzy partial order relation in a set to be a fuzzy partial order relation and find sufficient conditions for the inverse image of a fuzzy partial order relation in a set to be a fuzzy partial order relation.

DEFINITION 2.5. Let X and Y be sets and let $f : X \times X \to Y \times Y$ be a function. Let B be a fuzzy relation in Y. Then $f^{-1}(B)$ is a fuzzy relation in X defined by $f^{-1}(B)(x,y) = B(f(x,y))$. Let A be a fuzzy relation in X. Then $f(A)$ is a fuzzy relation in Y defined by

$$f(A)(p,q) = \begin{cases} \sup_{(a,b) \in f^{-1}(p,q)} A(a,b), & \text{if } f^{-1}(p,q) \neq \emptyset \\ 0, & \text{if } f^{-1}(p,q) = \emptyset. \end{cases}$$

THEOREM 2.6. *Let X and Y be sets and let B be a fuzzy partial order relation in Y. Let $\phi : X \times X \to Y \times Y$ be a map such that*

(1) $\phi_1(x,x) = \phi_2(x,x)$ *for all* $x \in X$,
(2) $\phi_1(x,y) = \phi_1(x,z)$ *for all* $x,y,z \in X$,
(3) $\phi_2(p,q) = \phi_2(r,q)$ *for all* $p,q,r \in X$,
(4) $p \neq q$ *implies* $\phi_1(p,q) \neq \phi_1(q,p)$ *(or* $p \neq q$ *implies* $\phi_2(p,q) \neq \phi_2(q,p)$*),*

where $\phi(x,y) = (\phi_1(x,y), \phi_2(x,y))$. *Then* $(X,\ \phi^{-1}(B))$ *is a fuzzy poset.*

Proof. Since $\phi_1(x,x) = \phi_2(x,x)$,

$$(\phi^{-1}(B))(x,x) = B(\phi(x,x)) = B(\phi_1(x,x), \phi_2(x,x)) = 1$$

for all $x \in X$. By (1), (2), and (3) of our hypothesis, $\phi_1(x,y) = \phi_1(x,x) = \phi_2(x,x) = \phi_2(y,x)$ for all $x,y \in X$.

Suppose $(\phi^{-1}(B))(x,y) > 0$ and $(\phi^{-1}(B))(y,x) > 0$.

Then

$$B(\phi(x,y)) = B(\phi_1(x,y),\ \phi_2(x,y)) > 0$$

and

$$B(\phi(y,x)) = B(\phi_1(y,x), \phi_2(y,x)) > 0.$$

Since $\phi_1(x,y) = \phi_2(y,x)$ for all $x, y \in X$,

$$B(\phi_1(x,y), \phi_2(x,y)) > 0$$

and

$$B(\phi_2(x,y), \phi_1(x,y)) > 0.$$

Since B is antisymmetric, $\phi_1(x,y) = \phi_2(x,y) = \phi_1(y,x) = \phi_2(y,x)$. By (4) of our hypothesis, $x = y$. Thus $\phi^{-1}(B)$ is antisymmetric.

$$\begin{aligned}(\phi^{-1}(B))(x,z) &= B(\phi(x,z)) = B(\phi_1(x,\ z),\ \phi_2(x,\ z)) \\ &\geq \sup_{y\in X} \min[B(\phi_1(x,z),\ y), B(y, \phi_2(x,z))].\end{aligned}$$

Since $\phi_1(x,y) = \phi_1(x,z)$ and $\phi_2(p,q) = \phi_2(r,q)$ by (3) and (4) of our hypothesis,

$$\begin{aligned}(\phi^{-1}(B))(x,\ z) &\geq \sup_{y\in X} \min[B(\phi_1(x,t),\ y), B(y,\ \phi_2(t,z))] \\ &\geq \sup_{t\in X} \min[B(\phi_1(x,t), \phi_2(x,t)), B(\phi_2(x,t),\ \phi_2(t,z))].\end{aligned}$$

Since $\phi_1(x,y) = \phi_2(y,x)$ for all $x,\ y \in X$,

$$(\phi^{-1}(B))(x,z) \geq \sup_{t\in X} \min[B(\phi_1(x,t), \phi_2(x,t)), B(\phi_1(t,x), \phi_2(t,z))]$$

Since $\phi_1(t,x) = \phi_1(t,z)$ by (2) of our hypothesis,

$$\begin{aligned}(\phi^{-1}(B))(x,z) &\geq \sup_{t\in X} \min[B(\phi(x,t)), B(\phi(t,z))] \\ &= \sup_{t\in X} \min[(\phi^{-1}(B))(x,t), (\phi^{-1}(B))(t,z)].\end{aligned}$$

□

THEOREM 2.7. *Let X and Y be sets and Let A be a fuzzy partial order relation in X. Let $\phi : X \times X \to Y \times Y$ be a map such that*

(1) *for each $y \in Y$, there exists $x \in X$ such that $\phi(x,x) = (y,y)$,*

(2) *for each $x, z \in X$, there exists $y \in Y$ such that $\phi(x,z) = (y,y)$.*

Then $(Y, \phi(A))$ is a fuzzy poset.

Proof. By (1) of our hypothesis,

$$(\phi(A))(y,y) = \sup_{(p,q)\in\phi^{-1}(y,y)} A(p,\ q) = 1$$

for all $y \in Y$.

If $p \neq q$, then $\phi^{-1}(p,q) = \emptyset$ by (2) of our hypothesis, and hence

$$(\phi(A))(p,q) = \sup_{(s,t)\in\phi^{-1}(p,q)} A(s,\ t) = 0.$$

By the contrapositive law, $(\phi(A))(p,q) > 0$ implies $p = q$. Thus $(\phi(A))(p,q) > 0$ and $(\phi(A))(q,p) > 0$ implies $p = q$. That is, $\phi(A)$ is antisymmetric. If $x = z$,

$$(\phi(A))(x,z) = \sup_{(s,t)\in\phi^{-1}(x,x)} A(s,t) = 1$$

and hence

$$(\phi(A))(x,z) \geq \sup_{y\in X} \min[(\phi(A))(x,y),\ (\phi(A))(y,z)].$$

Suppose $x \neq z$. Then $x \neq y$ or $z \neq y$ for all $y \in Y$. If $x \neq y$,

$$(\phi(A))(x,y) = \sup_{(s,t)\in\phi^{-1}(x,y)} A(s,t) = 0$$

by (2) of our hypothesis. If $y \neq z$,

$$(\phi(A)(y,z) = \sup\,(s,t) \in \phi^{-1}(y,z)\ A(s,t) = 0.$$

Thus $(\phi(A))(x,y) = 0$ or $(\phi(A))(y,z)$ for all $y \in Y$. That is,

$$\sup_{y\in Y} \min[(\phi(A))(x,y),\ (\phi(A))(y,z)] = 0.$$

Hence

$$(\phi(A))(x,z) \geq \sup y \in Y \min[(\phi(A))(x,y),(\phi(A))(y,z)].$$

□

3. Fuzzy lattices

In this section, we define a fuzzy lattice as a fuzzy partial order relation and develop some properties of fuzzy lattices.

Definition 3.1. Let (X, A) be a fuzzy poset and let $B \subseteq X$. An element $u \in X$ is said to be an *upper bound* for a subset B iff $A(b, u) > 0$ for all $b \in B$. An upper bound u_0 for B is the *least upper bound* of B iff $A(u_0, u) > 0$ for every upper bound u for B. An element $v \in X$ is said to be a *lower bound* for a subset B iff $A(v, b) > 0$ for all $b \in B$. A lower bound v_0 for B is the *greatest lower bound* of B iff $A(v, v_0) > 0$ for every lower bound v for B.

We denote the least upper bound of the set $\{x, y\}$ by $x \vee y$ and denote the greatest lower bound of the set $\{x, y\}$ by $x \wedge y$.

Definition 3.2. Let (X, A) be a fuzzy poset. (X, A) is a *fuzzy lattice* iff $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.

Example. Let $X = \{x, y, z\}$ and let $A : X \times X \to [0, 1]$ be a fuzzy relation such that $A(x, x) = A(y, y) = A(z, z) = 1$, $A(x, y) = A(x, z) = A(y, z) = 0$, $A(y, x) = 0.5$, $A(z, x) = 0.3$, and $A(z, y) = 0.2$. Then it is easily checked that A is a fuzzy partial order relation. Also $x \vee y = x$, $x \vee z = x$, $y \vee z = y$, $x \wedge y = y$, $x \wedge z = z$, and $y \wedge z = z$. Thus (X, A) is a fuzzy lattice.

Proposition 3.3. *Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then*

(1) $A(x, x \vee y) > 0$, $A(y, x \vee y) > 0$, $A(x \wedge y, x) > 0$, $A(x \wedge y, y) > 0$.
(2) $A(x, z) > 0$ *and* $A(y, z) > 0$ *implies* $A(x \vee y, z) > 0$.
(3) $A(z, x) > 0$ *and* $A(z, y) > 0$ *implies* $A(z, x \wedge y) > 0$.
(4) $A(x, y) > 0$ *iff* $x \vee y = y$.
(5) $A(x, y) > 0$ *iff* $x \wedge y = x$.
(6) *If* $A(y, z) > 0$, *then* $A(x \wedge y,\ x \wedge z) > 0$ *and* $A(x \vee y,\ x \vee z) > 0$.

Proof. (1), (2), and (3) are Straightforward.
(4) Suppose $A(x, y) > 0$. Since $A(y, y) = 1 > 0$, $A(x \vee y, y) > 0$ by (2). Since $A(y, x \vee y) > 0$ by (1), $x \vee y = y$ by the antisymmetry of A. Conversely, suppose $x \vee y = y$. Then $A(x, y) = A(x, x \vee y) > 0$ by (1).
(5) The proof is similar to that of (4).

(6) Suppose $A(y,z)>0$. Then

$$\begin{aligned} A(x\wedge y,\ z) &\geq \sup_{p\in X}\min[A(x\wedge y,\ p),A(p,z)] \\ &\geq \min\ [A(x\wedge y,\ y),A(y,z)]>0. \end{aligned}$$

Since $A(x\wedge y,\ x)>0$ by (1), $x\wedge y$ is a lower bound of $\{x,z\}$. Since $x\wedge z$ is the greatest lower bound of $\{x,z\}$, $A(x\wedge y,\ x\wedge z)>0$.

$$\begin{aligned} A(y,\ x\vee z) &\geq \sup_{p\in X}\min[A(y,\ p),A(p,x\vee z)] \\ &\geq \min\ [A(y,\ z),A(z,x\vee z)]>0. \end{aligned}$$

Since $A(x,\ x\vee z)>0$ by (1), $A(x\vee y,x\vee z)>0$ by (2). □

Proposition 3.4. *Let (X,A) be a fuzzy lattice and let $x,y,z\in X$. Then*

(1) $x\vee x=x$, $x\wedge x=x$.
(2) $x\vee y=y\vee x$, $x\wedge y=y\wedge x$.
(3) $(x\vee y)\vee z=x\vee(y\vee z)$, $(x\wedge y)\wedge z=x\wedge(y\wedge z)$.
(4) $(x\vee y)\wedge x=x$, $(x\wedge y)\vee x=x$.

Proof. (1) and (2) are straightforward.
(3) Since $A(x,x\vee(y\vee z))>0$ and

$$\begin{aligned} A(y,x\vee(y\vee z)) &\geq \sup_{k\in X}\min[A(y,k),A(k,x\vee(y\vee z))] \\ &\geq \min[A(y,y\vee z),A(y\vee z,x\vee(y\vee z))]>0 \end{aligned}$$

$A(x\vee y,x\vee(y\vee z))>0$ by (2) of Proposition 3.3. Since

$$\begin{aligned} A(z,x\vee(y\vee z)) &\geq \sup_{k\in X}\min[A(z,k),A(k,x\vee(y\vee z))] \\ &\geq \min[A(z,y\vee z),A(y\vee z,x\vee(y\vee z))]>0, \end{aligned}$$

$A((x\vee y)\vee z,x\vee(y\vee z))>0$ by (2) of Proposition 3.3. Similarly we may show $A(x\vee(y\vee z),(x\vee y)\vee z)>0$. By the antisymmetry of A, $(x\vee y)\vee z=x\vee(y\vee z)$. Similarly we may show $(x\wedge y)\wedge z=x\wedge(y\wedge z)$.

(4) Let $B = \{x \vee y, x\}$. Since $A(x, x \vee y) > 0$ and $A(x, x) = 1 > 0$, x is a lower bound of B. If z is a lower bound of B, then $A(z, x) > 0$. Thus x is the greatest lower bound of B. Hence $(x \vee y) \wedge x = x$. Similarly we may show $(x \wedge y) \vee x = x$. □

We now turn to a characterization of the relationship between a fuzzy lattice and its level set.

PROPOSITION 3.5. *Let* $B : X \times X \to [0, 1]$ *be a fuzzy relation and let* $B_p = \{(x, y) \in X \times X : B(x, y) \geq p\}$. *If* (X, B_p) *is a lattice for every* p *with* $0 < p \leq 1$, *then* (X, B) *is a fuzzy lattice.*

Proof. Let (X, B_p) be a lattice for every p with $0 < p \leq 1$. Then (X, B) is a fuzzy poset by Proposition 2.4. For $x, y \in X$, there exists $r \in X$ such that $(x, r) \in B_p$, $(y, r) \in B_p$, and $(r, u) \in B_p$ for every upper bound u for $\{x, y\}$. That is, there exists $r \in X$ such that $B(x, r) \geq p > 0$, $B(y, r) \geq p > 0$, and $B(r, u) \geq p > 0$ for every upper bound u for $\{x, y\}$. Thus there exists a least upper bound $r \in X$ of $\{x, y\}$. Similarly we may show that there exists a greatest lower bound $c \in X$ of $\{x, y\}$. Hence (X, B) is a fuzzy lattice. □

PROPOSITION 3.6. *Let* $B : X \times X \to [0, 1]$ *be a fuzzy relation and let* $B_p = \{(x, y) \in X \times X : B(x, y) \geq p\}$. *If* (X, B) *is a fuzzy lattice, then* (X, B_p) *is a lattice for some* $p > 0$.

Proof. Let (X, B) be a fuzzy lattice. Then B_p is a partial order relation for every p with $0 < p \leq 1$ by Proposition 2.4. Let $x, y \in X$ and let U be the set of all upper bounds for $\{x, y\}$ and L be the set of all lower bounds for $\{x, y\}$. Then there exists $r \in X$ such that $B(x, r) > 0$, $B(y, r) > 0$, and $B(r, u) > 0$ for all $u \in U$ and there exists $c \in X$ such that $B(c, x) > 0$, $B(c, y) > 0$, and $B(l, c) > 0$ for all $l \in L$. Let $p = \min[B(x, r), B(y, r), B(r, u), B(c, x), B(c, y), B(l, c)] > 0$. Then there exists $r \in X$ such that $B(x, r) \geq p > 0$, $B(y, r) \geq p > 0$, and $B(r, u) \geq p > 0$ for all $u \in U$ and there exists $c \in X$ such that $B(c, x) \geq p > 0$, $B(c, y) \geq p > 0$, and $B(l, c) \geq p > 0$ for all $l \in L$. That is, there exists $r \in X$ such that $(x, r) \in B_p$, $(y, r) \in B_p$, and $(r, u) \in B_p$ for all $u \in U$ and there exists $c \in X$ such that $(c, x) \in B_p$, $(c, y) \in B_p$, and $(l, c) \in B_p$ for all $l \in L$. Thus there exists a least upper bound $r \in X$ of $\{x, y\}$ and there exists a greatest lower bound

$c \in X$ of $\{x, y\}$ for some $p > 0$. Hence (X, B_p) is a lattice for some $p > 0$. □

We now turn to the characterizations of distributive fuzzy lattices and modular fuzzy lattices.

PROPOSITION 3.7. *(Distributive inequalities) Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then $A((x \wedge y) \vee (x \wedge z),\ x \wedge (y \vee z)) > 0$ and $A(x \vee (y \wedge z),\ (x \vee y) \wedge (x \vee z)) > 0$.*

Proof. Since $A(x \wedge y,\ y) > 0$ and $A(y,\ y \vee z) > 0$, $A(x \wedge y,\ y \vee z) > 0$. Since $A(x \wedge y,\ x) > 0$, $A(x \wedge y,\ x \wedge (y \vee z)) > 0$ by (3) of Proposition 3.3. Since $A(x \wedge z,\ z) > 0$ and $A(z,\ y \vee z) > 0$, $A(x \wedge z,\ y \vee z) > 0$. Since $A(x \wedge z,\ x) > 0$, $A(x \wedge z,\ x \wedge (y \vee z)) > 0$ by (3) of Proposition 3.3. Thus $x \wedge (y \vee z)$ is an upper bound of $\{x \wedge y,\ x \wedge z\}$. Since $(x \wedge y) \vee (x \wedge z)$ is the least upper bound of $\{x \wedge y,\ x \wedge z\}$, $A((x \wedge y) \vee (x \wedge z),\ x \wedge (y \vee z)) > 0$. Similarly, we may prove $A(x \vee (y \wedge z),\ (x \vee y) \wedge (x \vee z)) > 0$. □

DEFINITION 3.8. Let (X, A) be a fuzzy lattice. (X, A) is *distributive* iff $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.

From the distributive inequalities, (X, A) is distributive iff $A(x \wedge (y \vee z),\ (x \wedge y) \vee (x \wedge z)) > 0$ and $A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z)) > 0$.

PROPOSITION 3.9. *Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then*

$$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z) \Longleftrightarrow (x \vee y) \wedge (x \vee z) = x \vee (y \wedge z).$$

Proof. ($\Rightarrow$) By (4) of Proposition 3.4, $(x \vee y) \wedge x = x$. Thus $A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z)) = A([(x \vee y) \wedge x] \vee [(x \vee y) \wedge z],\ x \vee (y \wedge z)) = A(x \vee [z \wedge (x \vee y)],\ x \vee (y \wedge z)) = A(x \vee [(z \wedge x) \vee (z \wedge y)],\ x \vee (y \wedge z)) = A([x \vee (z \wedge x)] \vee (z \wedge y),\ x \vee (y \wedge z))$. Since $x \vee (z \wedge x) = x$ by (4) of Proposition 3.4, $A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z)) = A(x \vee (z \wedge y),\ x \vee (y \wedge z)) = A(x \vee (y \wedge z),\ x \vee (y \wedge z))$. Thus $A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z)) > 0$. Similarly we may show $A(x \vee (y \wedge z),\ (x \vee y) \wedge (x \vee z)) > 0$. Since A is antisymmetric, $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$.
($\Leftarrow$) $A((x \wedge y) \vee (x \wedge z),\ x \wedge (y \vee z)) = A([(x \wedge y) \vee x] \wedge [(x \wedge y) \vee z],\ x \wedge (y \vee z)) = A(x \wedge [z \vee (x \wedge y)],\ x \wedge (y \vee z)) = A(x \wedge [(z \vee x) \wedge (z \vee y)],\ x \wedge (y \vee z)) =$

$A([x \wedge (z \vee x)] \wedge (z \vee y),\ x \wedge (y \vee z)) = A(x \wedge (z \vee y),\ x \wedge (y \vee z)) = A(x \wedge (y \vee z),\ x \wedge (y \vee z))$. Thus $A((x \wedge y) \vee (x \wedge z),\ x \wedge (y \vee z)) > 0$. Similarly we may show $A(x \wedge (y \vee z),\ (x \wedge y) \vee (x \wedge z)) > 0$. Since A is antisymmetric, $(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$. □

Theorem 3.10. *Let (X, A) be a fuzzy totally ordered set. Then (X, A) is a distributive fuzzy lattice.*

Proof. Let (X, A) be a fuzzy totally ordered set and let $x, y \in X$. Then $A(x, y) > 0$ or $A(y, x) > 0$. Suppose $A(x, y) > 0$. Since $A(y, y) = 1 > 0$, y is an upper bound of $\{x, y\}$. Let u be an upper bound of $\{x, y\}$. Then $A(y, u) > 0$. Thus y is the least upper bound of $\{x, y\}$. Since $A(x, y) > 0$ and $A(x, x) = 1 > 0$, x is a lower bound of $\{x, y\}$. Let v be a lower bound of $\{x, y\}$. Then $A(v, x) > 0$. Thus x is the greatest lower bound of $\{x, y\}$. In case of $A(y, x) > 0$, we may show that x is the least upper bound of $\{x, y\}$ and y is the greatest lower bound of $\{x, y\}$. Hence (X, A) is a fuzzy lattice.
(i) First, we consider the case of $A(x, y) > 0$.
Suppose $A(x, y) > 0$. Then $x \wedge y = x$ by (5) of Proposition 3.3. Since $A(x \wedge (y \vee z),\ x) > 0$ by (1) of Proposition 3.3, $A(x \wedge (y \vee z),\ x \wedge y) > 0$. By (1) of Proposition 3.3, $A(x \wedge y,\ (x \wedge y) \vee (x \wedge z)) > 0$. $A(x \wedge (y \vee z), (x \wedge y) \vee (x \wedge z)) \geq \sup_{k \in X} \min\ [A(x \wedge (y \vee z),\ k),\ A(k, (x \wedge y) \vee (x \wedge z))] \geq \min\ [A(x \wedge (y \vee z),\ x \wedge y),\ A(x \wedge y, (x \wedge y) \vee (x \wedge z)] > 0$. By the distributive inequalities, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. By Proposition 3.9, $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$. Thus (X, A) is distributive.
(ii) We consider the case of $A(y, x) > 0$.
Suppose $A(y, x) > 0$. Then $x \vee y = x$ by (4) of Proposition 3.3. Thus $A((x \vee y) \wedge (x \vee z),\ x) = A(x \wedge (x \vee z),\ x) > 0$. By (1) of Proposition 3.3, $A(x,\ x \vee (y \wedge z)) > 0$. $A((x \vee y) \wedge (x \vee z),\ x \vee (y \wedge z)) \geq \sup_{k \in X} \min\ [A((x \vee y) \wedge (x \vee z),\ k),\ A(k,\ x \vee (y \wedge z))] \geq \min\ [A((x \vee y) \wedge (x \vee z),\ x),\ A(x,\ x \vee (y \wedge z))] > 0$. By the distributive inequalities, $(x \vee y) \wedge (x \vee z) = x \vee (y \wedge z)$. By Proposition 3.9, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Thus (X, A) is distributive. □

Proposition 3.11. *(Modular inequality) Let (X, A) be a fuzzy lattice and let $x, y, z \in X$. Then $A(x, z) > 0$ implies $A(x \vee (y \wedge z),\ (x \vee y) \wedge z) > 0$.*

Proof. Since $A(x,\ x \vee y) > 0$ and $A(x, z) > 0$, $A(x,\ (x \vee y) \wedge z) > 0$. Since $A(y \wedge z,\ y) > 0$ and $A(y,\ x \vee y) > 0$, $A(y \wedge z,\ x \vee y) > 0$. Since $A(y \wedge z,\ z) > 0$, $A(y \wedge z,\ (x \vee y) \wedge z) > 0$ by (3) of Proposition 3.3. Thus $A(x \vee (y \wedge z),\ (x \vee y) \wedge z) > 0$. □

Definition 3.12. A fuzzy lattice (X, A) is *modular* iff $A(x, z) > 0$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$ for $x, y, z \in X$.

By the modular inequality, a fuzzy lattice (X, A) is *modular* iff $A(x, z) > 0$ implies $A((x \vee y) \wedge z,\ x \vee (y \wedge z)) > 0$ for $x, y, z \in X$.

Proposition 3.13. *Let (X, A) be a distributive fuzzy lattice. Then (X, A) is modular.*

Proof. Since (X, A) is distributive, $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$. Thus $A((x \vee y) \wedge z, x \vee (y \wedge z)) = A((x \wedge z) \vee (y \wedge z), x \vee (y \wedge z))$. Since $A(x, z) > 0$, $x \wedge z = x$ by (5) of Proposition 3.3. Thus $A((x \vee y) \wedge z, x \vee (y \wedge z)) = A(x \vee (y \wedge z), x \vee (y \wedge z)) > 0$. Thus $(x \vee y) \wedge z = x \vee (y \wedge z)$. □

We now turn to the direct product of fuzzy lattices.

Definition 3.14. Let (P, A) and (Q, B) be fuzzy posets. The *direct product* PQ of P and Q is defined by $(PQ, A \times B)$, where $A \times B : PQ \to [0, 1]$ is a fuzzy relation defined by $(A \times B)((p_1, q_1), (p_2, q_2)) = \min\ [A(p_1, p_2), B(q_1, q_2)]$.

Theorem 3.15. *Let (P, A) and (Q, B) be fuzzy lattices. The the direct product $(PQ, A \times B)$ of (P, A) and (Q, B) is a fuzzy lattice.*

Proof. Let $(p_1, q_1), (p_2, q_2) \in PQ$. Then $(A \times B)((p_1, q_1), (p_1, q_1)) = \min[A(p_1, p_1), B(q_1, q_1)] = 1$. Suppose $(A \times B)((p_1, q_1), (p_2, q_2)) > 0$ and $(A \times B)((p_2, q_2), (p_1, q_1)) > 0$. Then $\min\ [A(p_1, p_2), B(q_1, q_2)] > 0$ and $\min[A(p_2, p_1), B(q_2, q_1)] > 0$. That is, $A(p_1, p_2) > 0$, $A(p_2, p_1) > 0$, $B(q_1, q_2) > 0$, and $B(q_2, q_1) > 0$. Thus $p_1 = p_2$ and $q_1 = q_2$, that is,

$(p_1, q_1) = (p_2, q_2)$.

$$\begin{aligned}
(A \times B)((p_1, q_1), (p_2, q_2)) &= \min[A(p_1, p_2), B(q_1, q_2)] \\
&\geq \min[\sup_{p \in P} \min(A(p_1, p), A(p, p_2)), \sup_{q \in Q} \min(B(q_1, q), B(q, q_2))] \\
&\geq \sup_{(p,q) \in PQ} \min[\min(A(p_1, p), A(p, p_2)), \min(B(q_1, q), B(q, q_2))] \\
&= \sup_{(p,q) \in PQ} \min[A(p_1, p), B(q_1, q), A(p, p_2), B(q, q_2)] \\
&= \sup_{(p,q) \in PQ} \min[\min(A(p_1, p), B(q_1, q)), \min(A(p, p_2), B(q, q_2))] \\
&= \sup_{(p,q) \in PQ} \min[(A \times B)((p_1, q_1), (p, q)), (A \times B)((p, q), (p_2, q_2))].
\end{aligned}$$

Thus PQ is a fuzzy partial order relation.
Let $(p_1, q_1), (p_2, q_2) \in PQ$. Then $(A \times B)((p_1, q_1), (p_1 \vee p_2, q_1 \vee q_2)) = \min\,[A(p_1, p_1 \vee p_2),\ B(q_1, q_1 \vee q_2)] > 0$ by (1) of Proposition 3.3. Similarly $(A \times B)((p_2, q_2), (p_1 \vee p_2, q_1 \vee q_2)) > 0$. Thus $(p_1 \vee p_2, q_1 \vee q_2)$ is an upper bound of $\{(p_1, q_1), (p_2, q_2)\}$. Let (s, t) be an upper bound of $\{(p_1, q_1), (p_2, q_2)\}$. Then $(A \times B)((p_1, q_1), (s, t)) > 0$ and $(A \times B)((p_2, q_2), (s, t)) > 0$. That is, $\min\,[A(p_1, s), B(q_1, t)] > 0$ and $\min\,[A(p_2, s), B(q_2, t)] > 0$. Since $A(p_1, s) > 0$ and $A(p_2, s) > 0$, $A(p_1 \vee p_2, s) > 0$ by (2) of Proposition 3.3. Since $B(q_1, t) > 0$ and $B(q_2, t) > 0$, $B(q_1 \vee q_2, t) > 0$ by (2) of Proposition 3.3. Thus $(A \times B)((p_1 \vee p_2, q_1 \vee q_2), (s, t)) = \min\,[A(p_1 \vee p_2, s),\ B(q_1 \vee q_2, t)] > 0$. That is, $(p_1 \vee p_2, q_1 \vee q_2)$ is the least upper bound of $\{(p_1, q_1), (p_2, q_2)\}$. That is, $(p_1, q_1) \vee (p_2, q_2) = (p_1 \vee p_2, q_1 \vee q_2)$. Similarly we may show $(p_1, q_1) \wedge (p_2, q_2) = (p_1 \wedge p_2, q_1 \wedge q_2)$. Hence $(PQ, A \times B)$ is a fuzzy lattice. □

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