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CONVERGENCE AND POWER SPECTRUM DENSITY OF ARIMA MODEL AND BINARY SIGNAL

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Abstract. We study the weak convergence of various models to Fractional Brownian motion. First, we consider arima process and ON/OFF source model which allows for long packet trains and long inter-train distances. Finally, we figure out power spectrum density as a Fourier transform of autocorrelation function of arima model and binary signal model.

1. Introduction

Random processes find a wide variety of applications. The most common use is as a model for noise in physical systems. A second class of applications concerns the modeling of random phenomena that are not noise but are nevertheless unknown to the system designer. An example would be a signal and image processing, digital control and communications([4], [9]). On the other hand, the various models for capturing the long-range dependent nature of network traffic is proposed and self-similarity and long range dependence have been observed in many time series, i.e. network traffic and finance([1], [2], [3], [5], [6]). In particular, fractional Brownian motion, ARIMA model and binary signal in modern packet network traffic has been the focus of much attention ([7]).

In this paper we consider arrival process based on autoregressive process and FARIMA process and show that the suitably scaled distributions of those processes converge to fractional Brownian motion in the sense of finite dimensional distributions. On the other hand, we consider

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idealized ON/OFF source model which allows for long packet trains and long inter-train distances. In particular, we figure out the coefficients of $B_H(t)$ and time t in the case of having Pareto distribution as ON/OFF periods. Finally, we has considered power spectrum densities of arima process and binary signal process.

In section 2, we define short range dependence, long range dependence, fractional Brownian motion, farima process and power spectrum denity. In section 3, we prove the weak convergence to Fractional Brownian motion of arima process. In section 4, we consider ON/OFF source model which allows for long packet trains and long inter-train distances. In section 5, we figure out power spectrum density of arima process and binary signal process.

2. Definition and preliminary

In this section we first define short range dependence, long range dependence, Fractional Brownian motion, FARIMA process and power spectrum density. Let $\tau_X(k)$ be the covariance of stationary stochastic process $X(t)$, i.e. $\tau_X(k) = \mathrm{Cov}(X(t), X(t+k))$.

Definition 2.1. A stationary stochastic process $X(t)$ exhibits *short range dependence* if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| < \infty$$

Definition 2.2. A stationary stochastic process $X(t)$ exhibits *long range dependence* if

$$\sum_{k=-\infty}^{\infty} |\tau_X(k)| = \infty$$

Definition 2.3. A stochastic process $\{B_H(t)\}$ is said to be a *Fractional Brownian motion*(FBM) with Hurst parameter H if

1. $B_H(t)$ has stationary increments
2. For $t > 0$, $B_H(t)$ is normally distributed with mean 0
3. $B_H(0) = 0$ $a.s.$
4. The increments of $B_H(t)$, $Z(j) = B_H(j+1) - B_H(j)$ satisfy

$$\tau_Z(k) = \frac{1}{2}\{|k+1|^{2H} + |k-1|^{2H} - 2k^{2H}\}$$

Definition 2.4. A stationary process X_t is called a *FARIMA(p, d, q) process* if

$$\phi(B)\nabla^d X_t = \theta(B)Z_t$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$, $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ and the coefficients $\phi_1, \cdots, \phi_p$ and $\theta_1, \cdots, \theta_q$ are constants,

$$\nabla^d = (1-B)^d = \sum_{i=0}^{\infty} b_i(-d)B^i$$

and B is the backward shift operator defined as $B^i X_t = X_{t-i}$ and

$$b_i(-d) = \prod_{k=1}^{i} \frac{k+d-1}{k} = \frac{\Gamma(i+d)}{\Gamma(d)\Gamma(i+1)}.$$

Now, we define a density for average power versus frequency for wide-sense stationary process.

Definition 2.5. Let $R_{XX}(\tau)$ be an autocorrelation function. Then we define the *power spectrum density(PSD)* $S_{XX}(\omega)$ to be its Fourier transform (if it exits), that is,

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}d\tau.$$

3. Convergence of ARIMA

Let $X^j(i)$ be the number of arrivals in the ith time unit of jth source. Let

$$X_M(i) = \sum_{j=1}^{M}(X^j(i) - E(X^j(i))),$$

and $\tau(k)$ denote the covariance of $X_1(i)$.

Lemma 3.1. ([6]) The stationary sequence

$$\frac{1}{M^{1/2}}X_M(i)$$

converges in the sense of finite dimensional distributions to $G_H(i)$, where $G_H(i)$ represents a stationary Gaussian process with covariance function of the same form as $\tau(k)$, as $M \to \infty$.

Lemma 3.2.
$$\lim_{T\to\infty}\lim_{M\to\infty}\frac{1}{T^H M^{1/2}}\sum_{i=0}^{[Tt]} X_M(i)$$
converges to $\{\sigma_0 B_H(t)|0 \le t \le 1\}$ in the sense of finite dimensional distributions.
(a) (Long Range dependence) If
$$\tau(k) \sim ck^{2H-2},\ \ c > 0 \ \text{ and } \ 1/2 < H < 1,$$
then $\sigma_0^2 = \dfrac{c}{H(2H-1)}$.
(b) If
$$\sum_{k=1}^{\infty}|\tau(k)| < \infty \ \text{ and } \ \sum_{k=1}^{\infty}\tau(k) = c > 0,$$
then $\sigma_0^2 = c$.
(c) (Short Range dependence)
$$\tau(k) \sim ck^{2H-2},\ \ c < 0 \ \text{ and } \ 0 < H < 1/2,$$
then $\sigma_0^2 = \dfrac{c}{H(2H-1)}$.

Proof. Set $Z_i = 1/M^{1/2}X_M(i)$. By Lemma 3.1, Z_i converges in the sense of finite dimensional distributions to $G_H(i)$ as M goes to infinity. By Theorem 7.2.11 of [7], the finite dimensional distributions of $T^{-H}\sum_{i=0}^{[Tt]} Z_i$ converges to those of $\{\sigma_0 B_H(t), 0 \le t \le 1\}$. □

We consider a FARIMA(p, d, q) which is both long range dependent and has heavy tails. $FARIMA(p,d,q)$ processes are capable of modeling both short and long range dependence in traffic models since the effect of d on distant samples decays hyperbolically as the lag increases while the effects of p and q decay exponentially.

Theorem 3.3 (FARIMA(0,d,0)). Let $X^i(j) = b_i(-d)a_{j-i}$. Then
$$\lim_{T\to\infty}\lim_{M\to\infty}\frac{1}{T^H M^{1/2}}\sum_{j=0}^{[Tt]}\sum_{i=1}^{M}(X^j(i)) = \sqrt{\frac{c}{H(2H-1)}}B_H(t).$$

Proof.
$$\tau(k) = \frac{(-1)^k(-2d)!}{(k-d)!(-k-d)!} \sim ck^{2d-1} \ \ as \ \ k \to \infty$$

where $H = d + 1/2, -1/2 < d < 1/2$ and $c = \dfrac{\Gamma(1-2d)\sin(\pi d)}{\pi}$. By Lemma 3.2, we get the result. $\square$

4. Convergence of binary signal

Suppose that there are M i.i.d. sources. Since each source sends its own sequence of packet trains, it has its own reward sequence $X^{(m)}(t)$. Therefore, the cumulative packet count at time t is

$$\sum_{m=1}^{M} X^{(m)}(t).$$

Rescaling time by a factor T , we consider the aggregated cumulative packet counts

$$X_M(Tt) = \int_0^{Tt} (\sum_{m=1}^{M} X^{(m)}(u))du$$

in the interval $[0, Tt]$.

LEMMA 4.1. The aggregate packet process $\{X_M(Tt), t \geq 0\}$ behaves statistically like

$$TM\frac{\mu_1}{\mu_1+\mu_2}t + T^H\sqrt{L(t)M}\,\sigma B_H(t)$$

for large M and T.

Proof. ([8], Theorem 1) $\square$

To specify the distributions of ON-period O_1 and OFF-periods O_2, let

$$\mu_1 = EO_1,\ \mu_2 = EO_2$$

and as $x \to \infty$, tailing distributions of O_1 , O_2 are

$$l_1 x^{-\alpha_1} L_1(x) \quad \text{and} \quad l_2 x^{-\alpha_2} L_2(x)$$

with $1 < \alpha_j < 2$, where is a constant $l_j > 0$ and $L_j > 0$ is a slowly varying function at infinity.

NOTATION. When $1 < \alpha_j < 2$, set

$$a_j = l_j(\Gamma(2-\alpha_j))/(\alpha_j - 1),$$

$$b = \lim_{t\to\infty} t^{\alpha_2-\alpha_1}\frac{L_1(t)}{L_2(t)}.$$

If $0 < b < \infty$ then set

$$\sigma^2 = \frac{2(\mu_2^2 a_1 b + \mu_1^2 a_2)}{(\mu_1+\mu_2)^3\Gamma(4-\alpha_{min})},$$

if $b = 0$ or $b = \infty$ then set

$$\sigma^2 = \frac{2\mu_{max}^2 a_{min}}{(\mu_1+\mu_2)^3\Gamma(4-\alpha_{min})}.$$

Suppose that ON/OFF periods O_j has the Pareto distribution, then

$$P(O_j > x) = K^{\alpha_j} x^{-\alpha_j} \quad \text{for} \quad x \geq K > 0.$$

Each periods has infinite variance in the case of $1 < \alpha_j < 2$.

THEOREM 4.2. Let O_j be ON/OFF-periods that has the Pareto distributions as above. Then, for large M and T, the aggregate packet process $\{X_M(Tt) \mid t \geq 0\}$ behaves statistically like

$$TM\frac{\alpha_1\alpha_2 - \alpha_1}{2\alpha_1\alpha_2 - \alpha_1 - \alpha_2}t + T^H\sigma B_H(t)$$

where, $H = (3-\alpha_{min})/2$.

Case 1. Suppose that O_j have the same distributions, i.e., $\alpha_1 = \alpha_2 = \alpha$, then

$$H = \frac{3-\alpha}{2}$$

and

$$\sigma^2 = \frac{K^{\alpha-1}\Gamma(2-\alpha)}{2\alpha\Gamma(4-\alpha)}$$

Case 2. If $\alpha_1 < \alpha_2$, then

$$H = \frac{3-\alpha_1}{2}$$

and

$$\sigma^2 = \frac{2K^2\alpha_1^2(\alpha_1-1)(\alpha_2-1)^3 a_{min}}{(2\alpha_1\alpha_2 K - \alpha_1 K - \alpha_2 K)^3}$$

Case 3. If $\alpha_1 > \alpha_2$, then

$$H = \frac{3-\alpha_2}{2}$$

and

$$\sigma^2 = \frac{2K^2\alpha_2^2(\alpha_2-1)(\alpha_1-1)^3 a_{min}}{(2\alpha_1\alpha_2 K - \alpha_1 K - \alpha_2 K)^3}$$

Proof. Since the expectation of the Pareto distribution is

$$\frac{\alpha_j K}{\alpha_j - 1}$$

for $j = 1, 2, \cdots$. By Lemma 4.1, the coefficient of time t is

$$\frac{\alpha_1\alpha_2 - \alpha_1}{2\alpha_1\alpha_2 - \alpha_1 - \alpha_2}.$$

Case 1. Since O_j have the same distributions, we get

$$\lim_{t\to\infty} t^{\alpha_2-\alpha_1} = 1.$$

And we know

$$\alpha_1 = \alpha_2 = K^\alpha \Gamma(2-\alpha)/(\alpha-1).$$

Thus, we get

$$\sigma^2 = \frac{K^{\alpha-1}\Gamma(2-\alpha)}{2\alpha\Gamma(4-\alpha)}$$

In the similar way, we can get Case 2 and Case 3. □

Let $X^j(t) = 1$ mean that there is a packet at time t and $X^j(t) = 0$ means that there is no packet. Viewing $X^j(t)$ as the reward at time t, we have a reward of 1 throughout an ON-period, then a reward of 0 throughout the following OFF-period, then 1 again, and so on.

THEOREM 4.3. Let $X^j(i)$ denote the increment process for the ith stationary binary sequence $X^j(t)$. Then

$$\lim_{T\to\infty}\lim_{M\to\infty}\frac{1}{T^H M^{1/2}}\left\{\sum_{j=0}^{[Tt]}\sum_{i=1}^{M}(X^j(i)) - \frac{\mu_1 M t}{\mu_1+\mu_2}\right\} = \sqrt{\frac{c}{H(2H-1)}}B_H(t).$$

Proof. We get

$$\tau(k) \sim ck^{2H-2},$$

as $k \to \infty$ and

$$E[X^j(i)] = \frac{\mu_1}{\mu_1+\mu_2}$$

if $E[\text{On period}] = \mu_1$ and $E[\text{Off period}] = \mu_2$. By Lemma 3.2, we get the result. □

5. Power spectrum density of ARIMA and binary signal

Using the backshift operator B, the $ARMA(p,q)$ model is expressed as

$$(1-\rho_1 B-\cdots-\rho_p B^p)Y_t=(1-\theta_1 B-\cdots-\theta_q B^q)e_t,$$

where $e_t \sim N(0,\sigma^2)$ is white noise. Replace B^j with $e^{i\omega j}$, getting

$$A(\omega)=1-\rho_1 e^{i\omega}-\rho_2 e^{i\omega 2}-\cdots-\rho_p e^{i\omega p}$$

on the autoregressive and

$$M(\omega)=1-\theta_1 e^{i\omega}-\theta_2 e^{i\omega 2}-\cdots-\theta_q e^{i\omega q}$$

on the moving average side. Start with the spectral density of e_t, which is $\sigma^2/2\pi$. The spectral density for $ARMA(p,q)$ process Y_t becomes

$$f_Y(\omega)=\frac{\sigma^2}{2\pi}\frac{M(\omega)M^*(\omega)}{A(\omega)A^*(\omega)},$$

where $A^*(\omega)$ and $M^*(\omega)$ are corresponding complex conjugate expressions of the complex polynomial $A(\omega)$ and $M(\omega)$.

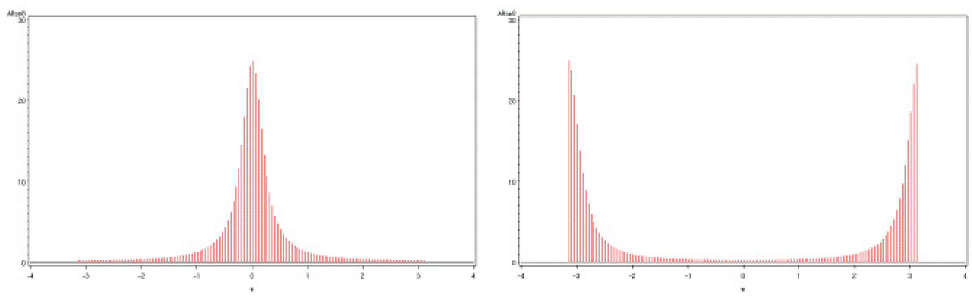

FIGURE 1. PSD of AR(0.8) and AR(−0.8)

For autoregressive order 1 series $AR(1)$, the theoretical spectral density is

$$f(\omega)=\frac{1}{2\pi(1+\rho^2-2\rho\cos(\omega))},$$

where ρ is the lag 1 autoregressive coefficient. Power spectrum density in the case of $\rho=0.8,\ -0.8$ is sketched in Figure 1.

For a moving average $MA(1)$ such as $X_t=e_t-\theta e_{t-1}$, the spectral density is

$$f_X(\omega)=\frac{1}{2\pi}(1+\theta^2-2\theta\cos(\omega)).$$

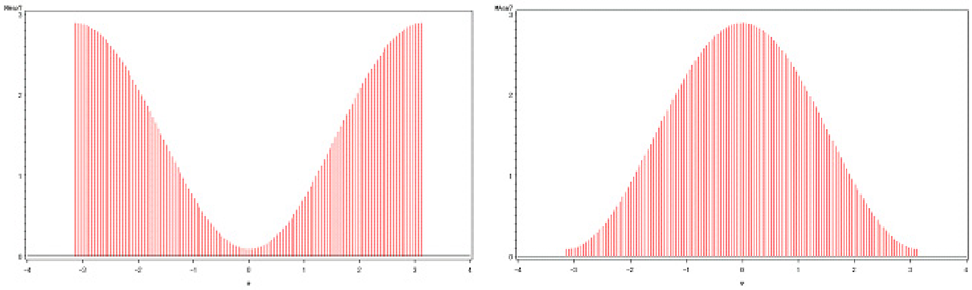

FIGURE 2. PSD of MA(0.7) and MA(−0.7)

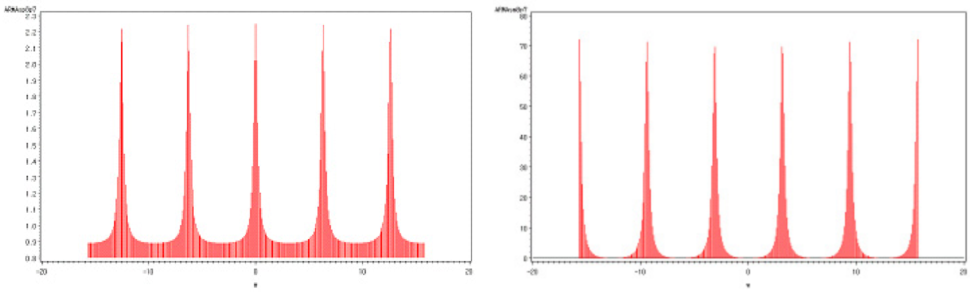

FIGURE 3. PSD of ARMA(0.8, 0.7) and ARMA(−0.8, 0.7)

Power spectrum density in the case of $\theta = 0.7, -0.7$ is sketched in Figure 2.

If Y_t has spectral density

$$f_Y(\omega) = \frac{1}{2\pi(1 + \rho^2 - 2\rho\cos(\omega))}$$

and is filtered to get $D_t = Y_t - \theta Y_{t-1}$, then the spectral density is

$$\begin{aligned} f_D(\omega) &= (1 + \theta^2 - 2\theta\cos(\omega)) f_Y(\omega) \\ &= \frac{1}{2\pi}(1 + \theta^2 - 2\theta\cos(\omega))/(1 + \rho^2 - 2\rho\cos(\omega)). \end{aligned}$$

Power spectrum density in the case of $\rho = 0.8, \theta = 0.7$ and $\rho = -0.8, \theta = 0.7$ is sketched in Figure 3.

On the other hand, we construct the RTS(random telegraph signal) on $t \geq 0$ as follows. Let $X(0) = \pm a$ with equal probability. Then take the Poisson arrival time sequence $T[n]$ and use it to switch the level of

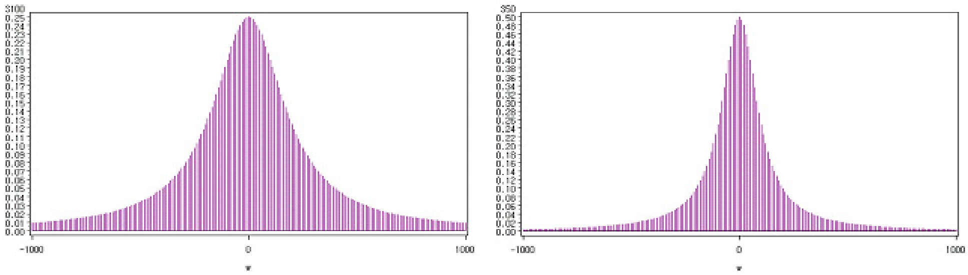

Figure 4. PSD of $\lambda = 100$ and $\lambda = 50$

the RTS, i.e., at $T[1]$ switch the sign of $X(t)$, and then at $T[2]$, and so forth. Clearly from the symmetry and the fact that the interarrival times $\tau[n]$ are stationary and form an independent random sequence, we must have $\mu_X(t) = 0$ and that $P_X(a) = P_X(-a) = 1/2$. Let $t_2 > t_1 > 0$, and consider

$$P_X(x_1, x_2) = P[X(t_1) = x_1, X(t_2) = x_2]$$

along with

$$P_X(x_2|x_1) = P[X(t_2) = x_2|X(t_1) = x_1].$$

Then the correlation function is

$$\begin{aligned} R_{XX}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \frac{1}{2}a^2(P_X(a|a) + P_X(-a|-a) - P_X(-a|a) - P_X(a|-a)). \end{aligned}$$

Hence, writing the average number of transitions per unit time as λ, and substituting $\tau = t_2 - t_1$, we get

$$R_{XX}(\tau) = a^2 e^{-\lambda\tau} \sum_{all\ k \geq 0} (-1)^k \frac{(\lambda\tau)^k}{k!} = a^2 e^{-2\lambda|\tau|}.$$

The power spectral density of the autocorrelation function $R_{XX}(\tau)$ is

$$S_{XX}(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2}.$$

Power spectrum density of RTS signal in the case of $\lambda = 100$ and $\lambda = 50$ is sketched in Figure 4.

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