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COVERING AND INTERSECTION CONDITIONS FOR PRIME IDEALS

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Abstract. Let D be an integral domain, P be a nonzero prime ideal of D, $\{P_\alpha | \alpha \in \mathcal{A}\}$ be a nonempty set of prime ideals of D, and $\{I_\beta | \beta \in \mathcal{B}\}$ be a nonempty family of ideals of D with $\cap_{\beta \in \mathcal{B}} I_\beta \neq (0)$. Consider the following conditions:

(i) If $P \subseteq \cup_{\alpha \in \mathcal{A}} P_\alpha$, then $P = P_\alpha$ for some $\alpha \in \mathcal{A}$;

(ii) If $\cap_{\beta \in \mathcal{B}} I_\beta \subseteq P$, then $I_\beta \subseteq P$ for some $\beta \in \mathcal{B}$.

In this paper, we prove that D satisfies (i) $\Leftrightarrow$ D is a generalized weakly factorial domain of $\dim(D) = 1 \Rightarrow D$ satisfies (ii) $\Leftrightarrow$ D is a weakly Krull domain of $\dim(D) = 1$. We also study the t-operation analogs of (i) and (ii).

1. Introduction

Let R be a commutative ring with identity, and let Spec(R) be the set of prime ideals of R. Consider the following property: (A) *If a prime ideal P of R is contained in $\cup_{\alpha \in \mathcal{A}} P_\alpha$, where $P_\alpha \in$ Spec(R) for each $\alpha \in \mathcal{A}$, then $P \subseteq P_\alpha$ for some $\alpha \in \mathcal{A}$* (we may assume that $P \neq (0)$ since the zero ideal of R is contained in any ideal). In [10, Theorem 1.1], the authors proved that if R is a Noetherian ring, then R satisfies (A) if and only if every prime ideal of R is the radical of a principal ideal. This result was completely generalized to arbitrary rings by Smith. In [11, Theorem], Smith proved that R satisfies (A) if and only if every prime ideal of R is the radical of a principal ideal.

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Let $\{P_\alpha\}_{\alpha\in\mathcal{A}}$ be a nonempty subset of Spec(R); then $S = R \setminus \cup_{\alpha\in\mathcal{A}} P_\alpha$ is a saturated multiplicative subset of R [9, Theorem 2]. So if I is an ideal of R with $I \subseteq \cup_{\alpha\in\mathcal{A}} P_\alpha$, then $I \cap S = \emptyset$, and hence there is a prime ideal P of R such that $I \subseteq P$ and $P \cap S = \emptyset$ (equivalently, $P \subseteq \cup_{\alpha\in\mathcal{A}} P_\alpha$). Thus the property (A) is equivalent to the following covering condition: *for any ideal I of R and any nonempty set $\{P_\alpha\}_{\alpha\in\mathcal{A}}$ of prime ideals of R, the inclusion $I \subseteq \cup_{\alpha\in\mathcal{A}} P_\alpha$ implies that $I \subseteq P_\alpha$ for some $\alpha \in \mathcal{A}$.* Also, the natural dual of the property (A) is the following intersection condition:

(#) *If $P \in Spec(R)$ and if $\{I_\alpha\}_{\alpha\in\mathcal{B}}$ is a nonempty family of ideals of R, then P contains $\cap_{\alpha\in S} I_\alpha$ only if P contains some I_α.*

Gilmer proved that R satisfies (#) if and only if R is zero dimensional and semi-quasilocal [5, Theorem 2]. For any $h \in R$, let $X_h = \{P \in \text{Spec}(R) | h \notin P\}$ and $V(h) = \text{Spec}(R) \setminus X_h$. In [6], we studied a ring R with the following property: $(**)$ *for any f and g_α of R, $X_f \subseteq \cup_\alpha X_{g_\alpha}$ implies $X_f \subseteq X_{g_\alpha}$ for some α.* In [7, Theorem], the second author proved that R satisfies $(**)$ if and only if Spec(R) is linearly ordered under inclusion, if and only if, for any f and h_β of R, $V(f) \subseteq \cup_\beta V(h_\beta)$ implies $V(f) \subseteq V(h_\beta)$ for some β.

In this paper, we continue our research on covering and intersection conditions for prime ideals. First, we consider the following conditions that are weaker than the poperties (A) and (#): Let P be a nonzero prime ideal of R, $\{P_\alpha | \alpha \in \mathcal{A}\}$ be a nonempty subset of Spec(R), and $\{I_\beta | \beta \in \mathcal{B}\}$ be a nonempty family of ideals of R with $\cap_{\beta\in\mathcal{B}} I_\beta \neq (0)$.

(A) If $P \subseteq \cup_{\alpha\in\mathcal{A}} P_\alpha$, then $P \subseteq P_\alpha$ for some $\alpha \in \mathcal{A}$.
(B) If $P \subseteq \cup_{\alpha\in\mathcal{A}} P_\alpha$, then $P = P_\alpha$ for some $\alpha \in \mathcal{A}$.
(C) If $\cap_{\beta\in\mathcal{B}} I_\beta \subseteq P$, then $I_\beta \subseteq P$ for some $\beta \in \mathcal{B}$.

Clearly, (B) implies (A); so if R is an integral domain, then R satisfies (B) if and only if $\dim(R) = 1$ and every prime ideal of R is the radical of a principal ideal (see Theorem 2.3). In Section 2, we prove that D satisfies (B) $\Leftrightarrow$ D is a generalized weakly factorial domain of $\dim(D) = 1$ $\Rightarrow$ D satisfies (C) $\Leftrightarrow$ D is a weakly Krull domain of $\dim(D) = 1$.

Next, in Section 3, we study the t-operation analogs of the properties (A), (B), and (C). Let D be an integral domain, P be a prime t-ideal of D, $\{P_\alpha | \alpha \in \mathcal{A}\}$ be a nonempty collection of prime t-ideals of D, and $\{I_\beta | \beta \in \mathcal{B}\}$ be a nonempty family of t-ideals of D with $\cap_{\beta\in\mathcal{B}} I_\beta \neq (0)$.

(A′) If $P \subseteq \cup_{\alpha\in\mathcal{A}} P_\alpha$, then $P \subseteq P_\alpha$ for some $\alpha \in \mathcal{A}$.

(B$'$) If $P \subseteq \cup_{\alpha \in \mathcal{A}} P_\alpha$, then $P = P_\alpha$ for some $\alpha \in \mathcal{A}$.
(C$'$) If $\cap_{\beta \in \mathcal{B}} I_\beta \subseteq P$, then $I_\beta \subseteq P$ for some $\beta \in \mathcal{B}$.

We show that D satisfies (B$'$) if and only if D is a GWFD and that D is a weakly Krull domain if and only if t-dim$(D) = 1$ and D satisfies (C$'$). As a corollary, we have that a PvMD D is a generalized Krull domain if and only if D satisfies (C$'$).

All rings considered in this paper are commutative rings with identity. Throughout this paper, we use the notations (#), (A), (B), (C), (A$'$), (B$'$), and (C$'$) to denote the covering and intersection conditions for prime ideals. Undefined notations and definitions are standard as in [4] and [9].

2. The conditions (A), (B) and (C)

Let R be a commutative ring with identity. Then $\dim(R)$ denotes the (Krull) dimension of R. It is clear that $\dim(R) = 0$ if and only if every prime ideal of R is maximal, while if R is an integral domain, then $\dim(R) = 1$ if and only if every nonzero prime ideal of R is maximal. We begin this section with the following lemma which is essential in the subsequent arguments.

Lemma 2.1. ([3, Theorem 2.1]) *Let I be a proper ideal of a ring R. If every minimal prime ideal of I is the radical of a finitely generated ideal, then I has only finitely many minimal prime ideals.*

In [5, Theorem 2], Gilmer showed that R satisfies (#) if and only if R is zero-dimensional and semi-quasilocal. So if R is an integral domain, then R satisfies (#) if and only if R is a field. We next give more equivalent conditions of (#).

Proposition 2.2. *The following statements are equivalent for a ring R.*

(1) *R satisfies* (#).
(2) *R satisfies* (B).
(3) *$\dim(R) = 0$ and R is semi-quasilocal.*
(4) *$\dim(R) = 0$ and every prime ideal of R is the radical of a finitely generated ideal.*
(5) *$\dim(R) = 0$ and every prime ideal of R is the radical of a principal ideal.*

Proof. (1) $\Leftrightarrow$ (3) [5, Theorem 2]. (2) $\Rightarrow$ (5) Clearly, every prime ideal of R is maximal; so $\dim(R) = 0$. Also, note that R satisfies (A), and thus every prime ideal of R is the radical of a principal ideal [11, Theorem]. (5) $\Rightarrow$ (4) Clear, (4) $\Rightarrow$ (3) Since $\dim(R) = 0$, each prime ideal of R is minimal over the zero ideal. Thus the result follows directly from Lemma 2.1. (3) $\Rightarrow$ (2) This is an immediate consequence of [9, Theorem 81]. □

Let $X^1(D)$ be the set of height-one prime ideals of an integral domain D. We say that D is a *weakly Krull domain* if $D = \cap_{P \in X^1(D)} D_P$ and the intersection has finite character. Also, D is a *generalized weakly factorial domain* (GWFD) if each nonzero prime ideal of D contains a principal primary ideal. It is known that if D is a GWFD, then D is a weakly Krull domain [2, Corollary 2.3].

In [6, page 682], we noted that R satisfies (B) if and only if every prime ideal of R is the radical of a principal ideal and every nonzero prime ideal is maximal and that if R is an integral domain of $\dim(R) = 1$, then (A) and (B) are the same conditions. We next study integral domains with the properties (B) and (C).

THEOREM 2.3. *Consider the following statements for an integral domain D.*

(1) *D is a semi-quasilocal domain of $\dim(D) = 1$.*
(2) *D satisfies* (B).
(3) *$\dim(D) = 1$ and every prime ideal of D is the radical of a principal ideal.*
(4) *$\dim(D) = 1$ and D is a GWFD.*
(5) *D satisfies* (C).
(6) *For any $0 \neq a \in D$, the quotient ring D/aD satisfies* (#).
(7) *$\dim(D) = 1$ and every prime ideal of D is the radical of a two generated ideal.*
(8) *$\dim(D) = 1$ and every prime ideal of D is the radical of a finitely generated ideal.*
(9) *$\dim(D) = 1$ and D is a weakly Krull domain.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9).

Proof. (1) $\Rightarrow$ (3) Let $\{P_1, \ldots, P_n\}$ be the set of nonzero prime ideals of D. Then, since $\dim(D) = 1$, we have $P_i \nsubseteq \cup_{j \neq i} P_j$ for each P_i [9, Theorem 81]. So if $a_i \in P_i \setminus (\cup_{j \neq i} P_j)$, then P_i is the unique prime ideal of D containing a_i, and thus $P_i = \sqrt{a_i D}$.

(2) $\Leftrightarrow$ (3) It is clear that if D satisfies (B), then $\dim(D) = 1$. Thus the result follows directly from [11, Theorem] since (A) and (B) are identical for one-dimensional integral domains.

(3) $\Leftrightarrow$ (4) [2, Theorem 2.2].

(3) $\Rightarrow$ (7) $\Rightarrow$ (8) Clear.

(5) $\Leftrightarrow$ (6) This follows from the fact that for any $0 \neq a \in D$, if $\{I_\beta | \beta \in \mathcal{B}\}$ is a set of ideals of D containing a, then $(\cap I_\beta)/aD = \cap(I_\beta/aD)$.

(6) $\Rightarrow$ (7) First, note that $\dim(D/aD) = 0$ for any $0 \neq a \in D$ by Proposition 2.2. Thus $\dim(D) = 1$. Next, let P be a nonzero prime ideal of D, and choose $0 \neq b \in P$. Then D/bD satisfies (#) by (6), and hence P/bD is the radical of a principal ideal of D/bD; so $P/bD = \sqrt{(b,c)/bD}$ for some $c \in P$ by Proposition 2.2. Thus $P = \sqrt{(b,c)}$.

(8) $\Rightarrow$ (6) Let $0 \neq a \in D$. Then $\dim(D/aD) = 0$. Note that each prime ideal of D/aD is of the form P/aD for some prime ideal P of D containing a. Hence every prime ideal of D/aD is the radical of a finitely generated ideal, and thus D/aD satisfies (#) by Proposition 2.2.

(8) $\Leftrightarrow$ (9) This follows directly from the definition of weakly Krull domains and Lemma 2.1. □

We next give two examples which show that the reverse implications of Theorem 2.3 do not hold.

Example 2.4. (1) Let $\mathbb{Z}$ be the ring of integers. Then $\dim(\mathbb{Z}) = 1$ and every prime ideal of $\mathbb{Z}$ is a principal ideal. However, $\mathbb{Z}$ is not semi-quasilocal.

(2) Let D be a Dedekind domain such that the divisor class group $Cl(D)$ of D is torsion-free (see [4, Theorems 45.8 and 45.9, Example 45.10] for the existence of such a Dedekind domain). Then, since D is an one-dimensional Noetherian domain, D satisfies the condition (8) of Theorem 2.3. However, since $Cl(D)$ is torsion-free, D has a prime ideal P such that P^n is not principal for all positive integers n. Note that if $P = \sqrt{aD}$, then $aD = P^k$ for some positive integer k [9, Theorem 97], a contradiction. Therefore, P is not the radical of a principal ideal, and hence D does not satisfy (B).

3. The conditions (A′), (B′) and (C′)

In this section, we study the t-operation analogs of the properties (A), (B) and (C). For this we first review some properties of the t-operation on an integral domain. Let D be an integral domain with quotient field K, and let I be a nonzero fractional ideal of D. Then $I^{-1} = \{x \in K | xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \cup\{J_v | J \subseteq I$ is a nonzero finitely generated ideal of $D\}$. The I is called a *t-ideal* if $I = I_t$, while a t-ideal I is called a *maximal t-ideal* if I is maximal among proper integral t-ideals of D. Let t-Max(D) be the set of maximal t-ideals of D. It is well known that t-Max$(D) \neq \emptyset$ when D is not a field; every maximal t-ideal is a prime ideal; every integral t-ideal is contained in a maximal t-ideal; every prime ideal minimal over a t-ideal is a t-ideal (in particular, height-one prime ideals are t-ideals); and $D = \cap_{P \in t\text{-Max}(D)} D_P$. We say that D has *t-dimension one*, denoted by t-dim$(D) = 1$, if every prime t-ideal of D is a maximal t-ideal, i.e., t-Max$(D) = X^1(D)$; so if t-dim$(D) = 1$, then $D = \cap_{P \in X^1(D)} D_P$.

Proposition 3.1. (1) *D satisfies* (A′) *if and only if every prime t-ideal of D is the radical of a principal ideal.*

(2) *D satisfies* (A) *if and only if D satisfies* (A′) *and every nonzero prime ideal of D is a t-ideal.*

Proof. (1) Suppose that there exists a prime t-ideal P of D such that $P \neq \sqrt{aD}$ for all $a \in P$. Then for each $a_\alpha \in P$, $\sqrt{a_\alpha D} \subsetneq P$, and hence there is a prime ideal P_α minimal over $a_\alpha D$ (hence P_α is a prime t-ideal) such that $P \nsubseteq P_\alpha$. So $P \subseteq \cup_{a_\alpha \in P} P_\alpha$ but $P \nsubseteq P_\alpha$ for all α. Thus if D satisfies (A′), then every prime t-ideal of D is the radical of a principal ideal. The converse is clear (or see the proof of [10, Theorem]).

(2) ($\Rightarrow$) Assume that D satisfies (A). Clearly, D satisfies (A′) and by [11, Theorem], each nonzero prime ideal P of D is the radical of a principal ideal. In particular, P is minimal over a principal ideal, and thus P is a t-ideal. ($\Leftarrow$) By (1), every prime t-ideal of D is the radical of a principal ideal; so every prime ideal of D is the radical of a principal ideal, and thus D satisfies (A) [11, Theorem]. $\square$

We next give the t-operation analog of Theorem 2.3, which also gives new characterizations of weakly Krull domains.

Theorem 3.2. *Consider the following statements for an integral domain D.*

(1) *D is a semi-quasilocal domain of* $\dim(D) = 1$.
(2) *D satisfies* (B′).
(3) *t-dim$(D) = 1$ and every prime t-ideal is the radical of a principal ideal.*
(4) *D is a GWFD.*
(5) *D is a weakly Krull domain.*
(6) *t-dim$(D) = 1$ and every prime t-ideal is the radical of a two generated ideal.*
(7) *t-dim$(D) = 1$ and every prime t-ideal is the radical of a finitely generated ideal.*
(8) *t-dim$(D) = 1$ and D satisfies* (C′).
(9) *t-dim$(D) = 1$ and for a prime t-ideal P of D and a nonempty collection $\{P_\alpha | \alpha \in \mathcal{A}\}$ of prime t-ideals of D with $\cap_{\alpha \in \mathcal{A}} P_\alpha \neq (0)$, $\cap_{\alpha \in \mathcal{A}} P_\alpha \subseteq P$ implies $P_\alpha \subseteq P$ for some $\alpha \in \mathcal{A}$.*

Then $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8) \Leftrightarrow (9)$.

Proof. $(1) \Rightarrow (2)$ This follows directly from [9, Theorem 81].

$(2) \Leftrightarrow (3)$ It is clear that if D satisfies (B′), then t-dim$(D) = 1$. Thus the result is an immediate consequence of Proposition 3.1(1) since (A′) and (B′) are the same conditions for integral domains with t-dimension one.

$(3) \Leftrightarrow (4)$ This is the $(1) \Leftrightarrow (3)$ of [2, Theorem 2.2].

$(4) \Rightarrow (5)$ [2, Corollary 2.3].

$(5) \Leftrightarrow (6)$ [2, Theorem 2.6].

$(6) \Rightarrow (7)$ Clear.

$(7) \Rightarrow (8)$ Let P be a prime t-ideal of D, and let $\{I_\alpha\}$ be a set of t-ideals of D such that $(0) \neq \cap_\alpha I_\alpha \subseteq P$. Let $I = \cap_\alpha I_\alpha$. Then I is a t-ideal [4, Proposition 32.2], and hence the number of prime t-ideals of D containing I is finite, say $P_1, \dots, P_n$, by (7) and Lemma 2.1.

Let $S = D \backslash \cup_{i=1}^n P_i$. Then $\dim(D_S/ID_S) = 0$ and $\{P_1 D_S/ID_S, \dots, P_n D_S/ID_S\}$ is the set of maximal ideals of the factor ring D_S/ID_S. So D_S/ID_S satisfies (#) by Proposition 2.2. Recall from [8, Proposition 2.8(3)] that

$$I_\alpha = \cap_{P \in X^1(D)} I_\alpha D_P = (\cap_{i=1}^n I_\alpha D_{P_i}) \cap D = I_\alpha D_S \cap D$$

since each I_α is a t-ideal of D, $(D_S)_{P_i D_S} = D_{P_i}$, and $\{P_1 D_S, \dots, P_n D_S\}$ is the set of maximal ideals of D_S. Hence $(\cap_\alpha I_\alpha D_S) \cap D = \cap_\alpha (I_\alpha D_S \cap D) = \cap_\alpha I_\alpha = I$, and thus $\cap_\alpha I_\alpha D_S = ID_S$ [4, Theorem 4.4(2)].

Clearly, $\cap_\alpha (I_\alpha D_S/ID_S) = (\cap_\alpha I_\alpha D_S)/ID_S$; so $\cap_\alpha (I_\alpha D_S/ID_S) = ID_S/ID_S \subseteq PD_S/ID_S$. Since D_S/ID_S satisfies (#), we have $I_\alpha D_S/ID_S \subseteq PD_S/ID_S$

for some α, and hence $I_\alpha = I_\alpha D_S \cap D \subseteq PD_S \cap D = P$. Thus D satisfies (C′).

(8) $\Rightarrow$ (9) Clear.

(9) $\Rightarrow$ (5) If t-dim$(D) = 1$, then $D = \cap_{P \in X^1(D)} D_P$. So it suffices to show that the set $\mathcal{S}(a) = \{P \in X^1(D) | a \in P\}$ is finite for each $0 \neq a \in D$. If $P \in \mathcal{S}(a)$, then, since each $Q \in \mathcal{S}(a)$ is a t-ideal of D and $0 \neq a \in \cap_{P \neq Q \in \mathcal{S}(a)} Q$, we have $\cap_{P \neq Q \in \mathcal{S}(a)} Q \nsubseteq P$ by (9). So we can choose $y_p \in (\cap_{P \neq Q \in \mathcal{S}(a)} Q) \setminus P$. Obviously, $(\{a, y_P | P \in \mathcal{S}(a)\}) \nsubseteq Q$ for all $Q \in X^1(D) = t$-Max(D), and hence $(\{a, y_P | P \in \mathcal{S}(a)\})_t = D$. Hence $(a, y_{P_1}, \ldots, y_{P_n})_t = D$ for some $P_1, \ldots, P_n \in \mathcal{S}(a)$. If $P \in \mathcal{S}(a) \setminus \{P_1, \ldots, P_n\}$, then $y_{P_i} \in P$, and hence $D = (a, y_{P_1}, \ldots, y_{P_n})_t \subseteq P_t = P \subsetneq D$, a contradiction. Thus $\mathcal{S}(a) = \{P_1, \ldots, P_n\}$. □

Remark 3.3. *(1) It is known that D is a weakly Krull domain if and only if t-dim$(D) = 1$ and for each $P \in X^1(D)$, $P = \sqrt{(a,b)}$ for some a, b such that $((a,b)(a,b)^{-1})_t = D$* [2, Theorem 2.6].

(2) Let V be a valuation domain of dim$(V) = 2$, and let $0 \neq P_0 \subsetneq P$ be the chain of prime ideals of V. Then V satisfies the second condition of the (9) of Theorem 3.2, but V does not satisfy (C′).

Recall that D is a *Prüfer v-multiplication domain* (PvMD) if every nonzero finitely generated ideal I of D is t-invertible, i.e., $(II^{-1})_t = D$. It is well known that D is a PvMD if and only if D_P is a valuation domain for all maximal t-ideals P of D [12, Theorem 5]. Also, recall that D is a *generalized Krull domain* if (i) $D = \cap_{P \in X^1(D)} D_P$, (ii) this intersection has finite character, and (iii) D_P is a rank one valuation domain for all $P \in X^1(D)$. It is obvious that a generalized Krull domain is a weakly Krull domain. Conversely, if D is a PvMD, then D is a generalized Krull domain if and only if D is a weakly Krull domain.

Corollary 3.4. *A PvMD D satisfies* (C′) *if and only if D is a generalized Krull domain.*

Proof. As we noted in the above paragraph, a PvMD D is a weakly Krull domain if and only if D is a generalized Krull domain; so by Theorem 3.2, it suffices to show that if D satisfies (C′), then t-dim$(D) = 1$.

Let P be a prime t-ideal of D. If ht$P \geq 2$, then there is a prime t-ideal P_0 of D such that $P_0 \subsetneq P$ (we may assume that there is no prime ideal properly between P_0 and P [9, Theorem 11] since every nonzero prime ideal of D contained in P is a t-ideal (cf. [8, Corollary 3.20])).

Let $a \in P$ such that $\sqrt{aD_P} = PD_P$, and let $I_a = aD_P \cap D$. Then I_a is a t-ideal of D [8, Lemma 3.17], $P_0 \subsetneq I_a$, and $\cap I_a = P_0$ (cf. [4, Theorem 17.3(d)]), a contradiction. Hence ht$P = 1$, and thus t-dim$(D) = 1$. $\square$

References

[1] D.D. Anderson, J.L. Mott, and M. Zafrullah, *Finite character representation for integral domains*, Bollettino U. M. I. (7) 6-B (1992), 613-630.

[2] D.F. Anderson, G.W. Chang, and J. Park, *Generalized weakly factorial domains*, Houston J. Math. **29** (2003), 1-13.

[3] G.W. Chang, *Weakly factorial rings with zero divisors*, Lecture Notes in Pure and Appl. Math., Marcel Dekker, **220** (2001), 119-131.

[4] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[5] R. Gilmer, *An intersection condition for prime ideals*, Lecture Notes in Pure and Appl. Math., Marcel Dekker, **189**(1997), 327-331.

[6] C.J. Hwang and G.W. Chang, *A note on coverings of prime ideals*, Comm. Korean Math. Soc. **14** (1999), 681-685.

[7] C.J. Hwang, *A covering condition for the prime spectrums*, East Asian Math. J.**21**(2005), 61-64.

[8] B.G. Kang, *Prüfer v-multiplication domains and the ring* $R[X]_{N_v}$, J. Algebra **123** (1989), 151-170.

[9] I. Kaplansky, *Commutative Rings*, revised edition, Univ. of Chicago Press, 1974.

[10] C.M. Reis and T.M. Viswanathan, *A compactness property for prime ideals in Noetherian rings*, Proc. Amer. Math. Soc. **25** (1970), 353-356.

[11] W. Smith, *A covering condition for prime ideals*, Proc. Amer. Math. Soc. **30** (1971), 451-452.

[12] M. Griffin, *Some results on v-multiplication rings*, Canad. Math. J. **19**(1967), 710-722.

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