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DIRICHLET-JORDAN THEOREM ON SIM SPACE

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ABSTRACT. We would like to propose Dirichlet-Jordan theorem on the space of summable in measure(SIM). Surely, this is a kind of extension of bounded variation([1, 4]), and considered as an application of fuzzy set such that α-cut is 0.

1.Introduction

It is well known fact that if a continuous function f is of bounded variation, then its Fourier series converges uniformly to f on a closed bounded set. This is the content of Dirichlet-Jordan theorem. The direction of research would be considered as follows. One is to extend this theorem to larger classes of functions([5-7], [9-10]), the other is summability([2, 8]). In the paper, we would like to investigate Dirichlet-Jordan theorem on SIM space, a kind of extension of bounded variation.

In ([3]), although we have introduced the definition of summable in measure, but in this paper, let us deal with the concept more precisely.

It is considered the following definition as modifying the concept of α-cut of Zadeh([11]). We would like to begin the discussion of SIM with definition. The following definition would be considered. Let f be an almost every continuous monotonic function on $[a, b]$ and λ is Lebesgue measure. And let us choose $\{I_k\}$ be any collection of non-overlapping subintervals of $[a, b]$ such that $[a, b] = \bigcup I_k$, where $I_k = [t_{k-1}, t_k]$, $f(I_k) = f(t_k) - f(t_{k-1})$. Then $V(f)$, the variation of f over $[a, b]$, is defined by

$$V(f) = \sup\{\sum_{k=1}^{n} |f(I_k)| : I_k = [t_{k-1}, t_k]\},$$

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for any $\{I_k\}$.

Now that, let us introduce the concept of summable in measure(SIM).

Definition 1.1. A function f is of summable in measure on $[a, b]$ (simply $f \in SIM$) if for a suitable positive number α,

$$\lambda\{I_k : V(f) \geq \alpha\} = 0$$

as $n \to \infty$.

It is considered as an application of fuzzy set such that α-cut is 0, and surely, the above f can be rewrite to $V(f) \leq \alpha$ *a.e.* as $n \to \infty$. Here, because of $\lambda\{I_k : V(f) = \alpha\} = 0$, it does not matter the existence of equality.

2.Dirichlet-Jordan theorem on the space of summable in measure

It is well known fact that if we define the convolution by

$$(f * g)(x) = \int f(x-y)g(y)dy$$

for $f, g \in SIM$, then the equality $f * g = g * f$ clearly holds.

The following proof centering on the existence of function g in BV so that $f = g$ a.e. for $f \in SIM$, and the rest is going with this, and by the property of almost every finite function, a function f in SIM is closed for four arithmetic operations.

Theorem 2.1. *If f is of summable in measure on $T = [-\pi, \pi]$, then its Fourier series converges uniformly to f.*

Proof. We would start by assume that a function f is of summable in measure on $T = [-\pi, \pi]$. Then the N-th order partial sum of the Fourier series of f is easily calculated as following;

$$s_N(x) = \sum_{n=-N}^{N} \int_T f(y)e^{-iny}\, dy\, e^{inx} = \int_T f(y) \sum_{n=-N}^{N} e^{in(x-y)}\, dy$$

$$= \int_T f(y) D_N(x-y)\, dy,$$

where D_N is the Dirichlet kernel. Thus, $S_N = f * D_N = D_N * f$ and using $D_N(y) = sin(N+\frac{1}{2})y/sin\frac{y}{2}$, we get

$$S_N(x) = \int_T D_N(y)f(x-y)\ dy = \int_T f(x-y)sin(N+\frac{1}{2})y/sin\frac{y}{2}\ dy.$$

Since $\int_T D_N(y)\ dy = 1$,

$$s_N(x) - f(x) = \int_T [f(x-y) - f(x)]sin(N+\frac{1}{2})y/sin\frac{y}{2}dy$$

$$= \frac{1}{2\pi}\int_0^\pi [f(x+y) - f(x)]sin(N+\frac{1}{2})y/sin\frac{y}{2}dy$$

$$+\frac{1}{2\pi}\int_0^\pi [f(x-y) - f(x)]sin(N+\frac{1}{2})y/sin\frac{y}{2}dy$$

for replaced y with $-y$ on the interval $(-\pi, 0)$. Implies,

$$s_N(x)-f(x) = \int_0^\pi \frac{f(x+y)+f(x-y)-2f(x)}{2\pi sin\frac{y}{2}}sin(N+\frac{1}{2})y\ dy.\cdots(*)$$

Since $f \in SIM$,

$$\frac{f(x+y)+f(x-y)-2f(x)}{2\pi sin\frac{y}{2}}sin(N+\frac{1}{2})y\cdots(**)$$

is of SIM. Next, let us show that a function $f \in SIM$ if and only if there is a function $g \in BV$ that is equal to f almost everywhere.

Let $f \in SIM$ and let $B = \{I_k : \sum_{k=1}^n |f(I_k)| = \infty\}$. Since $f \in SIM$, $\lambda(B) = 0$ as $n \to \infty$ and so, the function g defined by

$$g = f\chi_{B^c}$$

agrees with f almost everywhere. Since $g = f$ on

$$B^c = \{I_k : \sum |g(I_k)| < \infty\}$$

and B^c is of bounded variation, $g \in BV$ follows.

On the contrary, if there is a function g in BV that is equal to f almost everywhere, then $f \in SIM$ follows from next. Let $X = \{I_k : f(I_k) \neq g(I_k)\}$, and we assume that a function h defined by $h(I_n) = f(I_n) - g(I_n) = \infty$ if $I_n \in X$ and otherwise is 0. Then

$$\int h d\lambda = 0.$$

We would like to show about the above h in SIM. Let us put a set A by $A = \{I_k : \sum |h(I_k)| > \alpha\}$ and we assume that $\int \chi_A d\mu = 0$. Since $\chi_A \geq 0$, $\chi_A = |\chi_A|$ and so, $\int |\chi_A| d\mu = 0$. This implies, for every $\{I_k\}$, $\chi_A(I_k) = 0$ a.e.. Thus $I_k \notin A$ a.e. and so, $I_k \in A^c$ a.e.. Hence, we can insist on $\sum |h(I_k)| \leq \alpha$ a.e. and so

$$h \in SIM.$$

Since $g \in BV$ and $h(I_n) = f(I_n) - g(I_n)$ a.e., $f(I_n) \in SIM$ for each I_n and so $f \in SIM$.

Let us back to the point. Since $(**) \in SIM$, by the above, there is a function $h \in BV$ such that $(**) = h$ a.e.. Implies, $\int (**) = \int h$ and so, $\int h = s_N(x) - f(x)$ for some $h \in BV$. It is a well known fact that the integral of $(*)$ less than

$$\frac{3c\pi}{N + \frac{1}{2}}[1 + \log \frac{N + \frac{1}{2}}{2}]$$

for $s_N(x) - f(x) \in BV$, and surely, this has a limit of zero as $N \to \infty$, uniformly for all x. $\square$

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