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ON LEFT Γ-FILTERS OF Γ-*po*-SEMIGROUPS

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Abstract. We introduce the notions of a left(right) Γ-filter in a *po*-Γ-semigroups and give a characterization of a left(right) Γ-filter of a *po*-Γ-semigroups in term of right(left) prime Γ-ideals.

M. K. Sen and N. K. Saha([6]) introduced the notion of a Γ-semigroup. Y. I. Kwon and S. K. Lee([3]) introduced the notion of a *po*-Γ-semigroup which is a generalization of *po*-semigroups and Γ-semigroups. N. Kehayopulu([2]) gave the the characterization of the filter of S in term of the prime ideals. S. K. Lee and S. S. Lee([5]) introduced the notion of a left(right) filter in a *po*-semigroup and gave a characterization of the left(right)-filter of S in term of the right(left) prime ideals. Y. B. Jun([1]) introduce the notion of a Γ-filter in a *po*-Γ-semigroup and gave a characterization of a Γ-filter of a *po*-Γ-semigroup in terms of a prime Γ-ideal.

In this paper, we introduce the notion of a left(right) Γ-filter in a *po*-Γ-semigroup and gave a characterization of a left(right) Γ-filter of a *po*-Γ-semigroup in terms of a right(left) prime Γ-ideal.

A *po*-semigroup (: ordered semigroup) is an ordered set $(S, \leq)$ at same time a semigroup such that :

$$a \leq b \quad \Rightarrow \quad ca \leq cb \text{ and } ac \leq bc$$

for all $a, b, c \in S$.

Let S and Γ be nonempty sets. S is called a Γ-semigroup([6]) if

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1) $a\alpha b \in S$

2) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

A nonempty subset T of a Γ-semigroup S is said to be a sub-Γ-semigroup of S if $T\Gamma T \subseteq T$.

A *po*-Γ-semigroup is an ordered set $(S, \leq)$ at same time a Γ-semigroup such that :

$$a \leq b \quad \Rightarrow \quad ca \leq cb \text{ and } ac \leq bc$$

for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$(see [3]).

In this paper, S shall mean a *po*-Γ-semigroup unless otherwise specified.

Definition 1. [1]. A nonempty set A of S is called a *right* (resp. *left*) Γ-*ideal* of S if

(1.1) $A\Gamma S \subseteq A$ (resp. $S\Gamma A \subseteq A$)

(1.2) $a \in A, b \leq a$ for $b \in S$ imply $b \in A$.

If A is both a right and left Γ-ideal of S, we say that A is a Γ-ideal of S.

Definition 2. [4]. A subset T of S is called *prime* if $AB \subseteq T$ implies $A \subseteq T$ or $B \subseteq T$ for subsets A, B of S.

T is called a *prime right* (resp. *left*) *ideal* if T is prime as a right (resp. left) ideal. T is called a prime ideal if T is prime as an ideal.

Theorem 3. *Let A be a right (resp. left) Γ-ideal of S and let T be a sub-po-Γ-semigroup of S. If $A \cap T \neq \emptyset$, then $A \cap T$ is a right (resp. left) Γ-ideal of T.*

proof. Assume that A is a right Γ-ideal of S and $A \cap T \neq \emptyset$. Then we have

$$(A \cap T)\Gamma T \subseteq A\Gamma T \cap T\Gamma T \subseteq A \cap T$$

Let $a \in A \cap T$ and $b \in T$ be such that $b \leq a$. Since A is a right Γ-ideal of S and $a \in A$, we have $b \in A$. Thus $b \in A \cap T$. Therefore $A \cap T$ is a right Γ-ideal of T. □

Now we introduce the notion of left(right) filter in *po*-Γ-semigroups.

Definition 4.. A sub-Γ-semigroup F of S is called a *left* (resp. *right*) *filter* of S if

(4.1) $a\gamma b \in F$ for $a, b \in S \Rightarrow a \in F$ (resp. $b \in F$), (4.2) $a \in F$ and $a \leq c$ for $c \in S \Rightarrow c \in F$.

A sub-Γ-semigroup F of S is called a *filter* of S if F is a left and right filter of S.

We denote by $F_l(x)$ (resp. $F_r(x), F$) the left Γ-filter (resp. right Γ-filter, filter) generated by $x \in S$. Let $\mathcal{F}_l$ (resp. $\mathcal{F}_r$) and $\mathcal{F}$ be the equivalence relations on S denoted by

$$\mathcal{F}_l = \{(x,y) | F_l(x) = F_l(y)\}, \quad \mathcal{F}_r = \{(x,y) | F_r(x) = F_r(y)\}$$

$$\mathcal{F} = \{(x,y) | F(x) = Fy)\}.$$

THEOREM 5. *If $a \leq b$ for $a, b \in S$, then $(a, a\gamma b) \in \mathcal{F}_l$ (resp. $(a, a\gamma b) \in \mathcal{F}_r$) for every $\gamma \in \Gamma$.*

proof. Let $a, b \in S$ and $a \leq b$. Since $F_l(a)$ is a left Γ-filter generated by a. By (4.2), $b \in F_l(a)$. Since $F_l(a)$ is a sub-Γ-semigroup and $a, b \in F_l(a)$, we have $a\gamma b \in F(a)$ for $\gamma \in \Gamma$. Hence $F_l(a\gamma b) \subseteq F_l(a)$. On the other hand, $a\gamma b \in F_l(a\gamma b)$ implies $a \in F_l(a\gamma b)$ by (4.1). Thus $F_l(a) \subseteq F_l(a\gamma b)$. Therefore $F_l(a\gamma b) = F_l(a)$, and so $(a, a\gamma b) \in \mathcal{F}_l$. □

COROLLARY 6. ([1]). *If $a \leq b$ for $a, b \in S$, then $(a, a\gamma b) \in \mathcal{F}$ for every $\gamma \in \Gamma$.*

Now we give a characterization of a left (resp. right) Γ-filter of a *po*-Γ-semigroup in term of right (resp. left) prime Γ-ideal.

THEOREM 7. *Let S be a po-Γ-semigroup. Then the following conditions are equivalent.*

(i) F is a left (resp. right) Γ-filter of S.

(ii) $S \setminus F = \emptyset$ of $S \setminus F$ is a prime right (resp. left) Γ-ideal of S.

proof. (i) ⇒ (ii). Assume that $S \setminus F \neq \emptyset$. Let $x \in S \setminus F$ and $y \in S, \gamma \in \Gamma$. Then $x\gamma y \in S \setminus F$. This means that $(S \setminus F)\Gamma S \subseteq S \setminus F$. Indeed: If $x\gamma y \notin S \setminus F$, then $x\gamma y \in F$. Since F is a left Γ-filter, $x \in F$. It is impossible. Thus $x\gamma y \in S \setminus F$.

Now let $x \in S \setminus F$ and $y \in S$ be such that $y \leq x$. Then $y \in S \setminus F$. Indeed: If $y \notin S \setminus F$ i.e., $y \in F$, then $x \in F$ by (4.2). It is impossible. Thus $y \in S \setminus F$. Therefore $S \setminus F$ is a right Γ-ideal of S.

Next we shall prove that $S \setminus F$ is prime. Let $x\gamma y \in S \setminus F$ for $x, y \in S$ and every $\gamma \in \Gamma$. Suppose that $x \notin S \setminus F$ and $y \notin S \setminus F$. Then $x \in F$ and $y \in F$. Since F is a Γ-subsemigroup of S, $x\gamma y \in S$. It is impossible. Thus $x \in S \setminus F$ or $y \in S \setminus F$. Hence $S \setminus F$ is prime, and so $S \setminus F$ is a prime right γ-ideal.

(ii) $\Rightarrow$ (i) If $S \setminus F = \emptyset$, then $F = S$. Thus $F = S$ is a left filter of S.

Assume that $S \setminus F$ is a prime right ideal of S. Then F is a Γ-subsemigroup of S. Indeed : Suppose that $x\gamma y \notin F$ for $x, y \in F$ and every $\gamma \in \Gamma$. Then $xy \in S \setminus F$. Since $S \setminus F$ is prime, x and y are in $S \setminus F$. It is impossible. Thus $xy \in F$, and so F is a Γ-subsemigroup of S.

Let $x\gamma y \in F$ for $x, y \in F$ and every $\gamma \in \Gamma$. Then $x \in F$. Indeed : If $x \notin F$, then $x \in S \setminus F$ Since $S \setminus F$ is a prime right Γ-ideal of S, $x\gamma y \in (S \setminus F)\Gamma S \subseteq S \setminus F$. It is impossible. Thus $x \in F$.

Let $x \in F$ and $x \leq xy$ for $y \in S$. Then $y \in F$. Indeed : If $y \notin F$, then $y \in S \setminus F$. Since $S \setminus F$ is a right ideal of S, $x \in S \setminus F$. It is impossible. Thus $y \in F$. It follows that F is a left filter of s. $\square$

Corollary 8.([1]). *Let S be a po-Γ-semigroup. Then the following conditions are equivalent.*

(i) F is a Γ-filter of S.

(ii) $S \setminus F = \emptyset$ of $S \setminus F$ is a prime Γ-ideal of S.

References

1. Y. B. Jun, *Γ-filters and semilattice Γ-congruences in po-Γ-semigroups*, PU. M. A. **No. 2-3-4** (1977), 275-280.
2. N. Kehayopulu, *On left regular ordered semigroups*, Math. Japon., **35(6)** (1990), 1057-1060.
3. Y. I. Kwon and S. K. Lee, *Some special elements in ordered Γ-semigroups*, Kyungpook Math. J., **35(3)** (1996), 679-685.
4. S. K. Lee and Y. I. Kwon, *A generaization of the thm of Giri and Wazalwar*, Kyungpook Math. J., **37(1)** (1997), 109-111.
5. S. K. Lee and S. S. Lee, *Left(right) filters on po-semigroups*, Kangweon-Kyungki Math. J., **8(1)** (2000), 43-45.
6. M. K. Sen and N. K. Saha, *On Γ-semigroup*, Bull. Calcutta Math. Soc., **78** (1986), 180-186.

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