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BOUNDARY VALUE PROBLEM FOR A CLASS OF THE SYSTEMS OF THE NONLINEAR ELLIPTIC EQUATIONS

Tacksun Jung and Q-Heung Choi*

Abstract. We show the existence of at least two nontrivial solutions for a class of the systems of the nonlinear elliptic equations with Dirichlet boundary condition under some conditions for the nonlinear term. We obtain this result by using the variational linking theory in the critical point theory.

1. Introduction

In this paper we consider the multiplicity of solutions for a class of the systems of the nonlinear elliptic equations with Dirichlet boundary condition

$$\begin{aligned} -\Delta u_1 &= F_{u_1}(u_1, \ldots, u_n) \qquad \text{in } \Omega, \qquad (1.1)\\ -\Delta u_2 &= F_{u_2}(u_1, \ldots, u_n) \qquad \text{in } \Omega,\\ &\vdots \qquad \vdots \qquad \vdots\\ -\Delta u_n &= F_{u_n}(u_1, \ldots, u_n) \qquad \text{in } \Omega,\\ u_i(x) &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where Ω be a bounded subset of R^n with smooth boundary, $u_i(x) \in W_0^{1,2}(\Omega)$ and $F : R^n \to R$ be a C^2 function such that $F(0, \ldots, 0) = 0$. Let $u = (u_1, \ldots, u_n)$, $F_{u_i}(u_1, \ldots, u_n) = \frac{\partial F(u_1,\ldots,u_n)}{\partial u_i}$, $F_u(u) = \text{grad}F(u) = (F_{u_1}(u_1, \ldots, u_n), \ldots, F_{u_n}(u_1, \ldots, u_n))$ and $|\cdot|$ denote the Euclidean norm in R^n. We assume that F satisfies the following conditions:

$(F1)$ $\lim_{(u_1,\ldots,u_n)\to(0,\ldots,0)} \frac{F_{r_i}(u)}{|u_1|+\ldots+|u_n|} = 0.$

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*Corresponding author.

$(F2)$ $\lim_{|u_1|+\ldots+|u_n|\to\infty} \frac{F_{r_i}(u)}{|u_1|+\ldots+|u_n|} = \infty$, $i = 1, \ldots, n$.
$(F3)$ $u \cdot F_u(u) \geq \mu F(u)$ $\forall u$,
$(F4)$ $|F_{r_1}(r_1, \ldots, r_n)| + \ldots + |F_{r_n}(r_1, \ldots, r_n)| \leq \gamma(|r_1|^\nu + \ldots + |r_n|^\nu)$, $\forall r_1, \ldots, r_n$, where $\gamma \geq 0$, $\mu \in]2, 2^*[$, $\nu \leq 2^* - 1 - (2^* - \mu)(1 - \frac{2^{*\prime}}{2^*})$, $i = 1, \ldots, n$.
Some papers of Lee [13, 16, 17, 18] concerning the semilinear elliptic system and some papers of the other several authors [10, 15] have treated the system of this like nonlinear elliptic equations. System (1.1) can be rewritten by

$$-\Delta u = \nabla F(u), \qquad \text{in } \Omega,$$

$$u = 0 \qquad \text{on } \partial\Omega.$$

Let H be a cartesian product of the Sobolev spaces $W_0^{1,2}(\Omega, R)$, i.e., $H = W_0^{1,2}(\Omega, R) \times \cdots \times W_0^{1,2}(\Omega, R)$. We endow the Hilbert space H with the norm

$$\|u\|^2 = \sum_{i=1}^{n} \|u_i\|^2,$$

where $\|u_i\|^2 = \int_\Omega |\nabla u_i(x)|^2 dx$. In this paper we are looking for the weak solutions of system (1.1) in H, that is, $u = (u_1 \ldots, u_n) \in H$ such that

$$\int_\Omega [-\Delta u \cdot v] dx - \int_\Omega F_u(u) \cdot v = 0, \text{ for all } v \in H,$$

where $F_u(u) = \nabla F(u) = (F_{u_1}(u), \ldots, F_{u_n}(u))$.
Our main result is the following:

Theorem 1.1. *Assume that F satisfies the conditions $(F1) - (F4)$. Then system (1.1) has at least two nontrivial solutions.*

For the proof of Theorem 1.1 we approach the variational method. We use the variational linking theorem and the critical point theory for the definite functional. We study the topology and the geometry of the sublevels of I and find some linking inequalities, hence by the variational linking theorem we get the existence of at least two nontrivial solutions.

In section 2, we obtain some results on the operator $-\Delta$ on $W_0^{1,2}(\Omega)$, F and the functional I on H. n section 3, we recall the variation linking theorem, which plays a crucial role to prove the multiplicity result. In section 4, we prove Theorem 1.1.

2. Some results on $-\Delta$, F, I

Let $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$ be the eigenvalues of the eigenvalue problem for a single elliptic equation $-\Delta u = \lambda u$ with Dirichlet boundary condition and ϕ_k be the eigenfunction belonging to the eigenvalue λ_k, $k \geq 1$.

Since $|\lambda_i| \geq 1$ for all $i \geq 1$, we have the following lemma.

LEMMA 2.1. *Let $u \in W_0^{1,2}(\Omega, R)$ and $\|\cdot\|$ is a Sobolev norm. Then*
(i) $\|u\| \geq C\|u\|_{L^2(\Omega)}$ for some constant $C > 0$.
(ii) $\|u\| = 0$ if and only if $\|u\|_{L^2(\Omega)} = 0$.
(iii) $-\Delta u \in W_0^{1,2}(\Omega, R)$ implies $u \in W_0^{1,2}(\Omega, R)$.

Proof. (i) and (ii) can be checked easily.
(iii) Let λ_n be an eigenvalue of the eigenvalue problem for a single elliptic equation $-\Delta u = \lambda u$ in Ω with Dirichlet boundary condition. We note that $\{\lambda_n : \ |\lambda_n| < |c|\}$ is finite. Let us set $f = -\Delta u \in W_0^{1,2}(\Omega, R)$. Let f be expressed by

$$f = \sum h_n \phi_n.$$

Then

$$(-\Delta)^{-1} f = \sum \frac{1}{\lambda_n} h_n \phi_n.$$

Hence we have the inequality

$$\|(-\Delta)^{-1} f\|^2 = \sum \lambda_n^2 \frac{1}{\lambda_n^2} h_n^2 \leq \sum h_n^2,$$

which means that

$$\|(-\Delta)^{-1} f\| \leq \|f\|_{L^2(\Omega)}.$$

□

From Lemma 2.1, we have:

LEMMA 2.2. *Let $\nabla F(u) \in H = W_0^{1,2}(\Omega, R) \times \cdots \times W_0^{1,2}(\Omega, R)$. Then all the solutions of*

$$-\Delta u = \nabla F(u)$$

belong to H.

Now we return to the case of the system. We observe that by the following Proposition 2.1, the weak solutions of system (1.1) coincide with the critical points of the associated functional I

$$I \in C^{1,1}(H, R),$$

$$I(u)=\int_\Omega[\frac{1}{2}|\nabla u|^2-F(x,u)]dx, \tag{2.1}$$

where $u=(u_1,\dots,u_n)$ and $|\nabla u|^2=\sum_{i=1}^n|\nabla u_i|^2$, $n\geq 1$.

PROPOSITION 2.1. *Assume that the conditions $(F1)$-$(F4)$ hold. Then the functional $I(u)$ is continuous, Fréchet differentiable in H with Fréchet derivative*

$$\nabla I(u)v=\int_\Omega[(-\Delta u)\cdot v-F_u(u)\cdot v]dx.$$

Moreover $DI\in C$. That is $I\in C^1$.

Proof. For $u,v\in H$,

$$\begin{aligned}
&|I(u+v)-I(u)-\nabla I(u)v|\\
=\ &|\frac{1}{2}\int_\Omega(-\Delta u-\Delta v)\cdot(u+v)dx-\int_\Omega F(u+v)dx\\
&-\frac{1}{2}\int_\Omega(-\Delta u)\cdot udx+\int_\Omega F(u)dx-\int_\Omega(-\Delta u-F_u(u))\cdot vdx|\\
=\ &|\frac{1}{2}\int_\Omega[-\Delta u\cdot v-\Delta v\cdot u-\Delta v\cdot v]dx\\
&-\int_\Omega[F(u+v)-F(u)]dx-\int_\Omega[(-\Delta u-F_u(u))\cdot v]dx|.
\end{aligned}$$

We have

$$|\int_\Omega[F(u+v)-F(u)]dx|\leq|\int_\Omega[F_u(u)\cdot v+o(|v|)]dx|=O(|v|). \tag{2.2}$$

Thus we have

$$|I(u+v)-I(u)-\nabla I(u)v|=O(|v|^2). \tag{2.3}$$

Next we prove that $I(u)$ is continuous. For $u,v\in H$,

$$\begin{aligned}
|I(u+v)-I(u)|\ =\ &|\frac{1}{2}\int_\Omega(-\Delta u-\Delta v)\cdot(u+v)dx-\int_\Omega F(u+v)dx\\
&-\frac{1}{2}\int_\Omega(-\Delta u)\cdot udx+\int_\Omega F(u)dx|\\
=\ &|\frac{1}{2}\int_\Omega[(-\Delta u\cdot v-\Delta v\cdot u-\Delta v\cdot v)dx\\
&-\int_\Omega(F(u+v)-F(u))dx|\\
=\ &O(|v|).
\end{aligned}$$

Similarly, it is easily checked that $I \in C^1$. □

PROPOSITION 2.2. *Assume that F satisfies the conditions (F1)-(F4). Then there exist $a_0 > 0$, $b_0 \in R$ and $\mu > 2$ such that*

$$F(u) \geq a_0 |u|^\mu - b_0, \qquad \forall u. \tag{2.4}$$

Proof. Let u be such that $|u|^2 \geq R^2$. Let us set $\varphi(\xi) = F(\xi u)$ for $\xi \geq 1$. Then

$$\varphi(\xi)' = u \cdot F_u(\xi u) \geq \frac{\mu}{\xi} \varphi(\xi).$$

Multiplying by $\xi^{-\mu}$, we get

$$(\xi^{-\mu} \varphi(\xi))' \geq 0,$$

hence $\varphi(\xi) \geq \varphi(1)\xi^\mu$ for $\xi \geq 1$. Thus we have

$$F(u) \geq F\big(\frac{R|u|}{\sqrt{|u|^2}}\big)\big(\frac{\sqrt{|u|^2}}{R}\big)^\mu$$

$$\geq c_0 \big(\frac{\sqrt{|u|^2}}{R}\big)^\mu \geq a_0 |u|^\mu - b_0, \text{ for some } a_0, b_0,$$

where $c_0 = \inf\{F(u) | \; |u|^2 = R^2\}$. □

PROPOSITION 2.3. *Assume that F satisfies the conditions (F1)-(F4). Then*
if $\|u_n\| \to +\infty$ and

$$\frac{\int_\Omega u_n \cdot F_u(u_n) dx - 2 \int_\Omega F(u_n) dx}{\|u_n\|} \to 0,$$

then there exist $(u_{h_n})_n$ and $w \in H$ such that

$$\frac{\text{grad}(\int_\Omega F(u_{h_n}) dx)}{\|u_{h_n}\|} \to w \text{ and } \frac{u_{h_n}}{\|u_{h_n}\|} \rightharpoonup (0, \dots, 0).$$

Proof. By (F3) and Proposition 2.2, for $u \in H$,

$$\int_\Omega [u \cdot F_u(u)] dx - 2 \int_\Omega F(u) dx \geq$$

$$(\mu - 2) \int_\Omega F(u) dx \geq (\mu - 2)(a_0 \|u\|^\mu_{L^\mu} - b_1).$$

By (F4),

$$\|\text{grad}(\int_\Omega F(u) dx)\| \leq C' \| |u|^\nu \|_{L^{2^{*\prime}}}$$

for suitable constant C'. To get the conclusion it suffices to estimate $\|\frac{|u|^\nu}{\|u\|}\|_{L^{2^{*\prime}}}$ in terms of $\frac{\|u\|^\mu_{L^\mu}}{\|u\|}$. If $\mu \geq 2^{*\prime}\nu$, then this is an consequence of *Hölder* inequality. Next we consider the case $\mu < 2^{*\prime}\nu$. By the assumptions μ and ν,

$$\nu \leq 2^* - 1 - (2^* - \mu)(1 - \frac{2^{*\prime}}{2^*}). \tag{2.5}$$

By the standard interpolation arguments, it follows that $\|\frac{|u|^\nu}{\|u\|}\|_{L^{2^{*\prime}}} \leq C\big(\frac{\|u\|^\mu_{L^\mu}}{\|u\|}\big)^{\frac{\nu\alpha}{\mu}}\|u\|^\beta$, where α is such that $\frac{\alpha}{\mu} + \frac{1-\alpha}{2^*} = \frac{1}{2^{*\prime}\nu}$ $(\alpha > 0)$ and $\beta = (1-\alpha)\nu - 1 - \frac{\nu\alpha}{\mu}$. By (2.5), $\beta \leq 0$. Thus we prove the proposition. □

3. Recall of the variational linking theorem

Let H be an Hilbert space with a norm $\|\cdot\|$, $X \subset H$, $r > 0$, $\rho > 0$ and $e \in H\backslash X$, $e \neq 0$. Set:

$$B_r = \{w \in X|\ \|w\| \leq r\},$$

$$S_r = \{w \in X|\ \|w\| = r\},$$

$$\Delta_R(e, X) = \{\sigma e + w|\ \sigma \geq 0, w \in X : \|\sigma e + w\| \leq R\},$$

$$\Sigma_R(e, X) = \{\sigma e + w|\ \sigma \geq 0, w \in X : \|\sigma e + w\| = R\} \cup \{w|\ w \in X, \|w\| \leq R\}.$$

Now we recall the variation linking theorem in [21].

THEOREM 3.1. *(Variation linking theorem). Let H be an Hilbert space, which is topological direct sum of the subspaces H_1 and H_2. Let $I \in C^1(H, R)$. Moreover assume that*
(a) $\dim H_1 < +\infty$,
(b) *there exist $r > 0$, $R > 0$ and $e \in B_r \subset H_1$, $e \neq 0$ such that $r < R$ and*

$$\sup_{w \in S_r \subset H_1} I(w) < \inf_{w \in \Sigma_R(e, H_2)} I(w),$$

(c) $-\infty < a = \inf_{w \in \Delta_R(e,H_2)} I(w)$,
(d) $(P.S.)_c$ *condition holds for any $c \in [a, b]$, which $b = \sup_{w \in B_r \subset H_1} I(w)$. Then there exist at least two critical levels c_1 and c_2 for the functional I such that*

$$\inf_{\Delta_R(e,H_2)} I \leq c_1 \leq \sup_{S_r \subset H_1} I < \inf_{\Sigma_R(e,H_2)} I \leq c_2 \leq \sup_{B_r \subset H_1} I$$

4. Proof of theorem 1.1

Assume that F satisfies the conditions (F1)-(F4). From now on we shall show that $-I$ satisfies the variation linking theorem.

We have the following inequalities:

LEMMA 4.1. *Assume that F satisfies the conditions $(F1)-(F4)$. Let V_i be the finite dimensional subspace of $W_0^{1,2}(\Omega)$ spanned by eigenfunctions corresponding to the eigenvalues $\lambda < \lambda_k$, for some $k \geq 1$, $i = 1, \ldots, n$. Let us set*

$$V = V_1 \times \cdots \times V_n.$$

Then V is a subspace of H and $H = V \oplus V^{\perp}$. Let us set $X = V^{\perp}$. Then (i) there exist $r > 0$ and a ball $B_r \subset V$ with radius r such that

$$\sup_{u \in \partial B_r \subset V} (-I)(u) < 0$$

and
(ii) there exist $R > r$ and an element $e \in B_1 \subset V$, $e \neq 0$ such that

$$\inf_{u \in \Sigma_R(e,X)} (-I)(u) > 0, \qquad \inf_{u \in \Delta_R(e,X)} (-I)(u) = a > -\infty.$$

Proof. (i) By (F3) and (F4), $F(u) \leq a|u|^{\beta}$, $a > 0$ and $\beta > 2$. If $u \in V$, then we have that

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(u)dx \geq \frac{1}{2}\|u\|^2 - a\|u\|^{\beta}_{L^2(\Omega)}.$$

Since $\beta > 2$, there exist a small number $r > 0$ such that if $u \in \partial B_r \subset V$, then $\inf I(u) > 0$. Thus we have $\sup_{u \in \partial B_r}(-I)(u) < 0$.
(ii) By Proposition 2.2, there exist $a_0 > 0$, $b_0 \in R$ and $\mu > 2$ such that $F(u) \geq a_0|u|^{\mu} - b_0$, $\forall u$. Let us choose an element $e \in B_1 \subset V$ and $w \in X$. Then we have that

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\Omega} F(u)dx \leq \frac{1}{2}\sigma^2 + \frac{1}{2}\|w\|^2 - a_0\sigma^{\mu} - a_0\|w\|^{\mu}_{L^2(\Omega)} + b_1$$

for some $b_1 \in R$. Since $\mu > 2$ and $w \in X$, there exists $R > 0$ such that if $u = \sigma e + w \in \Sigma_R(e, X)$, then $\sup I(u) < 0$. Thus $\inf_{\Sigma_R(e,X)}(-I)(u) > 0$. Moreover if $u = \sigma e + w \in \Delta_R(e, X)$, then $\sup I(u) < \infty$. Thus $\inf_{\Delta_R(e,X)}(-I)(u) = a > -\infty$ □

LEMMA 4.2. *Assume that F satisfies the conditions $(F1)-(F4)$ hold. Then I satisfies the $(P.S.)_c$ condition for every real number $c \in R$.*

Proof. Let $c \in R$ and (h_n) be a sequence in N such that $h_n \to +\infty$, $(u_n)_n$ be a sequence such that

$$u_n = (u_1, \ldots, u_n) \in H, \forall n, I(u_n) \to c, \nabla I(u_n) \to 0.$$

We claim that $(u_n)_n$ is bounded. By contradiction we suppose that $\|u_n\| \to +\infty$ and set $\hat{u_n} = \frac{u_n}{\|u_n\|}$. Then

$$\langle \nabla I(u_n), \hat{u_n} \rangle = \langle \nabla I(u_n), \hat{u_n} \rangle = 2\frac{I(u_n)}{\|u_n\|} -$$

$$\frac{\int_\Omega F_u(u_n) \cdot u_n dx - 2\int_\Omega F(u_n)dx}{\|u_n\|} \longrightarrow 0.$$

Hence

$$\frac{\int_\Omega F_u(u_n) \cdot u_n dx - 2\int_\Omega F(u_n)dx}{\|u_n\|} \longrightarrow 0.$$

By Proposition 2.3,

$$\frac{\text{grad} \int_\Omega F(u_n)dx}{\|u_n\|} \qquad \text{converges}$$

and $\hat{u_n} \rightharpoonup 0$. We get

$$\frac{\nabla I(u_n)}{\|u_n\|} = -\Delta \hat{u_n} - \frac{\text{grad}(\int_\Omega F(u_n)dx)}{\|u_n\|} \longrightarrow 0,$$

so $-\Delta \hat{u_n}$ converges. Since $(\hat{u_n})_n$ is bounded and the inverse operator of $-\Delta$ is a compact mapping, up to subsequence, $(\hat{u_n})_n$ has a limit. Since $\hat{u_n} \rightharpoonup (0, \ldots, 0)$, we get $\hat{u_n} \to (0, \ldots, 0)$, which is a contradiction to the fact that $\|\hat{u_n}\| = 1$. Thus $(u_n)_n$ is bounded. We can now suppose that $u_n \rightharpoonup u$ for some $u \in H$. Since the mapping $u \mapsto \text{grad}(\int_\Omega F(u)dx)$ is a compact mapping, $\text{grad}(\int_\Omega F(u_n)dx) \longrightarrow \text{grad}(\int_\Omega F(u)dx)$. Thus $-\Delta u_n$ converges. Since the inverse operator of $-\Delta$ is a compact operator and $(u_n)_n$ is bounded, we deduce that, up to a subsequence, $(u_n)_n$ converges to some u strongly with $\nabla I(u) = \lim \nabla I(u_n) = 0$. Thus we prove the lemma. □

Proof of Theorem 1.1

By Proposition 2.1, $I \in C^1(H, R)$. Thus $-I \in C^1(H, R)$. We have $H = V \oplus X$ with $\dim V < \infty$. By Lemma 4.1, there exist $r > 0$, $R > r$, a ball $B_r \subset V$ with radius r and an element $e \in B_1 \subset V$, $e \neq 0$ such that $\sup_{u \in \partial B_r \subset V}(-I)(u) < 0$, $\inf_{u \in \Sigma_R(e,X)}(-I)(u) > 0$ and $\inf_{u \in \Delta_R(e,X)}(-I)(u) = a > -\infty$. Thus the condition (b) and (c) of

Theorem 3.1 for $-I$ is satisfied. By Lemma 4.2, $-I(u)$ satisfies the $(P.S.)_c$ condition for any $c \in R$, so the condition (d) of Theorem 3.1 for $-I$ is satisfied. By Theorem 3.1, there exist at least two nontrivial critical levels c_1 and c_2 for the functional $-I$ such that

$$\inf_{\Delta_R(e,X)}(-I) \leq c_1 \leq \sup_{\partial B_r \subset V}(-I) < \inf_{\Sigma_R(e,X)}(-I) \leq c_2 \leq \sup_{B_r \subset V}(-I).$$

Thus I has at least two nontrivial critical levels c_1 and c_2 for the functional I such that

$$\inf_{B_r \subset V} I \leq c_2 \leq \sup_{\Sigma_R(e,X)} I < \inf_{\partial B_r \subset V} I \leq c_1 \leq \sup_{\Delta_R(e,X)} I.$$

Thus we prove theorem.

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Department of Mathematics
Kunsan National University
Kunsan 573-701, Korea
E-mail: tsjung@kunsan.ac.kr

Department of Mathematics Education
Inha University
Incheon 402-751, Korea
E-mail: qheung@inha.ac.kr