STABILITY OF THE CAUCHY FUNCTIONAL EQUATION IN BANACH ALGEBRAS

JUNG RYE LEE* AND CHOOKIL PARK

Abstract. Using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras and of derivations on Banach algebras for the 3-variable Cauchy functional equation.

1. Introduction and preliminaries


Theorem 1.1. (Th. M. Rassias). Let \( f : E \rightarrow E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality

\[
\| f(x + y) - f(x) - f(y) \| \leq \epsilon (\| x \|^p + \| y \|^p)
\]

for all \( x, y \in E \), where \( \epsilon \) and \( p \) are constants with \( \epsilon > 0 \) and \( p < 1 \). Then the limit

\[
L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]
exists for all \( x \in E \) and \( L : E \to E' \) is the unique additive mapping which satisfies
\[
\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2p} \|x\|^p
\]
for all \( x \in E \). Also, if for each \( x \in E \) the function \( f(tx) \) is continuous in \( t \in \mathbb{R} \), then \( L \) is \( \mathbb{R} \)-linear.

The above inequality (1.1) that was introduced for the first time by Th.M. Rassias [30] for the proof of the stability of the linear mapping between Banach spaces has provided a lot of influence in the development of what is now known as a generalized Hyers-Ulam stability or as Hyers-Ulam-Rassias stability of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [8] extended the Hyers-Ulam stability by proving the following theorem in the spirit of Th.M. Rassias’ approach.

**Theorem 1.2.** [8] Let \( f : E \to E' \) be a mapping for which there exists a function \( \varphi : E \times E' \to [0, \infty) \) such that
\[
\tilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty,
\]
\[
\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)
\]
for all \( x, y \in E \). Then there exists a unique additive mapping \( T : E \to E' \) such that
\[
\|f(x) - T(x)\| \leq \frac{1}{2} \tilde{\varphi}(x, x)
\]
for all \( x \in E \).

**Theorem 1.3.** [29] Let \( X \) be a real normed linear space and \( Y \) a real complete normed linear space. Assume that \( f : X \to Y \) is an approximately additive mapping for which there exist constants \( \theta \geq 0 \) and \( p \in \mathbb{R} - \{1\} \) such that \( f \) satisfies inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \theta \cdot \|x\|^p \cdot \|y\|^p
\]
for all \( x, y \in X \). Then there exists a unique additive mapping \( L : X \to Y \) satisfying
\[
\|f(x) - L(x)\| \leq \frac{\theta}{|2p - 2|} \|x\|^p
\]
for all \( x \in X \). If, in addition, \( f : X \to Y \) is a mapping such that the transformation \( t \to f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \), then \( L \) is an \( \mathbb{R} \)-linear mapping.

The functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic function. A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [39] for mappings \( f : X \to Y \), where \( X \) is a normed space and \( Y \) is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain \( X \) is replaced by an Abelian group. Czerwik [6] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [2], [12], [14]–[27], [32]–[38]).

We recall a fundamental result in fixed point theory. The reader is referred to the book of D.H. Hyers, G. Isac and Th.M. Rassias [10] for an extensive account of fixed point theory with several applications.

Let \( X \) be a set. A function \( d : X \times X \to [0, \infty) \) is called a generalized metric on \( X \) if \( d \) satisfies

1. \( d(x, y) = 0 \) if and only if \( x = y \);
2. \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in X \).

**Theorem 1.4.** [4, 7, 28] Let \( (X, d) \) be a complete generalized metric space and let \( J : X \to X \) be a strictly contractive mapping with Lipschitz constant \( L < 1 \). Then for each given element \( x \in X \), either

\[
d(J^n x, J^{n+1} x) = \infty
\]

for all nonnegative integers \( n \) or there exists a positive integer \( n_0 \) such that

1. \( d(J^n x, J^{n+1} x) < \infty \), \( \forall n \geq n_0 \);
2. the sequence \( \{J^n x\} \) converges to a fixed point \( y^* \) of \( J \);
3. \( y^* \) is the unique fixed point of \( J \) in the set \( Y = \{y \in X \mid d(J^n x, y) < \infty\} \);
4. \( d(y, y^*) \leq \frac{1}{1-L} d(y, Jy) \) for all \( y \in Y \).
G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

This paper is organized as follows: In Section 2, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the 3-variable Cauchy functional equation.

In Section 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the 3-variable Cauchy functional equation.

Throughout this paper, assume that $A$ is a complex Banach algebra with norm $\| \cdot \|_A$ and that $B$ is a complex Banach algebra with norm $\| \cdot \|_B$.

2. Stability of homomorphisms in Banach algebras

For a given mapping $f : A \to B$, we define

$$D_\mu f(x, y, z) := \mu f(x + y + z) - f(\mu x) - f(\mu y) - f(\mu z)$$

for all $\mu \in T^1 := \{ \nu \in \mathbb{C} : |\nu| = 1 \}$ and all $x, y, z \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a homomorphism in Banach algebras if $H$ satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the functional equation $D_\mu f(x, y, z) = 0$.

**Theorem 2.1.** Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0, \infty)$ such that

\begin{align}
\|D_\mu f(x, y, z)\|_B & \leq \varphi(x, y, z), \\
\|f(xy) - f(x)f(y)\|_B & \leq \varphi(x, y, 0)
\end{align}

for all $\mu \in T^1$ and all $x, y, z \in A$. If there exists an $L < 1$ such that $\varphi(x, y, z) \leq 3L\varphi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3})$ for all $x, y, z \in A$, then there exists a unique homomorphism $H : A \to B$ such that

\begin{align}
\|f(x) - H(x)\|_B & \leq \frac{1}{3 - 3L} \varphi(x, x, x)
\end{align}

for all $x \in A$. 

Proof. Consider the set

\[ X := \{ g : A \to B \} \]

and introduce the generalized metric on \( X \):

\[ d(g, h) = \inf \{ C \in \mathbb{R}_+ : \| g(x) - h(x) \|_B \leq C\varphi(x, x, x), \quad \forall x \in A \}. \]

It is easy to show that \((X, d)\) is complete.

Now we consider the linear mapping \( J : X \to X \) such that

\[ Jg(x) := \frac{1}{3} g(3x) \]

for all \( x \in A \).

By Theorem 3.1 of [3],

\[ d(Jg, Jh) \leq Ld(g, h) \]

for all \( g, h \in X \).

Letting \( \mu = 1 \) and \( y = z = x \) in (2.1), we get

\[ \| f(3x) - 3f(x) \|_B \leq \varphi(x, x, x) \]  

for all \( x \in A \). So

\[ \| f(x) - \frac{1}{3} f(3x) \|_B \leq \frac{1}{3} \varphi(x, x, x) \]

for all \( x \in A \). Hence \( d(f, Jf) \leq \frac{1}{3} \).

By Theorem 1.4, there exists a mapping \( H : A \to B \) such that

(1) \( H \) is a fixed point of \( J \), i.e.,

\[ H(3x) = 3H(x) \]

for all \( x \in A \). The mapping \( H \) is a unique fixed point of \( J \) in the set

\[ Y = \{ g \in X : d(f, g) < \infty \}. \]

This implies that \( H \) is a unique mapping satisfying (2.5) such that there exists \( C \in (0, \infty) \) satisfying

\[ \| H(x) - f(x) \|_B \leq C\varphi(x, x, x) \]

for all \( x \in A \).

(2) \( d(J^n f, H) \to 0 \) as \( n \to \infty \). This implies the equality

\[ \lim_{n \to \infty} \frac{f(3^n x)}{3^n} = H(x) \]

for all \( x \in A \).
(3) \(d(f, H) \leq \frac{1}{1-L}d(f, Jf)\), which implies the inequality
\[
\begin{align*}
  d(f, H) &\leq \frac{1}{3-3L}.
\end{align*}
\]
This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.6) that
\[
\begin{align*}
  \|H(x + y + z) - H(x) - H(y) - H(z)\|_B &= \lim_{n \to \infty} \frac{1}{3^n} \|f(3^n(x + y + z)) - f(3^n x) - f(3^n y) - f(3^n z)\|_B \\
  &\leq \lim_{n \to \infty} \frac{1}{3^n} \varphi(3^n x, 3^n y, 3^n z) = 0
\end{align*}
\]
for all \(x, y, z \in A\). So
\[
(2.7) \quad H(x + y + z) = H(x) + H(y) + H(z)
\]
for all \(x, y, z \in A\). Letting \(z = 0\) in (2.7), we get
\[
H(x + y) = H(x) + H(y) + H(0) = H(x) + H(y)
\]
for all \(x, y \in A\).

Letting \(y = z = x\) in (2.1), we get
\[
\|\mu f(3x) - f(\mu 3x)\| \leq \varphi(x, x, x)
\]
for all \(\mu \in \mathbb{T}^1\) and all \(x \in A\). By a similar method to above, we get
\[
\mu H(3x) = H(3\mu x)
\]
for all \(\mu \in \mathbb{T}^1\) and all \(x \in A\). Thus one can show that the mapping \(H : A \to B\) is \(\mathbb{C}\)-linear.

It follows from (2.2) that
\[
\begin{align*}
  \|H(xy) - H(x)H(y)\|_B &= \lim_{n \to \infty} \frac{1}{9^n} \|f(9^n xy) - f(3^n x)f(3^n y)\|_B \\
  &\leq \lim_{n \to \infty} \frac{1}{9^n} \varphi(3^n x, 3^n y, 0) \\
  &\leq \lim_{n \to \infty} \frac{1}{3^n} \varphi(3^n x, 3^n y, 0) = 0
\end{align*}
\]
for all \(x, y \in A\). So
\[
H(xy) = H(x)H(y)
\]
for all \(x, y \in A\).

Thus \(H : A \to B\) is a homomorphism satisfying (2.3), as desired. \(\square\)
Corollary 2.2. Let $0 < r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that
\[
\|D_\mu f(x, y, z)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r),
\]
\[
\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r)
\]
for all $\mu \in T^1$ and all $x, y, z \in A$. Then there exists a unique homomorphism $H : A \to B$ such that
\[
\|f(x) - H(x)\|_B \leq \frac{3\theta}{3 - 3^r} \|x\|_A^r
\]
for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking
\[
\varphi(x, y, z) := \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)
\]
for all $x, y, z \in A$. Then $L = 3^{-r-1}$ and we get the desired result.

Theorem 2.3. Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0, \infty)$ satisfying (2.2). If there exists an $L < 1$ such that $\varphi(x, y, z) \leq \frac{1}{3}L \varphi(3x, 3y, 3z)$ for all $x, y, z \in A$, then there exists a unique homomorphism $H : A \to B$ such that
\[
\|f(x) - H(x)\|_B \leq \frac{L}{3 - 3^r} \varphi(x, x, x)
\]
for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that
\[
Jg(x) := 3g\left(\frac{x}{3}\right)
\]
for all $x \in A$.

It follows from (2.4) that
\[
\|f(x) - 3f\left(\frac{x}{3}\right)\|_B \leq \varphi\left(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}\right) \leq \frac{L}{3} \varphi(x, x, x)
\]
for all $x \in A$. Hence $d(f, Jf) \leq \frac{L}{3}$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that
(1) $H$ is a fixed point of $J$, i.e.,
\[
H(3x) = 3H(x)
\]
for all $x \in A$. The mapping $H$ is a unique fixed point of $J$ in the set
\[
Y = \{g \in X : d(f, g) < \infty\}.
\]
This implies that $H$ is a unique mapping satisfying (2.11) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x, x, x)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n \to \infty} 3^n f\left(\frac{x}{3^n}\right) = H(x)$$

for all $x \in A$.

(3) $d(f, H) \leq \frac{2}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{L}{3 - 3L},$$

which implies that the inequality (2.10) holds.

The rest of the proof is similar to the proof of Theorem 2.1.

**Corollary 2.4.** Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (2.8) and (2.9). Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{3\theta}{3^r - 3}\|x\|_A^r$$

for all $x \in A$.

**Proof.** The proof follows from Theorem 2.3 by taking

$$\varphi(x, y, z) := \theta\left(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r\right)$$

for all $x, y, z \in A$. Then $L = 3^{1-r}$ and we get the desired result.

**3. Stability of derivations on Banach algebras**

Note that a $C$-linear mapping $\delta : A \to A$ is called a *derivation* on $A$ if $\delta$ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the functional equation $D_\mu f(x, y, z) = 0$.

**Theorem 3.1.** Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^3 \to [0, \infty)$ such that

(3.1) $\|D_\mu f(x, y, z)\|_A \leq \varphi(x, y, z),$  
(3.2) $\|f(xy) - f(x)y - xf(y)\|_A \leq \varphi(x, y, 0)$
for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. If there exists an $L < 1$ such that $\varphi(x, y, z) \leq 3L\varphi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3})$ for all $x \in A$. Then there exists a unique derivation $\delta : A \to A$ such that
\[\|f(x) - \delta(x)\|_A \leq \frac{1}{3 - 3L}\varphi(x, x, x)\]
for all $x \in A$.

**Proof.** By the same reasoning as the proof of Theorem 2.1, there exists a unique $C$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by
\[\delta(x) = \lim_{n \to \infty} \frac{f(3^n x)}{3^n}\]
for all $x \in A$.

It follows from (3.2) that
\[\|\delta(xy) - \delta(x)y - x\delta(y)\|_A = \lim_{n \to \infty} \frac{1}{9^n}\|f(9^n xy) - f(3^n x) \cdot 3^n y - 3^n x f(3^n y)\|_A \leq \lim_{n \to \infty} \frac{1}{9^n}\varphi(3^n x, 3^n y, 0) \leq \lim_{n \to \infty} \frac{1}{3^n}\varphi(3^n x, 3^n y, 0) = 0\]
for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3).

**Corollary 3.2.** Let $0 < r < 1$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that
\[\|D_\mu f(x, y, z)\|_A \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r),\]
\[\|f(xy) - f(x)y - xf(y)\|_A \leq \theta(\|x\|_A^r + \|y\|_A^r)\]
for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique derivation $\delta : A \to A$ such that
\[\|f(x) - \delta(x)\|_A \leq \frac{3\theta}{3 - 3r}\|x\|_A^r\]
for all $x \in A$.

**Proof.** The proof follows from Theorem 3.1 by taking
\[\varphi(x, y, z) := \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)\]
for all $x, y, z \in A$. Then $L = 3^{-r - 1}$ and we get the desired result.
Theorem 3.3. Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^3 \to [0, \infty)$ satisfying (3.1) and (3.2). If there exists an $L < 1$ such that $\varphi(x, y, z) \leq \frac{1}{3} L \varphi(3x, 3y, 3z)$ for all $x, y, z \in A$, then there exists a unique derivation $\delta : A \to A$ such that

$$
\|f(x) - \delta(x)\|_A \leq \frac{L}{3 - 3L^r} \varphi(x, x, x)
$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1.

Corollary 3.4. Let $r > 2$ and $\theta$ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (3.4) and (3.5). Then there exists a unique derivation $\delta : A \to A$ such that

$$
\|f(x) - \delta(x)\|_A \leq \frac{3\theta}{3^r - 3} \|x\|_A^r
$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking

$$
\varphi(x, y, z) := \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)
$$

for all $x, y, z \in A$. Then $L = 3^{1-r}$ and we get the desired result.

References


Department of Mathematics
Daejin University
Kyeonggi 487-711, Republic of Korea
E-mail: jrlee@daejin.ac.kr

Department of Mathematics
Hanyang University
Seoul 133–791, Republic of Korea
E-mail: baak@hanyang.ac.kr