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STABILITY OF THE CAUCHY FUNCTIONAL EQUATION IN BANACH ALGEBRAS

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Abstract. Using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras and of derivations on Banach algebras for the 3-variable Cauchy functional equation.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [40] concerning the stability of group homomorphisms. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Th.M. Rassias [30] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1. (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

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exists for all $x \in E$ *and* $L : E \to E'$ *is the unique additive mapping which satisfies*

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. *Also, if for each* $x \in E$ *the function* $f(tx)$ *is continuous in* $t \in \mathbb{R}$, *then* L *is* $\mathbb{R}$*-linear.*

The above inequality (1.1) that was introduced for the first time by Th.M. Rassias [30] for the proof of the stability of the linear mapping between Banach spaces has provided a lot of influence in the development of what is now known as a *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [8] extended the Hyers-Ulam stability by proving the following theorem in the spirit of Th.M. Rassias' approach.

Theorem 1.2. [8] *Let* $f : E \to E'$ *be a mapping for which there exists a function* $\varphi : E \times E' \to [0, \infty)$ *such that*

$$\begin{aligned} \widetilde{\varphi}(x,y) : &= \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty, \\ \|f(x+y) - f(x) - f(y)\| &\leq \varphi(x,y) \end{aligned}$$

for all $x, y \in E$. *Then there exists a unique additive mapping* $T : E \to E'$ *such that*

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in E$.

Theorem 1.3. [29] *Let* X *be a real normed linear space and* Y *a real complete normed linear space. Assume that* $f : X \to Y$ *is an approximately additive mapping for which there exist constants* $\theta \geq 0$ *and* $p \in \mathbb{R} - \{1\}$ *such that* f *satisfies inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \cdot ||x||^{\frac{p}{2}} \cdot ||y||^{\frac{p}{2}}$$

for all $x, y \in X$. *Then there exists a unique additive mapping* $L : X \to Y$ *satisfying*

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|}||x||^p$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic function.* A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [39] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [6] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [2], [12], [14]–[27], [32]–[38]).

We recall a fundamental result in fixed point theory. The reader is referred to the book of D.H. Hyers, G. Isac and Th.M. Rassias [10] for an extensive account of fixed point theory with several applications.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [4, 7, 28] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \geq n_0$*;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

This paper is organized as follows: In Section 2, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the 3-variable Cauchy functional equation.

In Section 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the 3-variable Cauchy functional equation.

Throughout this paper, assume that A is a complex Banach algebra with norm $\|\cdot\|_A$ and that B is a complex Banach algebra with norm $\|\cdot\|_B$.

2. Stability of homomorphisms in Banach algebras

For a given mapping $f : A \to B$, we define

$$D_\mu f(x,y,z) := \mu f(x+y+z) - f(\mu x) - f(\mu y) - f(\mu z)$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x, y, z \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* in Banach algebras if H satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the functional equation $D_\mu f(x,y,z) = 0$.

THEOREM 2.1. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ such that*

$$\|D_\mu f(x,y,z)\|_B \leq \varphi(x,y,z), \tag{2.1}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \varphi(x,y,0) \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. If there exists an $L < 1$ such that $\varphi(x,y,z) \leq 3L\varphi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3})$ for all $x, y, z \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{1}{3-3L}\varphi(x,x,x) \tag{2.3}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \to B\}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \le C\varphi(x,x,x), \quad \forall x \in A\}.$$

It is easy to show that (X, d) is complete.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{3}g(3x)$$

for all $x \in A$.

By Theorem 3.1 of [3],

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in X$.

Letting $\mu = 1$ and $y = z = x$ in (2.1), we get

$$\|f(3x) - 3f(x)\|_B \le \varphi(x,x,x) \tag{2.4}$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{3}f(3x)\|_B \le \frac{1}{3}\varphi(x,x,x)$$

for all $x \in A$. Hence $d(f, Jf) \le \frac{1}{3}$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(3x) = 3H(x) \tag{2.5}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.5) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x,x,x)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(3^n x)}{3^n} = H(x) \tag{2.6}$$

for all $x \in A$.

(3) $d(f,H) \le \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,H) \le \frac{1}{3-3L}.$$

This implies that the inequality (2.3) holds.

It follows from (2.1) and (2.6) that

$$\begin{aligned} &\|H(x+y+z) - H(x) - H(y) - H(z)\|_B \\ &= \lim_{n\to\infty} \frac{1}{3^n} \|f(3^n(x+y+z)) - f(3^n x) - f(3^n y) - f(3^n z)\|_B \\ &\le \lim_{n\to\infty} \frac{1}{3^n} \varphi(3^n x, 3^n y, 3^n z) = 0 \end{aligned}$$

for all $x, y, z \in A$. So

$$H(x+y+z) = H(x) + H(y) + H(z) \tag{2.7}$$

for all $x, y, z \in A$. Letting $z = 0$ in (2.7), we get

$$H(x+y) = H(x) + H(y) + H(0) = H(x) + H(y)$$

for all $x, y \in A$.

Letting $y = z = x$ in (2.1), we get

$$\|\mu f(3x) - f(\mu 3x)\| \le \varphi(x,x,x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. By a similar method to above, we get

$$\mu H(3x) = H(3\mu x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.2) that

$$\begin{aligned} \|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} \frac{1}{9^n} \|f(9^n xy) - f(3^n x) f(3^n y)\|_B \\ &\le \lim_{n\to\infty} \frac{1}{9^n} \varphi(3^n x, 3^n y, 0) \\ &\le \lim_{n\to\infty} \frac{1}{3^n} \varphi(3^n x, 3^n y, 0) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Thus $H : A \to B$ is a homomorphism satisfying (2.3), as desired. □

Corollary 2.2. *Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\|D_\mu f(x,y,z)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \tag{2.8}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r) \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{3\theta}{3 - 3^r}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y,z) := \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)$$

for all $x, y, z \in A$. Then $L = 3^{r-1}$ and we get the desired result. □

Theorem 2.3. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^3 \to [0,\infty)$ satisfying (2.2). If there exists an $L < 1$ such that $\varphi(x,y,z) \leq \frac{1}{3}L\varphi(3x,3y,3z)$ for all $x, y, z \in A$, then there exists a unique homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{L}{3 - 3L}\varphi(x,x,x) \tag{2.10}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 3g(\frac{x}{3})$$

for all $x \in A$.

It follows from (2.4) that

$$\|f(x) - 3f(\frac{x}{3})\|_B \leq \varphi(\frac{x}{3}, \frac{x}{3}, \frac{x}{3}) \leq \frac{L}{3}\varphi(x,x,x)$$

for all $x \in A$. Hence $d(f, Jf) \leq \frac{L}{3}$.

By Theorem 1.4, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(3x) = 3H(x) \tag{2.11}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.11) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x, x, x)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 3^n f(\frac{x}{3^n}) = H(x)$$

for all $x \in A$.

(3) $d(f, H) \le \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, H) \le \frac{L}{3 - 3L},$$

which implies that the inequality (2.10) holds.

The rest of the proof is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.8) *and* (2.9). *Then there exists a unique homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{3\theta}{3^r - 3}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y, z) := \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)$$

for all $x, y, z \in A$. Then $L = 3^{1-r}$ and we get the desired result. □

3. Stability of derivations on Banach algebras

Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the functional equation $D_\mu f(x, y, z) = 0$.

Theorem 3.1. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^3 \to [0, \infty)$ such that*

$$\|D_\mu f(x, y, z)\|_A \le \varphi(x, y, z), \tag{3.1}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \le \varphi(x, y, 0) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. If there exists an $L < 1$ such that $\varphi(x, y, z) \leq 3L\varphi(\frac{x}{3}, \frac{y}{3}, \frac{z}{3})$ for all $x \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{1}{3 - 3L}\varphi(x, x, x) \tag{3.3}$$

for all $x \in A$.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(3^n x)}{3^n}$$

for all $x \in A$.

It follows from (3.2) that

$$\begin{aligned} &\|\delta(xy) - \delta(x)y - x\delta(y)\|_A \\ &\quad= \lim_{n\to\infty} \frac{1}{9^n}\|f(9^n xy) - f(3^n x) \cdot 3^n y - 3^n x f(3^n y)\|_A \\ &\quad\leq \lim_{n\to\infty} \frac{1}{9^n}\varphi(3^n x, 3^n y, 0) \leq \lim_{n\to\infty} \frac{1}{3^n}\varphi(3^n x, 3^n y, 0) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). □

Corollary 3.2. *Let $0 < r < 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that*

$$\|D_\mu f(x, y, z)\|_A \leq \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r), \tag{3.4}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \leq \theta(\|x\|_A^r + \|y\|_A^r) \tag{3.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \leq \frac{3\theta}{3 - 3^r}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y, z) := \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)$$

for all $x, y, z \in A$. Then $L = 3^{r-1}$ and we get the desired result. □

Theorem 3.3. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^3 \to [0, \infty)$ satisfying* (3.1) *and* (3.2). *If there exists an $L < 1$ such that $\varphi(x, y, z) \leq \frac{1}{3}L\varphi(3x, 3y, 3z)$ for all $x, y, z \in A$, then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{L}{3-3L}\varphi(x, x, x)$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1. □

Corollary 3.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.4) *and* (3.5). *Then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \leq \frac{3\theta}{3^r - 3}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y, z) := \theta(\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)$$

for all $x, y, z \in A$. Then $L = 3^{1-r}$ and we get the desired result. □

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