A STUDY ON GENERALIZED QUASI-CLASS A OPERATORS

GEON-HO KIM AND IN HO JEON*

ABSTRACT. In this paper, we consider the operator $T$ satisfying
$T^*k(|T^2| - |T|^2)T^k \geq 0$ and prove that if the operator is injective
and has the real spectrum, then it is self-adjoint.

1. Introduction

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on a
Hilbert space $\mathcal{H}$. Recall ([2]) that $T \in \mathcal{L}(\mathcal{H})$ is called $p$-hyponormal if
$(T^*T)^p \geq (TT^*)^p$ for $p \in (0, 1]$, and $T$ is called paranormal if $||T^2x|| \geq
||Tx||^2$ for all unit vector $x \in \mathcal{H}$. Following [3] and [2] we say that
$T \in \mathcal{L}(\mathcal{H})$ belongs to class $A$ if $|T^2| \geq |T|^2$. We shall denote classes of
$p$-hyponormal operators, paranormal operators, and class $A$ operators
by $\mathcal{H}(p)$, $\mathcal{P}\mathcal{N}$, and $\mathcal{A}$, respectively. It is well known that

\begin{equation}
\mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{P}\mathcal{N}.
\end{equation}

In [5] Jeon and Kim considered an extension of the notion of class $A$
operators; we say that $T \in \mathcal{L}(\mathcal{H})$ is quasi-class $A$ if

$T^*|T^2|T \geq T^*|T|^2T$.

We shall denote the set of quasi-class $A$ operators by $\mathcal{Q}A$. As shown in
[5], the class of quasi-class $A$ operators properly contains classes of class
$A$ operators, i.e., the following inclusions holds;

\begin{equation}
\mathcal{H}(p) \subset \mathcal{A} \subset \mathcal{Q}A.
\end{equation}

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*Corresponding author.
In view of inclusions (1.1) and (1.2), it seems reasonable to expect that the operators in class QA are paranormal. But there exists an example [5] that one would be wrong in such an expectation.

Now, we consider a following generalization of quasi-class A operators in [10].

**Definition 1.1.** We say that $T \in \mathcal{L}(\mathcal{H})$ is of quasi-class $(A, k)$ class if

$$T^*k(|T^2| - |T|^2)T^k \geq 0 \quad \text{for } k \in \mathbb{N}$$

We denote the spectrum and the closure of numerical range of an operator $T \in \mathcal{L}(\mathcal{H})$ by $\sigma(T)$ and $\overline{W(T)}$, respectively.

In 1966, I. H. Sheth [9] showed that if $T$ is a hyponormal operator and $S^{-1}TS = T^*$ for any operator $S$, where $0 \notin \overline{W(S)}$, then $T$ is self-adjoint, and then I. H. Kim [7] extended this result of Sheth to the class of $p$-hyponormal operators. Very recently, Jeon, Kim, Tanahashi and Uchiyama [6] also extended this result to the class of quasi-class A operators as follows.

**Proposition 1.2 ([6], Theorem 2.6).** If $T$ is a quasi-class A operator and $S$ is an arbitrary operator for which $0 \notin \overline{W(S)}$ and $ST = T^*S$, then $T$ is self-adjoint.

The aim of this paper is to extend this result to more generalized quasi-class A operators (i.e., quasi-class $(A, k)$ operators) as follows.

**Theorem 1.3.** Let $T$ be of injective quasi-class $(A, k)$ with the real spectrum. Then $T$ is self-adjoint.

In [11], J.P. Williams showed that if $T \in \mathcal{L}(\mathcal{H})$ is any operator such that $ST = T^*S$, where $0 \notin \overline{W(S)}$, then the spectrum of $T$ is real. So, for a $T \in \mathcal{L}(\mathcal{H})$, the condition that

there exists an operator $S$ such that $ST = T^*S$, where $0 \notin \overline{W(S)}$

is stronger than that the spectrum of $T$ is real, which shows that the above Theorem extends Proposition 1.2. under the injectiveness of $T$.

2. Proofs

In this section we give a proof of Theorem 1.3, modifying arguments used in proofs of [6]. We need some lemmas.
**Lemma 2.1.** Let $T$ be of quasi-class $(A,k)$. Then the following assertions hold:

1. Assume that $\operatorname{ran}^k T$ is not dense, and decompose
   $$ T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker T^{k*} $$
   where $\overline{\operatorname{ran}(T^k)}$ is the closure of $\operatorname{ran}^k T$. Then $T_1$ is of class $A$, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

2. The restriction $T|_M$ to an invariant subspace $M$ of $T$ is also of quasi-class $(A,k)$.

**Lemma 2.2.** Let $T \in \mathcal{L}(\mathcal{H})$ be a class $A$ operator. Then we have an inequality

\begin{equation}
\| |T^2| - |T|^2 \| \leq \| |T|U|T| - |T|U^*|T| \| \leq \frac{1}{\pi} \operatorname{meas} \sigma(T),
\end{equation}

where $T = U|T|$ is the polar decomposition of $T$.

**Lemma 2.3.** Let $T \in \mathcal{L}(\mathcal{H})$ be a class $A$ operator with the real spectrum. Then $T$ is self-adjoint.

**Proof.** Since $T$ is of class $A$ and it has the real spectrum, from (2.1), we have $|T^2| = |T|^2$. Now let

$$ T = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \quad \text{on} \quad \overline{\operatorname{ran}(T)} \oplus \ker(T^*) $$

be a $2 \times 2$ matrix representation of $T$, and let $P$ be the orthogonal projection onto $\overline{\operatorname{ran}(T)}$. Then since

$$ |T^2| - |T|^2 = 0 \Rightarrow T^*(T^*T - TT^*)T = 0, $$

we have $P(T^*T - TT^*)P = 0$. Therefore, by simple calculation, $A^*A - AA^* = BB^*$ and hence $A$ is hyponormal. Since the spectrum of $A$ is contained in the spectrum of $T$, it is also real. Thus $A$ is self-adjoint and $B = 0$, which implies that $T$ is self-adjoint.

**Proof of Theorem 1.3.** If $T$ is of quasi-class $(A,k)$ and the range of $T^k$ is dense, then $T$ is of class $A$ from Lemma 2.1. Hence Theorem 1.3 is reduced to Lemma 2.3. Assume that the range of $T^k$ is not dense. From Lemma 2.1 we have a decomposition

$$ T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\operatorname{ran}(T^k)} \oplus \ker T^{k*}. $$
Then $T_1$ is of class $A$ and $T_3^k = 0$. Since the spectrum of $T_1$ is contained in the spectrum of $T$, $T_1$ is self-adjoint by Lemma 2.3. Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection onto $\text{ran}(T^k)$. Then

$$Q|T|^2Q = QT^*TQ = \begin{pmatrix} T_1^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and so we may write

$$|T|^2 = \begin{pmatrix} T_1^2 & C \\ C^* & D \end{pmatrix}.$$

On the other hand, let $|T| = \begin{pmatrix} E & F \\ F^* & G \end{pmatrix}$. Then we have

$$\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = (Q|T|^2Q)^{\frac{1}{2}} \geq Q|T|Q = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q(T^*T)^{\frac{1}{2}}Q \geq Q(T^*QT)^{\frac{1}{2}}Q = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $E = T_1$ and $|T| = \begin{pmatrix} T_1 & F \\ F^* & G \end{pmatrix}$. By straight forward calculation we have

$$\begin{pmatrix} T_1^2 & T_1T_2 \\ T_2^*T_1 & |T_2|^2 + |T_3|^2 \end{pmatrix} = |T|^2 = \begin{pmatrix} T_1^2 + FF^* & T_1F + FG \\ F^*T_1 + GF^* & F^*F + G^2 \end{pmatrix},$$

which implies that $F = 0$ and $T_1T_2 = 0$. Since $T_1$ is injective, $T_2 = 0$. Thus $\text{ran}(T^k)$ and $\text{ker} T^k$ are reducing subspaces. Since $T$ is injective, $T_3$ is also injective. Therefore we have that $T_3 = 0$. Hence $T$ is self-adjoint. □

References


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Department of Industrial Management
Ansan College of Technology
425-792, Korea
E-mail: ghok6096@act.ac.kr

Department of Mathematics Education
Seoul National University of Education
Seoul 137-742, Korea
E-mail: jihmath@snue.ac.kr