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AT LEAST FOUR SOLUTIONS TO THE NONLINEAR ELLIPTIC SYSTEM

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Abstract. We prove the existence of multiple solutions (ξ, η) for perturbations of the elliptic system with Dirichlet boundary condition

$$\begin{aligned} A\xi + g_1(\xi + 2\eta) &= s\phi_1 + h \quad \text{in} \quad \Omega, \\ A\eta + g_2(\xi + 2\eta) &= s\phi_1 + h \quad \text{in} \quad \Omega, \end{aligned} \tag{0.1}$$

where $lim_{u\to\infty}\frac{g_j(u)}{u} = \beta_j$, $lim_{u\to-\infty}\frac{g_j(u)}{u} = \alpha_j$ are finite and the nonlinearity $g_1 + 2g_2$ crosses eigenvalues of A.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R^n}$ with smooth boundary $\partial\Omega$. In this paper we investigate the existence of solutions (ξ, η) for the nonlinear elliptic system with Dirichlet boundary condition

$$\begin{aligned} A\xi + g_1(\xi + 2\eta) &= s\phi_1 + h \quad \text{in} \quad \Omega, \\ A\eta + g_2(\xi + 2\eta) &= s\phi_1 + h \quad \text{in} \quad \Omega, \\ \xi = 0, \qquad \eta &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $lim_{u\to\infty}\frac{g_j(u)}{u} = \beta_j$, $lim_{u\to-\infty}\frac{g_j(u)}{u} = \alpha_j$ are finite and the nonlinearity $g_1 + 2g_2$ crosses eigenvalues of A. Here A denote the differential operator $A = \sum_{1\le i,j\le n}\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$ with $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$.

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In [4] the authors investigate multiplicity of solutions of the nonlinear elliptic equation with Dirichlet boundary condition

$$
\begin{aligned}
&Au + g(u) = f(x) \quad \text{in} \quad \Omega,\\
&u = 0 \quad \text{on} \quad \partial\Omega,
\end{aligned}
\tag{1.2}
$$

where the semilinear term $g(u) = bu^+ - au^-$ and A is a second order linear elliptic differential operator and a mapping from $L^2(\Omega)$ into itself with compact inverse, with eigenvalues $-\lambda_i$, each repeated according to its multiplicity,

$$
0 < \lambda_1 < \lambda_2 < \lambda_3 \leq \cdots \leq \lambda_i \leq \cdots \to \infty.
$$

Here the source term f is generated by the eigenfunctions of the second order elliptic operator with Dirichlet boundary condition.

In [5, 7, 8], the authors have investigated multiplicity of solutions of (1.2) when the forcing term f is supposed to be a multiple of the first eigenfunction and the nonlinearity $-(bu^+ - au^-)$ crosses eigenvalues. In [4], the authors investigated a relation between multiplicity of solutions and source terms of (1.1) when the forcing term f is supposed to be spanned two eigenfunction ϕ_1, ϕ_2 and the nonlinearity $-(bu^+ - au^-)$ crosses two eigenvalues λ_1, λ_2.

Equation (1.2) and the following type nonlinear equation with Dirichlet boundary condition was studied by many authors:

$$
\begin{aligned}
&Lu = b[(u+2)^+ - 2] \quad \text{in} \quad \Omega,\\
&u = 0 \quad \text{on} \quad \partial\Omega.
\end{aligned}
\tag{1.3}
$$

In [9] Lazer and McKenna point out that this kind of nonlinearity $b[(u+2)^+ - 2]$ can furnish a model to study traveling waves in suspension bridges. So the nonlinear equation with jumping nonlinearity have been extensively studied by many authors. For fourth elliptic equation Tarantello [14] , Micheletti and Pistoia [12][13] proved the existence of nontrivial solutions used degree theory and critical points theory separately. For one-dimensional case Lazer and McKenna [10] proved the existence of nontrivial solution by the global bifurcation method. For this jumping nonlinearity we are interest in the multiple nontrivial solutions of the equation.

The organization of this paper is as following. In section 2, we have a concern with the multiplicity of solutions and source terms of a nonlinear elliptic equation when the nonlinearity crosses eigenvalues. We investigate the uniqueness and multiplicity of solutions for the single nonlinear elliptic equation. In section 3, we investigate the existence of multiple solutions (ξ, η) for the elliptic system with Dirichlet boundary condition when the nonlinearity crosses the eigenvalues of the elliptic operator.

2. Appendix: The nonlinear elliptic equation

Let L be the linear partial differential operator. Many authors ([3,4,5,6,8]) have studied multiplicity of solutions of the following type nonlinear problem

$$Lu + bu^+ - au^- = g \quad \text{in } L^2(\Omega). \tag{2.1}$$

In this section we investigate the existence of solutions for the nonlinear elliptic system with Dirichlet boundary condition

$$Au + f(u) = s\phi_1(x) + h(x). \tag{2.2}$$

Here we assume that $lim_{u\to\infty}\frac{f(u)}{u} = \beta$, $lim_{u\to-\infty}\frac{f(u)}{u} = \alpha$, and $\alpha < \lambda_1 < \lambda_2 < \beta < \lambda_3$ and $\|h\| = 1$.

Let us denote an element u, in $H_0 \subset L^2(\Omega)$, as $u = \sum h_j\phi_j$ and we define a subspace H of H_0 as

$$H = \{u \in H_0 : \sum |\lambda_j| h_j^2 < \infty\}.$$

Then this is a complete normed space with a norm $\|u\| = (\sum |\lambda_{mn}| h_{mn}^2)^{\frac{1}{2}}$. If $f \in H_0$ and a, b are not eigenvalues of L, then every solution in H_0 of $Lu + bu^+ - au^- = f$ belongs to H (cf. [2]).

Let V be the two dimensional subspace of $L^2(\Omega)$ spanned by $\{\phi_1, \phi_2\}$ and W be the orthogonal complement of V in $L^2(\Omega)$. Let P be an orthogonal projection $L^2(\Omega)$ onto V. Then every element $u \in H$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I - P)u$. Hence equation (2.1) is equivalent to a system

$$Lw + (I - P)(b(v + w)^+ - a(v + w)^-) = 0, \tag{2.3}$$

$$Lv + P(b(v + w)^+ - a(v + w)^-) = s_1\phi_1 + s_2\phi_2. \tag{2.4}$$

Lemma 2.1. *For every $v = c_1\phi_1 + c_2\phi_2$, there exists a constant $d > 0$ such that*

$$(\Phi(v), \phi_1) \geq d|c_2|.$$

Since the subspace V is spanned by $\{\phi_1, \phi_2\}$ and $\phi_1(x) > 0$ in Ω, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 : c_1 \geq 0, |c_2| \leq kc_1\}$$

for some $k > 0$ so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 : c_1 \leq 0, |c_2| \leq k|c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.

We set the cones C_2, C_4 as follows

$$C_2 = \{c_1\phi_1 + c_2\phi_2 : c_2 \geq 0,\ c_2 \geq k|c_1|\},$$

$$C_4 = \{c_1\phi_1 + c_2\phi_2 : c_2 \leq 0,\ c_2 \leq -k|c_1|\}.$$

Then the union of four cones C_i $(1 \leq i \leq 4)$ is the space V.

We depended very heavily on the exact form of the piecewise nonlinearity. In this section, we return to the study of equation for large t. If we were content to consider the case where $f'(s) < \lambda_3$ for all s, then the task would be fairly simple. We would show that the reduced two-dimensional picture remained largely unchanged as c_1 and c_2 were made very large. However, we regard that restriction as somewhat artificial, restrictions on the derivative being only used previously for obtaining upper bounds in the number of solutions.

On the other hand, if we abandon the restriction $f' < \lambda_3$, then we cannot just consider the two dimensional reduction any since no longer would there be a unique solution to equation and thus no reduced problem.

Our plan is the following; we first convert the two dimensional statements into degree theoretic statements in the space $L^2(\Omega)$, and then show that these can be perturbed to give the result for the nonlinear equation with large t.

Our first lemma is a degree theoretic interpretation of Theorem.

Recall that ϕ_1, ϕ_2 satisfy $\phi_1(x) - \epsilon_0|\phi_2(x)| \geq 0$ for all $x \in \Omega$. Also recall that if $\Phi : PH \to PH$ is defined then there exists $d > 0$ satisfying the conditions of Lemma.

The map $\Phi : PH \to PH$ takes value ϕ_1, once in each of the four different regions of the plain. The next lemma gives information on the degree of the map in these regions.

We define $F : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$F(s_1, s_2) = (t_1, t_2) \quad \text{if} \quad \mathrm{v} = \mathrm{s}_1\phi_1 + \mathrm{s}_2\phi_2, \quad \Phi(\mathrm{v}) = \mathrm{t}_1\phi_1 + \mathrm{t}_2\phi_2.$$

LEMMA 2.2. *Let* $p = (1, 0)$. *Let* r *be so large that* $r > 1, r(b - \lambda_1) > 1, r(\lambda_a) > 1, r\epsilon_0 > 1$ *and* $rd\epsilon_0 > 1$, *where* d *and* ϵ_0 *cone from section* 1. *Let*

$$\begin{aligned} D_1 &= \{(s_1, s_2) \mid 0 < s_1 < r, |s_2| < \epsilon_0\epsilon_1\} \\ D_2 &= \{(s_1, s_2) \mid |s_1| \leq r, \epsilon|s_1| < s_2 < \epsilon_0 r\} \\ D_3 &= \{(s_1, s_2) \mid -r < s_1 < 0, |s_2| < \epsilon|s_1|\} \\ D_4 &= \{(s_1, s_2) \mid |s_1| \leq r, -\epsilon_0 r < -\epsilon_0|s_1|\} \end{aligned}$$

If $deg(F, D_k, p)$ *denotes the Brower degree of* F *with respect to* D_k *and* p *for* $1 \leq k \leq 4$, *then* $d(F, D_k, p)$ *is defined for* $1 \leq k \leq 4$ *and*

$$\mathrm{deg(F, D_k, p) = (-1)^{k+1}}.$$

Proof. First consider D_1. If $(s_1, s_2) \in \overline{D}$ and $v = s_1\phi_1 + s_2\phi_2$, then $\theta(v) = 0$. On D_1, the map $F(S_1, S_2)$ is given by $F(s_1, s_2) = ((b - \lambda_1)s_1, (b - \lambda_2)s_2)$. Since $1 < r(b - \lambda_1)$ the equation $F(s_1, s_2) = p$ has the unique solution $(s_1, s_2) = ((b - \lambda_1)^{-1}, 0)$. Since the determinant of the linear diagonal map is positive, we have

$$\mathrm{deg(F, D_1, p) = 1}.$$

In the case of D_3, we have the diagonal map with two negative entries, $(a - \lambda_1), (a - \lambda_2)$ and the determinant is also positive neat the unique solution in this region given by $((a - \lambda_1)^{-1}, 0)$, so again $\mathrm{deg(F, D_3, p) = 1}$.

Now consider D_2. The boundary of D_2 consists of three line segments;

(i) a ray in the first quadrant R_1.

(ii) a tay in the second quadrant R_2.

(iii) a line segment L of $s_2 = \epsilon_0 r$, parallel to the s_1 axis.

As we observed in the proof of Theorem , the image of R_1 under F will be a straight line segment in the fourth quadrant, the image of R_2 will be to the right of the line $s_1 = 1$, by virtue of the requirement.

Now consider the linear map $u \to Bu$, where B is given by

$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The image of R_1 under B, BR_1, will be a straight line in the first quadrant, so if $0 \leq \lambda \leq 1$, we have

$$\lambda Bs + (1 - \lambda)F(s) \neq p, \quad s = (s_1, s_2) \in R_1.$$

The image of the ray R_2 under B is in the fourth quadrant and again we have

$$\lambda Bs + (1-\lambda)F(s) \neq p, \quad s = (s_1, s_2) \in R_2.$$

Finally, if $s \in L$, then $s_2 = \epsilon_0 r > 1$ so

$$Bs \in \{(s_1, s_2) \mid s_1 > 1\}$$

and thus $\lambda Bs + (1-\lambda)F(S) \neq p$ for $s \in L$. By the usual homotopy argument,

$$\mathrm{deg(F, D_2, p) = deg(B, D_2, p)}.$$

But we know that $Bs - p$ has exactly one zero in D_2 and the sign of the determinant of B is -1. Thus

$$\mathrm{deg(F, D_2, p)} = -1.$$

The proof for D_4 is similar so we leave it as an exercise for the reader. □

Using the definition of the degree of a mapping on an arbitrary finite dimensional space, we obtain, letting $V = PH$.

Lemma 2.3. *If for* $1 \leq k \leq 4$,

$$U_k = \{v \in V \mid v = s_1\phi_1 + s_2\phi_2, (s_1, s_2) \in D_k\}$$

and $T : V \to V$ *is defined by*

$$Tv = PA^{-1}(b(v + \theta(v))^+ - a(v + \theta(v))^-)$$

then

$$\mathrm{deg}(\mathrm{I + T, U_k}, -\frac{\phi_1}{\lambda_1}) = (-1)^{\mathrm{k+1}}.$$

We have now calculated the degree of the two dimensional map on the various regions. But we remind ourselves that the two dimensional map is obtained from the infinite dimensional map by using the contraction fixed point theorem. Our aim now is to perturb the equation

$$Au + bu^+ - au^- = s\phi_1.$$

To do this, and arrive at the full non-linearity f equation, we could proceed in two ways. We could restrict the class of f under discussion so that they satisfied $f' \leq \lambda_3 - \epsilon$. Then each perturbed problem could be reduced to a two dimensional problem which could be viewed, for large s, as a perturbation of the piecewise linear problem.

The reader will soon see that this would be extremely restrictive. What we do instead is to deduce, from our knowledge of the two dimensional degree, a result on the degree of the associated map on the infinite dimensional space. This can then be perturbed by small perturbations, which perturbations need only be continuous.

Let $Nu = A^{-1}(bu^+ - au^-)$.

LEMMA 2.4. *Let $U_k, 1 \le k \le 4$, and T be as in the preceding lemma. If $r_2 > 0$ is sufficiently large, and for $1 \le k \le 4$*

$$Y_k = \{u \in L^2(\Omega) \mid Pu \in U_k, \|(I-P)u\| < r_2\}.$$

then the Leray-Schauder degree $d(I+N, Y_k, -\frac{\phi_1}{\lambda_1})$ is defined and

$$d(I+N, Y_k, -\frac{\phi_1}{\lambda_1}) = d(I+T, U_k, \frac{\phi_1}{\lambda_1}) = (-1)^{k+1}.$$

Proof. The proof of this lemma comes in several steps. First, we observe that there exists $r_1 > 0$ such that if $v \in \overline{U}_k, 1 \le k \le 4$, and $w = (1-s)(I-P)N(v+w)$, then $\|w\| < r_1$. This is because, as already observed, the map $w \to (1-s)(I-P)N(v+w)$ is a contraction on $(I-P)H$, for sone fixed k. Define $h_1 : Y_k \times [0,1] \to L^2$ by

$$h_1(u,s) = (I-P)N(v+w) + PN(v+w+w(\theta(v)-w)),$$

where $v = Pu, w = (I-P)u$. We obtain

$$u + h_1(u,s) \ne -\frac{\phi_1}{\lambda_1} \quad \text{for} \quad (\mathrm{u},\mathrm{s}) \in \partial \mathrm{Y_k} \times [0,1].$$

There are two possibilities to consider in $u \in \partial Y_k$. One is that $u = v + w$ with $v \in \partial Y_k, \|w\| < r_2, s \in [0,1]$, and $u + h_1(u,s) = -\frac{\phi_1}{\lambda_1}$. In this case,

$$w + (I-P)N(v+w) = 0$$

and

$$v + PN(v+w+s(\theta(v)-w)) = -\frac{\phi_1}{\lambda_1}.$$

The first of these implies $w = \theta(v)$, and the second implies $v + PN(v + \theta(v)) = v + N(v) = -\frac{\phi_1}{\lambda_1}$, which contradicts the fact that $v \in \partial U_k$.

Now suppose $n \in U_k, w \in (I-P)H, \|w\| = r_2$. If $0 \le s \le 1$ and $u + h_1(u,s) = -\frac{\phi_1}{\lambda_1}$, then

$$w + (I-P)N(v+w) = 0,$$

So $w = \theta(v)$ and $\|w\| \le r_1 < r_2$, which is a contradiction. This shows that $u + h_1(u,s) \ne \frac{\phi_1}{\lambda_1}$ for all $(u,s) \in \partial Y_k \times [0,1]$, and it follows by homotopy invariance of degree that

$$d(I+N, Y_k, -\frac{\phi_1}{\lambda_1}) = d(I+h_1(\cdot,1), Y_k, -\frac{\phi_1}{\lambda_1}).$$

Now let $h_2 : Y_k \times [0,1] \to L^2(\Omega)$ be defined by

$$h_2(u,s) = (I-s)(I-P)N(u) + PN(v+\theta(v)), \quad v = Pu.$$

If $v \in \partial U_k, w \in (I-P)H, 0 \le s \le 1, u = v+w$ and $u + h_2(u,s) = \frac{\phi_1}{\lambda_1}$ then

$$v + T(v) = v + PN(v+\theta(v)) = P(u + h_2(u,s)) = -\frac{\phi_1}{\lambda_1},$$

which contradicts the fact that there are no solutions if $v \in \partial U_k$. If $u = v+w, v \in U_k, w \in (I-P)H, \|w\| = r_2$, then

$$0 = (I-P)(u + h_2(u,s)) = w + (I-s)(I-P)N(v+w),$$

which would imply that $\|w\| < r_2$, which is a contradiction. Therefore, $u + h_2(u,s) \ne \frac{\phi_1}{\lambda_1}$, for $(u,s) \in \partial Y_k \times [0,1]$. Since $h_1(u,1) = h_2(u,0)$, we infer by homotopy invariance that

$$d(I+N, Y_k, -\frac{\phi_1}{\lambda_1}) = d(I+h_2(\cdot,1), Y_k, -\frac{\phi_1}{\lambda_1}).$$

Let B be the open ball of radius r_2 in $(I-P)H$. If $u \in \overline{Y}_k, v = Pu, w = (I-P)u$, then $u + h_2(u,1) = v + PN(v+\theta(v)) + w$.

Thus we see that the map $u \to u + h_2(u,1)$ is uncoupled on $PH \oplus (I-P)H$ and is the identity on $(I-P)H$. Therefore by the product property of degree,

$$d(I+N, Y_k, -\frac{\phi_1}{\lambda_1}) = d(I+T, U_k, -\frac{\phi_1}{\lambda_1}) = (-1)^{K+1}.$$

This concludes the proof of Lemma □

Remark. What we have just proved can be put into an abstract context. Assume one has an operator equation

$$Au + N(u) = 0$$

on a Hilbert space H.Assume that there exists P, commuting with L, so that

$$H = PH \oplus (I-P)H, \quad u \in H, \quad v = Pu, \quad w = (I-P)u,$$

and $Lu + N(u) = 0$ is equivalent to

$$\begin{aligned} &i) \quad Lw + (I-P)N(v+w) = 0 \\ &ii) \quad Lv + PN(v+w) = 0. \end{aligned}$$

Assume that for fixed v, i) may be solved uniquely and continuously for $w = \theta(v)$ and that for bounded v, there exists an a priori bound for $\theta(v)$. Then

Lemma 2.5. *(The Prism Lemma) Given a bounded region $U \subseteq PH$ such that*

$$v + A^{-1}PN(v+\theta(v)) = 0$$

has no solution on ∂U, and $r > 0$ so that $v \in \overline{U}, Aw + (1-s)N(v+w) = 0, 0 \le s \le 1$, imply $\|w\| < r$, then if $Y = \{u : Pu \in U, \|(I-P)u\| \le r\}$, we have

$$d(w + L^{-1}PN(v+\theta(v)), U, 0) = d(u + L^{-1}N(u), Y, 0).$$

Finally, having proved Lemma 3, we are in a position to produce solutions to the semi linear problem

$$Au + f(u) = s\phi_1 + h_1(x)$$

instead of the piecewise linear one,

$$Au + bu^+ - au^- = \phi_1, \quad a < \lambda_1, \quad \lambda_1 < b < \lambda_2.$$

Then, as before, if $f_1(u) = bu^+ - au^-$, we have

$$f(\zeta) = f_1(\zeta) + f_0(\zeta) \quad \text{with} \quad \lim_{|\zeta|\to\infty} \frac{\mathrm{f}_0(\zeta)}{\zeta} = 0.$$

We rewrite $Au + f(u) = s\phi_1 + h_1(x)$ as

$$Az = f_1(z) + \frac{f_0(sz)}{s} = \phi_1(x) + \frac{h(x)}{s}.$$

Let

$$N_s(z) = A^{-1}(f_1(z) + \frac{f_0(sz)}{s} - \frac{h}{s})$$

and let

$$N(z) = A^{-1}(f_1(z)).$$

Then it is easy to verify that

$$lim_{s\to\infty}\|N(z) - N_s(z)\| = 0$$

uniformly for z in bounded subsets of $L^2(\Omega)$.

Finally, we have everything in place to prove the main result of this lecture.

THEOREM 2.1. *Assume that* $lim_{u\to\infty}\frac{f(u)}{u} = \beta$, $lim_{u\to-\infty}\frac{f(u)}{u} = \alpha$, *and* $\alpha < \lambda_1 < \lambda_2 < \beta < \lambda_3$. *Let* $s > 0$ *and* $\|h\| be finite$. *Then there exists* s_0 *so that* $s \geq s_0$ *implies that*

$$Au + f(u) = s\phi_1(x) + h(x)$$

has at least four solutions.

Proof. We have established that

$$z + N(z) = -\frac{\phi_1}{\lambda_1} \quad \text{forall} \quad \mathrm{z} \in \partial\mathrm{Y_k}, \quad 1 \leq \mathrm{k} \leq 4.$$

Since ∂Y_k is closed and bounded, and N is continuous and compact, there exists $\eta > 0$ such that

$$\|z + N(x) + \frac{\phi_1}{\lambda_1}\| \geq \eta \quad \text{if} \quad \eta \in \partial\mathrm{Y_k}.$$

Now choose s_0 so that

$$\|N_s(z) - N(z)\| < \frac{\eta}{2} \quad \text{for} \quad \mathrm{z} \in \partial\mathrm{Y_k}, \quad 1 \leq \mathrm{k} \leq 4.$$

Then

$$\|z + N(z) + (1-\lambda)(N_s(z) - N(z)) + \frac{\phi_1}{\lambda_1}) \geq \frac{\eta}{2}$$

for $0 \leq \lambda \leq 1$, from which we conclude

$$d(I + N_s, Y_s, -\frac{\phi_1}{\lambda_1}) = d(I + N, Y_k, -\frac{\phi_1}{\lambda_1}) = (-1)^{k+1}, \quad 1 \leq k \leq 4.$$

This proved the theorem. since we have at least one solution in $Y_k, 1 \leq k \leq 4$. □

3. The nonlinear elliptic system

In this section we investigate the existence of solutions (ξ, η) for the nonlinear elliptic system with Dirichlet boundary condition

$$\begin{aligned} &A\xi + g_1(\xi + 2\eta) = s\phi_1 + h \quad \text{in} \quad \Omega, \\ &A\eta + g_2(\xi + 2\eta) = s\phi_1 + h \quad \text{in} \quad \Omega, \\ &\xi = 0, \qquad \eta = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.1}$$

where $lim_{u\to\infty}\frac{g_j(u)}{u} = \beta_j$, $lim_{u\to-\infty}\frac{g_j(u)}{u} = \alpha_j$ are finite. Here A denote the differential operator $A = \sum_{1\le i,j\le n}\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j})$ with $a_{ij} = a_{ji} \in C^\infty(\bar{\Omega})$.

We suppose that the nonlinearity $g_1 + 2g_2$ crosses eigenvalues of A.

LEMMA 3.1. *(cf. [11])Assume that* $lim_{u\to\infty}\frac{f(u)}{u} = \beta$, $lim_{u\to-\infty}\frac{f(u)}{u} = \alpha$, *and* $\alpha < \lambda_1 < \beta < \lambda_2$. *Let* $s > 0$ *and* $\|h\|$ *be finite. Then there exists* s_0 *so that* $s \ge s_0$ *implies that*

$$Au + f(u) = s\phi_1(x) + h(x)$$

has at least two solutions.

THEOREM 3.1. *Assume that* g_1, g_2 *satisfy*
(i) $\frac{g_1}{g_2} = \gamma \ne 0$ *and* $2 + \gamma \ne 0$
(ii) $\alpha_1 + 2\alpha_2 < \lambda_1 < \beta_1 + 2\beta_2 < \lambda_2$.
Let $s > 0$ *and* $\|h\|$ *be finite. Then there exists* s_0 *so that* $s \ge s_0$ *implies that system (3.1) has at least two solutions.*

Proof. From problem (3.1) we get that $A\xi - (s\phi_1 + h) = \gamma(A\eta - (s\phi_1 + h))$. For any $F \in H_0$ the elliptic problem

$$\begin{aligned} &Lu = F \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{3.2}$$

has a unique solution. If u_1 is a solution of $Au = (1-\gamma)(s\phi_1 + h)$, then the solution (ξ, η) of problem (3.1) satisfies

$$\xi - \gamma\eta = u_1. \tag{A}$$

On the other hand, from problem (3.1) we get the equation

$$\begin{aligned} &A(\xi + 2\eta) + g_1(\xi + 2\eta) + 2g_2(\xi + 2\eta) = 3(s\phi_1 + h) \quad \text{in} \quad \Omega, \\ &\xi = 0, \quad \eta = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.3}$$

Put $w = \xi + 2\eta$. Then the above equation is equivalent to

$$\begin{aligned} &Aw + g_1 w + 2g_2 w = 3s\phi_1 + 3h \quad \text{in} \quad \Omega, \\ &w = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.4}$$

Since $\alpha_1 + 2\alpha_2 < \lambda_1 < \beta_1 + 2\beta_2 < \lambda_2$, by Lemma 3.1 the above equation has at least two solutions, say w_1, w_2. Hence we get the solutions (ξ, η)

of problem (3.1) from the following systems:

$$\xi - \gamma\eta = u_1, \qquad (3.5)$$
$$\xi + 2\eta = w_1,$$

$$\xi - \gamma\eta = u_1, \qquad (3.6)$$
$$\xi + 2\eta = w_2.$$

Since $2+\gamma \neq 0$, the above systems (3.5) have unique solutions. Therefore there exists s_0 so that $s \geq s_0$ implies that system (3.1) has at least two solutions. □

THEOREM 3.2. *Assume that g_1, g_2 satisfy*
(i) $\frac{g_1}{g_2} = \gamma \neq 0$ and $2 + \gamma \neq 0$
(ii) $\alpha_1 + 2\alpha_2 < \lambda_1 < \lambda_2 < \beta_1 + 2\beta_2 < \lambda_3$.
Let $s > 0$ and $\|h\|$ be finite. Then there exists s_0 so that $s \geq s_0$ implies that system (3.1) has at least four solutions.

Proof. (From problem (3.1) we get that $A\xi - (s\phi_1 + h) = \gamma(A\eta - (s\phi_1 + h))$. For any $F \in H_0$ the elliptic problem

$$Lu = F \quad \text{in} \quad \Omega, \qquad (3.7)$$
$$u = 0 \quad \text{on} \quad \partial\Omega$$

has a unique solution. If u_1 is a solution of $Au = (1 - \gamma)(s\phi_1 + h)$, then the solution (ξ, η) of problem (3.1) satisfies

$$\xi - \gamma\eta = u_1. \qquad (A)$$

On the other hand, from problem (3.1) we get the equation

$$A(\xi + 2\eta) + g_1(\xi + 2\eta) + 2g_2(\xi + 2\eta) = 3(s\phi_1 + h) \quad \text{in} \quad \Omega, \qquad (3.8)$$
$$\xi = 0, \quad \eta = 0 \quad \text{on} \quad \partial\Omega.$$

Put $w = \xi + 2\eta$. Then the above equation is equivalent to

$$Aw + g_1 w + 2g_2 w = 3s\phi_1 + 3h \quad \text{in} \quad \Omega, \qquad (3.9)$$
$$w = 0 \quad \text{on} \quad \partial\Omega.$$

Since $\alpha_1 + 2\alpha_2 < \lambda_1 < \beta_1 + 2\beta_2 < \lambda_2$, by Lemma 3.1 the above equation has at least two solutions, say w_1, w_2, w_3, w_4. Hence we get the solutions (ξ, η) of problem (3.1) from the following systems:

$$\xi - \gamma\eta = u_1, \qquad (3.10)$$
$$\xi + 2\eta = w_1,$$

$$\begin{aligned} \xi - \gamma\eta &= u_1, \\ \xi + 2\eta &= w_2, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \xi - \gamma\eta &= u_1, \\ \xi + 2\eta &= w_3, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \xi - \gamma\eta &= u_1, \\ \xi + 2\eta &= w_4. \end{aligned} \tag{3.13}$$

Since $2+\gamma \neq 0$, the above systems (3.5) have unique solutions. Therefore there exists s_0 so that $s \geq s_0$ implies that system (3.1) has at least four solutions. □

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