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WEAK CONVERGENCE OF AN ITERATIVE METHOD FOR EQUILIBRIUM PROBLEMS AND RELATIVELY NONEXPANSIVE MAPPINGS

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Abstract. The purpose of this paper is to consider an iterative method for an equilibrium problem and a family relatively nonexpansive mappings. Weak convergence theorems are established in uniformly smooth and uniformly convex Banach spaces.

1. Introduction

Let E be a real Banach space, E^* the dual space of E and C a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. The equilibrium problem is to find $p \in C$ such that

$$f(p, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, we use $EP(f)$ to denote the solution set of the equilibrium problem (1.1). That is,

$$EP(f) = \{p \in C : f(p, y) \geq 0, \quad \forall y \in C\}$$

Given a mapping $S : C \to E^*$, let $f(x, y) = \langle Sx, y - x\rangle$, $x, y \in C$. Then $p \in EP(f)$ if and only if

$$\langle Sp, y - p\rangle \geq 0, \quad \forall y \in C.$$

That is, p is a solution of the above variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (1.1); see [5, 9-11, 15].

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Recall that a mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

A point $x \in C$ is a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T. We use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

The topic on common fixed points of a family nonexpansive mapping is hot, see, for example [7, 8, 11, 16 - 18] and the references therein. Finding an optimal point in the intersection of the fixed point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. The problem of finding an optimal point that minimizes a given cost function over common fixed point set of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance; see, for example, [23] and the reference therein.

For equilibrium problem, many authors considered the problem of finding a common element of the fixed point set of nonexpansive mappings and solution set of the equilibrium problem (1.1) based on iterative methods. Weak and strong convergence theorems are established in Hilbert spaces and Banach spaces; see, for example, [8, 16, 20] and the references therein. Relatively nonexpansive mappings as a important generalization of nonexpansive mappings have been studied by many authors; see [13, 14, 16, 20]. In this paper, we consider an iterative method which was introduced by Takahashi and Zembayashi [20] for the equilibrium problem (1.1) and common fixed point problem of a family of relatively nonexpansive mappings. A weak convergence theorem is established in uniformly smooth and uniformly convex Banach spaces.

2. Preliminaries

Let E be a Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that if E is uniformly smooth if and only if E^* is uniformly convex.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \text{for } x, y \in E. \tag{2.2}$$

Observe that, in a Hilbert space H, (2.2) is reduced to $\phi(x,y) = \|x-y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y\in C}\phi(y,x)$$

existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, for example, [1,2,6,21]). In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that

$$(\|y\| - \|x\|)^2 \le \phi(y,x) \le (\|y\| + \|x\|)^2, \quad \forall x, y \in E. \tag{2.3}$$

Remark 2.1. *If E is a reflexive, strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$.*

This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. *From the definition of* J, *one has* $Jx = Jy$. *Therefore, we have* $x = y$; *see [6, 21] for more details.*

Let C be a nonempty closed convex subset of E and T a mapping from C into itself. A point p in C is said to be an asymptotic fixed point of T [19] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is said to be relatively nonexpansive [3,4] if $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mappings was studied in [3, 4].

For the equilibrium problem (1.1), Let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0, \forall x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0, \forall x, y \in C$;

(A3)

$$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y), \forall x, y, z \in C;$$

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Recently, Takahashi and Zembayshi [20] introduced an iterative method for finding a common element in the fixed point set of a relatively nonexpansive mapping and in the solution set of the equilibrium problem (1.1). weak convergence theorems are established in a Banach space. To be more precise, they obtained the following result.

THEOREM 2.1 (Thoerem TZ). *Let* E *be a uniformly smooth and uniformly convex Banach space, and let* C *be a nonempty closed convex subset of* E. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* $(A1)$-$(A4)$ *and let* T *be a relatively nonexpansive mapping from* C *into itself such that* $F(T) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by* $u_1 \in E$,

(2.4)
$$\begin{cases} x_n \in C \textit{ such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n) \end{cases}$$

for every $n \geq 1$, *where* J *is the duality mapping on* E, $\{\alpha_n\} \subset [0, 1]$ *satisfies* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ *and* $\{r_n\} \subset [a, \infty)$ *for some* $a > 0$. *If* J *is weakly sequentially continuous, then* $\{x_n\}$ *converges weakly to* $z \in F(T) \cap EP(f)$, *where* $z = \lim_{n\to\infty} \Pi_{F(T)\cap EP(f)} x_n$.

In order to our main results, we need the following lemmas.

LEMMA 2.2. *([2]) Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \forall y \in C.$$

LEMMA 2.3. *([14]) Let E be a strictly convex and smooth Banach space, C a nonempty closed convex subset of E and $T : C \to C$ a relatively nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

LEMMA 2.4. *([12]) Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and $g(\|x-y\|) \leq \phi(x, y)$ for all $x, y \in B_r$.*

LEMMA 2.5. *([5]) Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

LEMMA 2.6. *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Define a mapping $T_r : E \to C$ by*

$$T_r x = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle, \quad \forall y \in C\}.$$

Then the following conclusions hold:

(1) *T_r is single-valued;*
(2) *T_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle$$

(3) *$F(T_r) = EP(f)$;*
(4) *T_r is relatively nonexpansive;*
(5)

$$\phi(q, S_r x) + \phi(S_r x, x) \leq \phi(q, x), \quad \forall q \in F(T_r);$$

(6) *$EP(f)$ is closed and convex.*

LEMMA 2.7. *([22]) Let $p > 1$ and $s > 0$ be two fixed real numbers. Then a Banach space E is uniformly convex if and only if there exists a continuous strictly increasing convex functiong $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p - w_p(\lambda)g(\|x-y\|)$$

for all $x, y \in B_s(0) = \{x \in E : \|x\| \le s\}$ and $\lambda \in [0,1]$, where $w_p(\lambda) = \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

LEMMA 2.8. *Let E be a uniformly convex Banach space, $s > 0$ a positive number and $B_s(0)$ a closed ball of E. There exits a continuous, strictly increasing and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\sum_{i=1}^{N} \alpha_i x_i\|^2 \le \sum_{i=1}^{N} \alpha_i \|x_i\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|)$$

for all $x_1, x_2, \ldots, x_N \in B_s(0) = \{x \in E : \|x\| \le s\}$ and $\alpha_1, \alpha_2, \ldots, \alpha_N \in [0,1]$ such that $\sum_{i=1}^{N} \alpha_i = 1$.

Proof. We prove it by inductions. For $N = 2$, we from Lemma 2.7 see that Lemma 2.8 holds. For $N = j$, where $j \ge 2$ is some integer, suppose that Lemma 2.8 holds. We see that Lemma 2.7 still holds for $N = j+1$. Indeed, from Lemma 2.7, we see that

$$\begin{aligned}
&\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_j x_j + \alpha_{j+1} x_{j+1}\|^2 \\
&= \|(1-\alpha_{j+1})\Big(\frac{\alpha_1}{1-\alpha_{j+1}}x_1 + \frac{\alpha_2}{1-\alpha_{j+1}}x_2 + \cdots + \frac{\alpha_j}{1-\alpha_{j+1}}x_j\Big) + \alpha_{j+1}x_{j+1}\|^2 \\
&\le (1-\alpha_{j+1})\|\frac{\alpha_1}{1-\alpha_{j+1}}x_1 + \frac{\alpha_2}{1-\alpha_{j+1}}x_2 + \cdots + \frac{\alpha_j}{1-\alpha_{j+1}}x_j\|^2 + \alpha_{j+1}\|x_{j+1}\|^2 \\
&\alpha_j(1-\alpha_{j+1})g\Big(\|\big(\frac{\alpha_1}{1-\alpha_{j+1}}x_1 + \frac{\alpha_2}{1-\alpha_{j+1}}x_2 + \cdots + \frac{\alpha_j}{1-\alpha_{j+1}}x_j\big) - x_{j+1}\|\Big) \\
&\le (1-\alpha_{j+1})\Big(\frac{\alpha_1}{1-\alpha_{j+1}}\|x_1\|^2 + \frac{\alpha_2}{1-\alpha_{j+1}}\|x_2\|^2 + \cdots + \frac{\alpha_j}{1-\alpha_{j+1}}\|x_j\|^2 \\
&- \frac{\alpha_1\alpha_2}{(1-\alpha_{j+1})(1-\alpha_{j+1})}g(\|x_1 - x_2\|)\Big) + \alpha_{j+1}\|x_{j+1}\|^2 \\
&= \alpha_1\|x_1\|^2 + \alpha_2\|x_2\|^2 + \cdots + \alpha_j\|x_j\|^2 + \alpha_{j+1}\|x_{j+1}\|^2 - \frac{\alpha_1\alpha_2}{1-\alpha_{j+1}}g(\|x_1 - x_2\|) \\
&\le \alpha_1\|x_1\|^2 + \alpha_2\|x_2\|^2 + \cdots + \alpha_j\|x_j\|^2 + \alpha_{j+1}\|x_{j+1}\|^2 - \alpha_1\alpha_2 g(\|x_1 - x_2\|).
\end{aligned}$$

This completes the proof. □

3. Main results

THEOREM 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, C a nonempty closed convex subset of E and f a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T_i : C \to C$ be a relatively nonexpansive mappings for every $i \in \{1, 2, \dots, N\}$. Assume that $\mathcal{F} = \cap_{i=1}^{N} F(T_i) \cap EP(f)$ is nonempty. Let $\{r_n\}$ be a sequence in $[a, \infty)$ for some $a > 0$ and $\{\alpha_{n,0}\}$, $\{\alpha_{n,1}\}, \dots,$ and $\{\alpha_{n,N}\}$ are sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated by $u_1 \in E$,*

$$\begin{cases} x_n \in C \text{ such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}\big(\alpha_{n,0}Jx_n + \sum_{i=1}^{N} \alpha_{n,i}JT_ix_n\big) \end{cases}$$

for every $n \geq 1$, where J is the duality mapping on E. Assume that the control sequences satisfy the following restrictions:

(a) $\alpha_{n,0} + \alpha_{n,1} + \cdots + \alpha_{n,N} = 1$, $\forall n \geq 0$;
(b) $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$, $\forall i = 1, 2, \dots, N$.

If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in \mathcal{F}$, where $z = \lim_{n\to\infty} \Pi_{\mathcal{F}}x_n$.

Proof. First, we show that the sequence $\{x_n\}$ is bounded. Fixing $p \in \mathcal{F}$, we see that

$$\begin{aligned} &\phi(p, x_{n+1}) = \phi(p, T_{r_n}u_{n+1}) \leq \phi(p, u_{n+1}) \\ &= \phi\Big(p, J^{-1}\big(\alpha_{n,0}Jx_n + \sum_{i=1}^{N}(\alpha_{n,i}JT_ix_n)\big)\Big) \\ &= \|p\|^2 - 2\langle p, \alpha_{n,0}Jx_n + \sum_{i=1}^{N}(\alpha_{n,i}JT_ix_n)\rangle + \|\alpha_{n,0}Jx_n + \sum_{i=1}^{N}(\alpha_{n,i}JT_ix_n)\|^2 \\ &\leq \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{N}\big(\alpha_{n,i}\langle p, JT_ix_n\rangle\big) + \alpha_{n,0}\|x_n\|^2 \\ &\qquad + \sum_{i=1}^{N}\big(\alpha_{n,i}\|T_ix_n\|^2\big) \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&= \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{N}\big(\alpha_{n,i}\langle p, JT_ix_n\rangle\big) + \alpha_{n,0}\|x_n\|^2 \\
&\qquad\qquad + \sum_{i=1}^{N}\big(\alpha_{n,i}\|T_ix_n\|^2\big) \\
&= \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N}\big(\alpha_{n,i}\phi(p, T_ix_n)\big) \\
&\le \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N}\big(\alpha_{n,i}\phi(p, x_n)\big) = \phi(p, x_n).
\end{aligned}
$$

It follows that $\lim_{n\to\infty}\phi(p, x_n)$ exists. It follows from (2.3) that $\{x_n\}$ is bounded. Let $r = \sup_{n\ge1}\{\|x_n\|, \|T_1x_n\|, \ldots, \|T_Nx_n\|\}$. From Lemma 2.8, we see that

(3.2)

$$
\begin{aligned}
&\phi(p, x_{n+1}) = \phi(p, T_{r_n}u_{n+1}) \le \phi(p, u_{n+1}) \\
&= \phi\Big(p, J^{-1}\big(\alpha_{n,0}Jx_n + \sum_{i=1}^{N}(\alpha_{n,i}JT_ix_n)\big)\Big) \\
&= \|p\|^2 - 2\langle p, \alpha_{n,0}Jx_n + \sum_{i=1}^{N}(\alpha_{n,i}JT_ix_n)\rangle + \|\alpha_{n,0}Jx_n + \sum_{i=1}^{N}(\alpha_{n,i}JT_ix_n)\|^2 \\
&\le \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{N}\big(\alpha_{n,i}\langle p, JT_ix_n\rangle\big) + \alpha_{n,0}\|Jx_n\|^2 \\
&\quad + \sum_{i=1}^{N}\big(\alpha_{n,i}\|JT_ix_n\|^2\big) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|) \\
&= \|p\|^2 - 2\alpha_{n,0}\langle p, Jx_n\rangle - 2\sum_{i=1}^{N}\big(\alpha_{n,i}\langle p, JT_ix_n\rangle\big) + \alpha_{n,0}\|x_n\|^2 \\
&\quad - + \sum_{i=1}^{N}\big(\alpha_{n,i}\|T_ix_n\|^2\big)\alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|) \\
&= \alpha_{n,0}\phi(p, x_n) + \sum_{i=1}^{N}\big(\alpha_{n,i}\phi(p, T_ix_n)\big) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|)
\end{aligned}
$$

$$\le \alpha_{n,0}\phi(p,x_n) + \sum_{i=1}^{N} \big(\alpha_{n,i}\phi(p,x_n)\big) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|)$$
$$= \phi(p,x_n) - \alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|).$$

It follows that

$$\alpha_{n,0}\alpha_{n,1}g(\|Jx_n - JT_1x_n\|) \le \phi(p,x_n) - \phi(p,x_{n+1}). \tag{3.3}$$

This implies that $\lim_{n\to\infty} g(\|Jx_n - JT_1x_n\|) = 0$. From the property of g, we can obtain that

$$\lim_{n\to\infty} \|Jx_n - JT_1x_n\| = 0 \tag{3.4}$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain that

$$\lim_{n\to\infty} \|x_n - T_1x_n\| = 0. \tag{3.5}$$

Repeating (3.2)-(3.5), we can obtain that

$$\lim_{n\to\infty} \|x_n - T_ix_n\| = 0, \quad \forall i \in \{1,2,\ldots,N\}. \tag{3.6}$$

Since $\{x_n\}$ is bounded, we may, without loss of generality, assume that subsequence $\{x_n\}$ converges weakly to $p \in C$. In view of definition of relatively nonexpansive mappings, we see that $p \in \cap_{i=1}^N \widetilde{F}(T_i) = \cap_{i=1}^N F(T_i)$.

Next, we show that $p \in EP(f)$. Let $s = \sup_{n\ge1}\{\|x_n\|, \|u_n\|\}$. From Lemma 2.6, we see that there exists a continuous, strictly increasing and convex function g_1 with $g_1(0) = 0$ such that

$$g_1(\|x-y\|) \le \phi(x,y)$$

for all $x, y \in B_s$. Putting $x_n = T_{r_n}u_n$ and from Lemma 2.4, and (3.1), we have

$$g_1(\|x_n - u_n\|) \le \phi(x_n,u_n) \le \phi(q,u_n) - \phi(q,x_n) \le \phi(q,x_{n-1}) - \phi(q,x_n),$$

where $q \in \mathcal{F}$. Noticing that $\lim_{n\to\infty} \phi(p,x_n)$ exists, we have

$$\lim_{n\to\infty} g_1(\|x_n - u_n\|) = 0.$$

It follows from the property of g_1 that $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain that

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.7}$$

From the assumption $r_n \geq a$, we see that

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0. \tag{3.8}$$

By virtue of $x_n = T_{r_n} u_n$, we obtain that

$$f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C.$$

from the (A2), we see that

$$\begin{aligned} \|y - x_n\| \frac{\|Jx_n - Ju_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \\ &\geq -f(x_n, y) \\ &\geq f(y, x_n), \quad \forall y \in C. \end{aligned}$$

Letting $n \to \infty$ in above inequality and from (A4), we have

$$f(y, p) \leq 0, \quad \forall y \in C.$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)p$. Noticing that $y, p \in C$, we obtain that $y_t \in C$, which yields that $f(y_t, p) \leq 0$. It follows from (A1) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

That is,

$$f(y_t, y) \geq 0.$$

Let $t \downarrow 0$, from (A3), we obtain that $f(p, y) \geq 0$, for $\forall y \in C$. This implies that $p \in EP(f)$. This shows that $p \in \mathcal{F}$.

Define $y_n = \Pi_{\mathcal{F}} x_n$ for all $n \geq 1$. From (3.1), we see that

$$\phi(y_n, x_{n+1}) \leq \phi(y_n, x_n). \tag{3.9}$$

It follows from Lemma 2.3 that

$$\begin{aligned} \phi(y_{n+1}, x_{n+1}) &= \phi(\Pi_F x_{n+1}, x_{n+1}) \\ &\leq \phi(y_n, x_{n+1}) - \phi(y_n, \Pi_F x_{n+1}) \\ &= \phi(y_n, x_{n+1}) - \phi(y_n, y_{n+1}) \\ &\leq \phi(y_n, x_{n+1}). \end{aligned}$$

From (3.9), we have

$$\phi(y_{n+1}, x_{n+1}) \leq \phi(y_n, x_n),$$

which yields that $\{\phi(y_n, x_n)\}$ is a convergence sequence. It also follows from (3.9) that

$$\phi(y_n, x_{n+m}) \leq \phi(y_n, x_n), \quad \forall m \geq 1. \tag{3.10}$$

From $y_{n+m} = \Pi_F x_{n+m}$ and Lemma 2.6, we have

$$\phi(y_n, y_{n+m}) + \phi(y_{n+m}, x_{n+m}) \leq \phi(y_n, x_{n+m}) \leq \phi(y_n, x_n),$$

which yields that

$$\phi(y_n, y_{n+m}) \leq \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m}).$$

Let $r = \sup_{n\geq 1}\{\|y_n\|\}$. Noticing that Lemma 2.4, we see that there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$ such that $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$. Therefore, we obtain that

$$g(\|y_n - y_{n+m}\|) \leq \phi(y_n, y_{n+m}) \leq \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m}).$$

Since $\{\phi(y_n, x_n)\}$ is a convergent sequence, from the property of g, we obtain that $\{y_n\}$ is a Cauchy sequence. Since $\mathcal{F}$ is closed, we see that $\{y_n\}$ converges strongly to $z \in \mathcal{F}$.

On the other hand, noticing that $p \in \mathcal{F}$, $y_n = \Pi_{\mathcal{F}} x_n$ and Lemma 2.2, we have

$$\langle y_n - p, Jx_n - Jy_n\rangle \geq 0.$$

Since J is weakly sequentially continuous, letting $n \to \infty$, we obtain that

$$\langle z - p, Jp - Jz\rangle \geq 0.$$

From the monotonicity of J, we see that

$$\langle z - p, Jp - Jz\rangle \leq 0.$$

Noticing that E is uniformly convex, we have $z = p$. This completes the proof. □

Remark 2.2. If $T_i = T$ for each $i \in \{1, 2, \ldots, N\}$, then Theorem 3.1 is reduced to Theorem TZ.

For the special case, letting $N = 2$, we have the following results.

Corollary 2.3. Let E be a uniformly smooth and uniformly convex Banach space, C a nonempty closed convex subset of E and f a bi-function from $C \times C$ to $\mathbb{R}$ satisfying $(A1)$-$(A4)$. Let $T : C \to C$ and $S : C \to C$ be two relatively nonexpansive mappings such that $F(T) \cap F(S)$ is nonempty. Let $\{r_n\}$ be a sequence in $[a, \infty)$ for some

$a > 0$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$. Let $\{x_n\}$ be a sequence generated by $u_1 \in E$,

$$\begin{cases} x_n \in C \textit{ such that } f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}\big(\alpha_{n,0}Jx_n + \beta_n JTx_n + \gamma_n JSx_n\big) \end{cases}$$

for every $n \geq 1$, where J is the duality mapping on E. Assume that the control sequences satisfy the following restrictions:

(a) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 0$;
(b) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$ and $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$

If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $z \in F(T) \cap F(S)$, where $z = \lim_{n\to\infty} \Pi_{F(T)\cap F(S)}x_n$.

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