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A STUDY ON THE RECURRENCE RELATIONS AND VECTORS X_λ, S_λ AND U_λ IN $g - ESX_n$

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Abstract. The manifold $g - ESX_n$ is a generalized n-dimensional Riemannian manifold on which the differential geometric structure is imposed by the unified field tensor $g_{\lambda\mu}$ through the ES-connection which is both Einstein and semi-symmetric. In this paper, we investigate the properties of the vectors X_λ, S_λ and U_λ of $g - ESX_n$, with main emphasis on the derivation of several useful generalized identities involving it.

1. Introduction

Manifolds with recurrent connections have been studied by many authors, such as Chung, Datta, E.M. Patterson, M.Pravanovitch, Singal, and Takano, etc(refer to [3] and [4]). Examples of such manifolds are those of recurrent curvature, Ricci-recurrent manifolds, and bi-recurrent manifolds.

In this paper, we introduce a new concept of semi-symmetric connection ${\Gamma_\lambda}^\nu{}_\mu$ on a generalized n-dimensional Riemannian manifold X_n and study its recurrence relations in the first. In the second, we investigate the properties of the vectors X_λ, S_λ and U_λ of $g - ESX_n$.

The main purpose of the present paper is to obtain several basic identities satisfied by the vectors X_λ, S_λ and U_λ and recurrence relations in $g - ESX_n$ which is both semi-symmetric and Einstein .

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2. Preliminaries

This section is a brief collection of basic concepts, results, and notations needed in subsequent considerations. They are due to Chung ([3], 1963), Hwang ([2], 1988), and Mishra([7], 1959) mostly due to [6].

(a) generalized n-dimensional Riemannian manifold X_n

Let X_n be a generalized n-dimensional Riemannian manifold referred to a real coordinate system x^ν, which obeys the coordinate transformations $x^\nu \to x^{\nu'}$ for which

$$det(\frac{\partial x'}{\partial x}) \neq 0 \tag{2.1}$$

In $n-g-UFT$ the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \qquad \text{where} \tag{2.2a}$$

$$\mathfrak{g} = det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = det(k_{\lambda\mu}) \tag{2.2b}$$

In virtue of $(2.2b)$ we may define a unique tensor $h^{\lambda\nu}$ by

$$h_{\lambda\mu}h^{\lambda\nu} = \delta_\mu^\nu \tag{2.3}$$

which together with $h_{\lambda\mu}$ will serve for raising and/or lowering indices of tensors in X_n in the usual manner. There exists a unique tensor $*g^{\lambda\nu}$ satisfying

$$g_{\lambda\mu}{}^*g^{\lambda\nu} = g_{\mu\lambda}{}^*g^{\nu\lambda} = \delta_\mu^\nu \tag{2.4}$$

It may be also decomposed into its symmetric part ${}^*h_{\lambda\mu}$ and skew-symmetric part ${}^*k_{\lambda\mu}$:

$${}^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu} \tag{2.5}$$

The manifold X_n is connected by a general real connection $\Gamma_\lambda{}^\nu{}_\mu$ with the following transformation rule:

$$\Gamma_{\lambda'}{}^{\nu'}{}_{\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\lambda'}}\frac{\partial x^\gamma}{\partial x^{\mu'}}\Gamma_\beta{}^\alpha{}_\gamma + \frac{\partial^2 x^\alpha}{\partial x^{\lambda'}\partial x^{\mu'}}\right) \tag{2.6}$$

It may also be decomposed into its symmetric part $\Lambda_\lambda{}^\nu{}_\mu$ and its skew-symmetric part $S_{\lambda\nu}{}^\nu$, called the torsion of $\Gamma_\lambda{}^\nu{}_\mu$:

$$\Gamma_\lambda{}^\nu{}_\mu = \Lambda_\lambda{}^\nu{}_\mu + S_{\lambda\mu}{}^\nu; \quad \Lambda_\lambda{}^\nu{}_\mu = \Gamma_{(\lambda}{}^\nu{}_{\mu)}; \quad S_{\lambda\mu}{}^\nu = \Gamma_{[\lambda}{}^\nu{}_{\mu]} \tag{2.7}$$

A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be Einstein if it satisfies the following system of Einstein's equations:

(2.8a) $\partial_\omega g_{\lambda\mu} - \Gamma_\lambda{}^\alpha{}_\omega g_{\alpha\mu} - \Gamma_\omega{}^\alpha{}_\mu g_{\lambda\alpha} = 0,$ or equivalently

(2.8b) $D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^\alpha g_{\lambda\alpha}$

where D_ω is the symbolic vector of the covariant derivative with respect to $\Gamma_\lambda{}^\nu{}_\mu$. In order to obtain $g_{\lambda\mu}$ involved in the solution for $\Gamma_\lambda{}^\nu{}_\mu$ in (2.8), certain conditions are imposed. These conditions may be condensed to

$$S_\lambda = S_{\lambda\alpha}{}^\alpha = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu} Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0 \tag{2.9}$$

where Y_λ is an arbitrary vector, and

$$R_{\omega\mu\lambda}{}^\nu = 2(\partial_{[\mu}\Gamma_{|\lambda|}{}^\nu{}_{\omega]} + \Gamma_\alpha{}^\nu{}_{[\mu}\Gamma_{|\lambda|}{}^\alpha{}_{\omega]}), \qquad R_{\mu\lambda} = R_{\alpha\mu\lambda}{}^\alpha \tag{2.10}$$

If the system (2.8) admits a solution $\Gamma_\lambda{}^\nu{}_\mu$, it must be of the form (Hlavatý, 1957)

$$\Gamma_\lambda{}^\nu{}_\mu = \left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\} + S_{\lambda\mu}{}^\nu + U^\nu{}_{\lambda\mu} \tag{2.11}$$

where $U^\nu{}_{\lambda\mu} = 2h^{\nu\alpha} S_{\alpha(\lambda}{}^\beta k_{\mu)\beta}$ and $\left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\}$ are Christoffel symbols defined by $h_{\lambda\mu}$

(b) Some notations and results The following quantities are frequently used in our further considerations:

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}} \tag{2.12}$$

$$K_p = k_{[\alpha_1}{}^{\alpha_1} k_{\alpha_2}{}^{\alpha_2} \cdots k_{\alpha_p]}{}^{\alpha^p}, \quad (p = 0, 1, 2, \cdots) \tag{2.13}$$

$$^{(0)}k_\lambda{}^\nu = \delta_\lambda^\nu, \quad ^{(p)}k_\lambda{}^\nu = k_\lambda{}^\alpha \; ^{(p-1)}k_\alpha{}^\nu \quad (p = 1, 2, \cdots) \tag{2.14}$$

In X_n it was proved in [3] that

(2.15)

$$K_0 = 1, \quad K_n = k \quad \text{if} \quad n \quad \text{is even}, \quad \text{and} \quad K_p = 0 \quad \text{if} \quad p \quad \text{is odd}$$

$$\mathfrak{g} = \mathfrak{h}(1 + K_1 + K_2 + \cdots + K_n) \quad \text{or} \quad g = 1 + K_1 + K_2 + \cdots + K_n \tag{2.16}$$

$$\sum_{s=0}^{n-\sigma} K_s \ {}^{(n-s+p)}k_\lambda{}^\nu = 0 \qquad (p = 01, 2, \cdots) \tag{2.17}$$

We also use the following useful abbreviations for an arbitrary vector Y, for $p = 1, 2, 3, \cdots$:

$$^{(p)}Y_\lambda =^{(p-1)} k_\lambda{}^\alpha Y_\alpha \tag{2.18}$$

$$^{(p)}Y^\nu =^{(p-1)} k^\nu{}_\alpha Y^\alpha \tag{2.19}$$

(c) n-dimensional ES manifold ESX_n

In this subsection, we display an useful representation of the ES-connection in n-g-UFT.

Definition 2.1. *A connection $\Gamma_\lambda{}^\nu{}_\mu$ is said to be semi-symmetric if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form*

$$S_{\lambda\mu}{}^\nu = 2\delta^\nu_{[\lambda} X_{\mu]} \tag{2.20}$$

for an arbitrary non-null vector X_μ. A connection which is both semi-symmetric and Einstein is called a ES-connection. An n-dimensional generalized Riemannian manifold X_n, on which the differential geometric structure is imposed by $g_{\lambda\mu}$ by means of a ES-connection, is called an n-dimensional ES-manifold. We denote this manifold by $g - ESX_n$ in our further considerations.

Theorem 2.2. *Under the condition* (2.20), *the system of equations* (2.8) *is equivalent to*

$$\Gamma_\lambda{}^\nu{}_\mu = \left\{ \begin{matrix} \nu \\ \lambda\mu \end{matrix} \right\} + 2k^\nu_{(\lambda} X_{\mu)} + 2\delta^\nu_{[\lambda} X_{\mu]} \tag{2.21}$$

Proof. Substituting (2.20) for $S_{\lambda\mu}{}^\nu$ into (2.11), we have the representation (2.21). □

3. Properties of the vectors X_λ, S_λ and U_λ

This section is concerned with identities satisfied by the vectors X_λ, given by (2.19) and the vectors S_λ in (2.7) and

$$U_\lambda = U^\alpha{}_{\lambda\alpha} \tag{3.1}$$

THEOREM 3.1. *In $g-ESX_n$ under present conditions, the following recurrence relation hold:*

$$\sum_{s=0}^{n-\sigma} K_s \ {}^{(n-s+p)}k_\lambda{}^\nu = 0 \qquad (p=0,1,2,\cdots) \tag{3.2}$$

where

$$\sigma = \begin{cases} 0 & if\, n \ \ is\ even \\ 1 & if\, n \ \ is\ odd \end{cases}$$

Proof. The relation (3.2) is a direct consequence of (2.17) and (2.18). □

THEOREM 3.2. *In $g-ESX_n$, the vectors S_λ and U_λ are given by*

$$S_\lambda = (1-n)X_\lambda \tag{3.3}$$

$$U_\lambda = \frac{1}{2}\partial_\lambda ln\mathfrak{g} \tag{3.4}$$

Proof. Putting $\mu=\nu$ in (2.20), we have (3.3). In order to prove (3.4), consider the following Einstein's equations (2.8a). Multiplying ${}^*g^{\lambda\mu}$ to both sides of (2.8a) and making use of (2.4), we have

$$\partial_\omega ln\mathfrak{g} - \Gamma_\alpha{}^\alpha{}_\omega - \Gamma_\omega{}^\alpha{}_\alpha = 0 \tag{3.5}$$

or equivalently

$$\partial_\omega ln\mathfrak{g} + 2S_\omega - 2\Gamma_\omega{}^\alpha{}_\alpha = 0 \tag{3.6}$$

On the other hand, in virtue of classical result

$$\left\{ \begin{array}{c} \alpha \\ \omega\alpha \end{array} \right\} = \frac{1}{2}\partial_\omega ln\mathfrak{h} \tag{3.7}$$

the result (2.2) gives

$$\Gamma_\omega{}^\alpha{}_\alpha = \frac{1}{2}\partial_\omega ln\mathfrak{h} + S_\omega + U_\omega \tag{3.8}$$

The relation (3.4) immediately follows from (3.6) and (3.8). □

THEOREM 3.3. *In $g-ESX_n$, the following relations hold for $p,q = 1,2,3,\cdots$:*

$${}^{(p+1)}S_\lambda = (1-n)^{(p)}U_\lambda \tag{3.9}$$

$${}^{(p)}U_\alpha{}^{(q)}X^\alpha = 0 \quad if \quad p+q-1 \quad is \quad odd \tag{3.10}$$

Proof. The relation (3.9) is a direct consequence of (3.3), (3.4), and (2.18). Making use of (3.9), the relation

$$ {}^{(p)}U_\alpha{}^{(q)}X^\alpha = {}^{(p+1)}\, X_\alpha{}^{(q)}X^\alpha = (-1)^{q(p+q-1)} k_{\alpha\beta} X^\alpha X^\beta \tag{3.11} $$

follows. The statement (3.10) may be proved from (3.11), since ${}^{(p+q-1)}k_{\alpha\beta}$ is skew-symmetric if $p+q-1$ is odd. □

THEOREM 3.4. *In $g-ESX_n$, the following relations hold:*

$$ D_\lambda X_\mu = \nabla_\lambda X_\mu \tag{3.12} $$

$$ D_{[\lambda} X_{\mu]} = \nabla_{[\lambda} X_{\mu]} = \partial_{[\lambda} X_{\mu]} \tag{3.13} $$

$$ \nabla_{[\lambda} U_{\mu]} = 0, \qquad D_{[\lambda} U_{\mu]} = 2U_{[\lambda} X_{\mu]} = 2^{(2)} X_{[\lambda} X_{\mu]} \tag{3.14} $$

where ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols defined by $h_{\lambda\mu}$.

Proof. In virtue of (2.18) and Theorem 2.2, the relation (3.11) follows as in the following way:

$$ \begin{aligned} D_\lambda X_\mu &= \nabla_\lambda X_\mu - X_\alpha S_{\mu\lambda}{}^\alpha - X_\alpha U^\alpha{}_{\mu\lambda} \\ &= \nabla_\lambda X_\mu - 2X_{[\mu} X_{\lambda]} + h_{\mu\lambda}(k_{\alpha\beta} X^\alpha X^\beta) \\ &= \nabla_\alpha X_\mu \end{aligned} $$

The relation (3.13) are direct consequences of (3.12). Since $\partial_{[\lambda} U_{\mu]} = 0$ in virtue of the (3.4), we have the first relation of (3.14). Similarly, the second relation of (3.14) may be proved in virtue of (3.4). □

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