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SOME RESULTS IN FUZZY ALMOST r-M CONTINUOUS FUNCTIONS ON FUZZY r-MINIMAL STRUCTURES

Won Keun Min* and Myeong Hwan Kim

Abstract. We introduce the concept of fuzzy almost r-M continuous function on fuzzy r-minimal structures and investigate characterizations and properties for such a function.

1. Introduction

The concept of fuzzy set was introduced by Zadeh [7]. Chang [1] defined fuzzy topological spaces using fuzzy sets. In [2], Chattopadhyay, Hazra and Samanta introduced a smooth fuzzy topological space which is a generalization of fuzzy topological space. In [4], Yoo et al. introduced the concept of fuzzy r-minimal space which is an extension of the smooth fuzzy topological space. The concept of fuzzy r-M continuity was also introduced and investigated in [4]. They [4,5] studied the concepts of fuzzy r-minimal compactness, almost fuzzy r-minimal compactness and nearly fuzzy r-minimal compactness. In [6], Min introduced and studied the concept of fuzzy almost r-M continuity which is a generalization of fuzzy r-M continuity. In this paper, we investigate the relationships between fuzzy almost$r$-M continuous functions and several types of fuzzy r-minimal compactness.

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*Corresponding author.

2. Preliminaries

Let I be the unit interval $[0,1]$ of the real line. A member A of I^X is called a fuzzy set of X. By $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$ we denote constant maps on X with value 0 and 1, respectively. For any $A \in I^X$, A^c denotes the complement $\tilde{\mathbf{1}} - A$. All other notations are standard notations of fuzzy set theory.

An *fuzzy point* x_α in X is a fuzzy set x_α defined as follows

$$x_\alpha(y) = \begin{cases} \alpha \text{ if } y = x \\ 0 \text{ if } y \neq x. \end{cases}$$

A fuzzy point x_α is said to belong to a fuzzy set A in X, denoted by $x_\alpha \in A$, if $\alpha \leq A(x)$ for $x \in X$.

A fuzzy set A in X is the union of all fuzzy points which belong to A.

Let $f : X \to Y$ be a function and $A \in I^X$ and $B \in I^Y$. Then $f(A)$ is a fuzzy set in Y, defined by

$$f(A)(y) = \begin{cases} \sup\limits_{z \in f^{-1}(y)} A(z), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

for $y \in Y$ and $f^{-1}(B)$ is a fuzzy set in X, defined by $f^{-1}(B)(x) = B(f(x)), x \in X$.

A *smooth topology* [2,3] on X is a map $\mathcal{T} : I^X \to I$ which satisfies the following properties:

(1) $\mathcal{T}(\tilde{\mathbf{0}}) = \mathcal{T}(\tilde{\mathbf{1}}) = 1$.
(2) $\mathcal{T}(A_1 \cap A_2) \geq \mathcal{T}(A_1) \wedge \mathcal{T}(A_2)$.
(3) $\mathcal{T}(\cup A_i) \geq \wedge \mathcal{T}(A_i)$.

The pair $(X, \mathcal{T})$ is called a *smooth topological space.*

Definition 2.1. ([4]) Let X be a nonempty set and $r \in (0,1] = I_0$. A fuzzy family $\mathcal{M} : I^X \to I$ on X is said to have a *fuzzy r-minimal structure* if the family

$$\mathcal{M}_r = \{A \in I^X \mid \mathcal{M}(A) \geq r\}$$

contains $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$.

Then the $(X, \mathcal{M})$ is called a *fuzzy r-minimal space* (simply r-FMS) if $\mathcal{M}$ has a fuzzy r-minimal structure. Every member of $\mathcal{M}_r$ is called a *fuzzy r-minimal open* set. A fuzzy set A is called a *fuzzy r-minimal closed* set if the complement of A (simply, A^c) is a fuzzy r-minimal open set.

Let $(X, \mathcal{M})$ be an r-FMS and $r \in I_0$. The fuzzy r-minimal closure and the fuzzy r-minimal interior of A [4], denoted by $mC(A, r)$ and $mI(A, r)$, respectively, are defined as

$$mC(A, r) = \cap\{B \in I^X : B^c \in \mathcal{M}_r \text{ and } A \subseteq B\},$$

$$mI(A, r) = \cup\{B \in I^X : B \in \mathcal{M}_r \text{ and } B \subseteq A\}.$$

THEOREM 2.2. ([4]) Let $(X, \mathcal{M})$ be an r-FMS and A, B in I^X.

(1) $mI(A, r) \subseteq A$ and if A is a fuzzy r-minimal open set, then $mI(A, r) = A$.
(2) $A \subseteq mC(A, r)$ and if A is a fuzzy r-minimal closed set, then $mC(A, r) = A$.
(3) If $A \subseteq B$, then $mI(A, r) \subseteq mI(B, r)$ and $mC(A, r) \subseteq mC(B, r)$.
(4) $mI(A, r) \cap mI(B, r) \supseteq mI(A \cap B, r)$ and $mC(A, r) \cup mC(B, r) \subseteq mC(A \cup B, r)$.
(5) $mI(mI(A, r), r) = mI(A, r)$ and $mC(mC(A, r), r) = mC(A, r)$.
(6) $\tilde{\mathbf{1}} - mC(A, r) = mI(\tilde{\mathbf{1}} - A, r)$ and $\tilde{\mathbf{1}} - mI(A, r) = mC(\tilde{\mathbf{1}} - A, r)$.

Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be two r-FMS's. Then a function $f : X \to Y$ is said to be

(1) *fuzzy r-M continuous* [4] if for every fuzzy r-minimal open set A in Y, $f^{-1}(A)$ is fuzzy r-minimal open in X,
(2) *fuzzy r-M open* [4] if for every fuzzy r-minimal open set G in X, $f(G)$ is fuzzy r-minimal open in Y.
(3) *fuzzy almost r-M continuous* [7] if for fuzzy point x_α in X and each fuzzy r-minimal open set V containing $f(x_\alpha)$, there is a fuzzy r-minimal open set U containing x_α such that $f(U) \subseteq mI(mC(V, r), r)$.

THEOREM 2.3. ([7]) Let $f : X \to Y$ be a function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. Then the following statements are equivalent:

(1) f is fuzzy almost r-M continuous.

(2) $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B, r), r)), r)$ for each fuzzy r-minimal open set B of Y.

(3) $mC(f^{-1}(mC(mI(F, r), r)), r) \subseteq f^{-1}(F)$ for each fuzzy r-minimal closed set F in Y.

Let X be a nonempty set and $\mathcal{M} : I^X \to I$ a fuzzy family on X. The fuzzy family $\mathcal{M}$ is said to have the property $(\mathcal{U})$ [4] if for $A_i \in \mathcal{M}$ $(i \in J)$,

$$\mathcal{M}(\cup A_i) \geq \wedge \mathcal{M}(A_i).$$

Theorem 2.4. ([4]) Let $(X, \mathcal{M})$ be an r-FMS with the property $(\mathcal{U})$. Then

(1) $mI(A, r) = A$ if and only if A is a fuzzy r-minimal open set for $A \in I^X$.

(2) $mC(A, r) = A$ if and only if A is a fuzzy r-minimal closed set for $A \in I^X$.

Corollary 2.5. ([7]) Let $f : X \to Y$ be a function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_Y$ has property $(\mathcal{U})$, then the following statements are equivalent:

(1) f is fuzzy almost r-M continuous.
(2) $f^{-1}(B) \subseteq mI(f^{-1}(mI(mC(B, r), r)), r)$ for each fuzzy r-minimal open set B of Y.
(3) $f^{-1}(mI(B, r)) \subseteq mI(f^{-1}(mI(mC(mI(B, r), r), r)), r)$ for each $B \subseteq Y$.
(4) $mCl(f^{-1}(mC(mI(mC(B, r), r), r)), r) \subseteq f^{-1}(mC(B, r))$ for each $B \subseteq Y$.

3. Fuzzy almost r-M continuous functions and fuzzy r-minimal compactness

We recall that the following notions introduced in [5]: Let $(X, \mathcal{M})$ be an r-FMS and $\mathcal{A} = \{A_i \in I^X : i \in J\}$. $\mathcal{A}$ is called a *fuzzy r-minimal cover* if $\cup\{A_i : i \in J\} = \tilde{\mathbf{1}}$. It is a *fuzzy r-minimal open cover* if each A_i is a fuzzy r-minimal open set. A subcover of a fuzzy r-minimal cover $\mathcal{A}$ is a subfamily of it which also is a fuzzy r-minimal cover. A fuzzy set A in X is said to be *fuzzy r-minimal compact* (resp. *almost fuzzy r-minimal compact*, *nearly fuzzy r-minimal compact*) if every fuzzy r-minimal open cover $\mathcal{A} = \{A_i \in \mathcal{M}_r : i \in J\}$ of A, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{i \in J_0} A_i$ (resp. $A \subseteq \cup_{i \in J_0} mC(A_i, r)$, $A \subseteq \cup_{i \in J_0} mI(mC(A_i, r), r)$).

Theorem 3.1. Let $f : X \to Y$ be a fuzzy almost r-M continuous function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If A is a fuzzy r-minimal compact set in X and $\mathcal{M}_X$ has property $(\mathcal{U})$, then $f(A)$ is a nearly fuzzy r-minimal compact set.

Proof. Let $\{B_i \in I^Y : i \in J\}$ be a fuzzy r-minimal open cover of $f(A)$ in Y. Then from fuzzy almost r-M continuity, we have $f^{-1}(B_i) \subseteq$

$mI(f^{-1}(mI(mC(B_i, r), r)), r)$ for each $i \in J$. And since $\mathcal{M}_X$ has property $(\mathcal{U})$, by Theorem 2.3 and Theorem 2.4, $\{mI(f^{-1}(mI(mC(B_i, r), r)), r) : i \in J\}$ is a fuzzy r-minimal open cover of A in X. By the fuzzy r-minimal compactness, there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that

$$A \subseteq \cup_{i \in J_0} mI(f^{-1}(mI(mC(B_i, r), r)), r) \subseteq f^{-1}(mI(mC(B_i, r), r)).$$

Hence $f(A) \subseteq \cup_{i \in J_0} mI(mC(B_i, r), r)$. □

COROLLARY 3.2. Let $f : X \to Y$ be a fuzzy r-M continuous function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If A is a fuzzy r-minimal compact set in X and $\mathcal{M}_X$ has property $(\mathcal{U})$, then $f(A)$ is a nearly fuzzy r-minimal compact set.

Proof. Since every fuzzy r-M continuous function is fuzzy almost r-M continuous function, it is obtained by Theorem 3.1. □

THEOREM 3.3. ([4]) Let $f : X \to Y$ be a function on two r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. Then
(1) f is fuzzy r-M open.
(2) $f(mI(A), r) \subseteq mI(f(A), r)$ for $A \in I^X$.
(3) $mI(f^{-1}(B), r) \subseteq f^{-1}(mI(B), r)$ for $B \in I^Y$.
Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3).

THEOREM 3.4. Let $f : X \to Y$ be a function on two r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If $\mathcal{M}_X$ and $\mathcal{M}_Y$ have the property $(\mathcal{U})$, then the following are equivalent:
(1) f is fuzzy r-M open.
(2) $f(mI(A), r) \subseteq mI(f(A), r)$ for $A \in I^X$.
(3) $mI(f^{-1}(B), r) \subseteq f^{-1}(mI(B), r)$ for $B \in I^Y$.

Proof. It follows from Theorem 2.4 and Theorem 3.3. □

THEOREM 3.5. Let $f : X \to Y$ be a fuzzy almost r-M continuous and fuzzy r-M open function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If A is an almost fuzzy r-minimal compact set and if $\mathcal{M}_X$ has the property $(\mathcal{U})$, then $f(A)$ is almost fuzzy r-minimal compact set.

Proof. Let $\{B_i \in I^Y : i \in J\}$ be a fuzzy r-minimal open cover of $f(A)$ in Y. Then by the property $(\mathcal{U})$, $\{mI(f^{-1}(mI(mC(B_i, r), r)), r) : i \in J\}$ is a fuzzy r-minimal open cover of A in X. So there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that

$$A \subseteq \cup_{i \in J_0} mC(mI(f^{-1}(mI(mC(B_i, r), r)), r), r).$$

From Theorem 3.3, it follows

$$\begin{aligned} A &\subseteq \cup_{i\in J_0} mC(mI(f^{-1}(mI(mC(B_i,r),r)),r),r) \\ &\subseteq \cup_{i\in J_0} mC(f^{-1}(mI(mC(B_i,r),r)),r) \\ &\subseteq \cup_{i\in J_0} f^{-1}(mC(mI(mC(B_i,r),r),r)) \\ &\subseteq \cup_{i\in J_0} f^{-1}(mC(B_i,r)). \end{aligned}$$

Hence $f(A) \subseteq \cup_{i\in J_0} mC(B_i,r)$. □

Corollary 3.6. Let $f : X \to Y$ be a fuzzy r-M continuous and fuzzy r-M open function between r-FMS's $(X,\mathcal{M}_X)$ and $(Y,\mathcal{M}_Y)$. If A is an almost fuzzy r-minimal compact set and if $\mathcal{M}_X$ has the property $(\mathcal{U})$, then $f(A)$ is almost fuzzy r-minimal compact.

Theorem 3.7. Let $f : X \to Y$ be a fuzzy almost r-M continuous function between r-FMS's $(X,\mathcal{M}_X)$ and $(Y,\mathcal{M}_Y)$. If A is an almost fuzzy r-minimal compact set and if $\mathcal{M}_X$ and $\mathcal{M}_Y$ have the property $(\mathcal{U})$, then $f(A)$ is almost fuzzy r-minimal compact set.

Proof. Let $\{B_i \in I^Y : i \in J\}$ be a fuzzy r-minimal open cover of $f(A)$ in Y. Then by the property $(\mathcal{U})$, $\{mI(f^{-1}(mI(mC(B_i,r),r)),r) : i \in J\}$ is a fuzzy r-minimal open cover of A in X. So there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that

$$A \subseteq \cup_{i\in J_0} mC(mI(f^{-1}(mI(mC(B_i,r),r)),r),r).$$

From Corollary 2.5, it follows

$$\begin{aligned} A &\subseteq \cup_{i\in J_0} mC(mI(f^{-1}(mI(mC(B_i,r),r)),r),r) \\ &\subseteq \cup_{i\in J_0} mC(f^{-1}(mI(mC(B_i,r),r)),r) \\ &\subseteq \cup_{i\in J_0} mC(f^{-1}(mC(mI(mC(B_i,r),r),r)),r) \\ &\subseteq \cup_{i\in J_0} f^{-1}(mC(B_i,r)). \end{aligned}$$

Hence $f(A) \subseteq \cup_{i\in J_0} mC(B_i,r)$. □

Corollary 3.8. Let $f : X \to Y$ be a fuzzy r-M continuous function between r-FMS's $(X,\mathcal{M}_X)$ and $(Y,\mathcal{M}_Y)$. If A is an almost fuzzy r-minimal compact set and if $\mathcal{M}_X$ and $\mathcal{M}_Y$ have the property $(\mathcal{U})$, then $f(A)$ is almost fuzzy r-minimal compact.

Theorem 3.9. Let $f : X \to Y$ be a fuzzy almost r-M continuous and fuzzy r-M open function between r-FMS's $(X,\mathcal{M}_X)$ and $(Y,\mathcal{M}_Y)$.

If A is a nearly fuzzy r-minimal compact set and if $\mathcal{M}_X$ has the property $(\mathcal{U})$, then $f(A)$ is a nearly fuzzy r-minimal compact set.

Proof. Let $\{B_i \in I^Y : i \in J\}$ be a fuzzy r-minimal open cover of $f(A)$ in Y. Then $\{mI(f^{-1}(mI(mC(B_i, r), r)), r) : i \in J\}$ is a fuzzy r-minimal open cover of A in X. So there exists $J_0 = \{j_1, j_2, \cdots, j_n\} \subseteq J$ such that $A \subseteq \cup_{i\in J_0} mI(mC(mI(f^{-1}(mI(mC(B_i, r), r)), r), r), r)$ by the nearly fuzzy r-minimal compactness. From Theorem 3.3, it follows,

$$\begin{aligned} A &\subseteq \cup_{i\in J_0} mI(mC(mI(f^{-1}(mI(mC(B_i, r), r)), r), r), r) \\ &\subseteq \cup_{i\in J_0} mI(mC(f^{-1}(mI(mI(mC(B_i, r), r), r)), r), r) \\ &\subseteq \cup_{i\in J_0} mI(f^{-1}(mC(mI(mC(B_i, r), r), r)), r) \\ &\subseteq \cup_{i\in J_0} mI(f^{-1}(mC(B_i, r)), r) \\ &\subseteq \cup_{i\in J_0} f^{-1}(mI(mC(B_i, r), r)). \end{aligned}$$

Hence $f(A) \subseteq \cup_{i\in J_0} mI(mC(B_i, r), r)$. □

Corollary 3.10. Let $f : X \to Y$ be a fuzzy almost r-M continuous and fuzzy r-M open function between r-FMS's $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. If A is a nearly fuzzy r-minimal compact set and if $\mathcal{M}_X$ has the property $(\mathcal{U})$, then $f(A)$ is a nearly fuzzy r-minimal compact set.

References

[1] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24**(1968), 182–190.

[2] K. C. Chattopadhyay, R. N. Hazra, and S. K. Samanta, *Gradation of openness : Fuzzy topology*, Fuzzy Sets and Systems, **49**(1992), 237–242.

[3] A. A. Ramadan, *Smooth topological spaces*, Fuzzy Sets and Systems, **48**(1992), 371–375.

[4] Y. H. Yoo, W. K. Min and J. I. Kim. *Fuzzy r-Minimal Structures and Fuzzy r-Minimal Spaces*, Far East J. Math. Sci., **33**(2)(2009), 193-205.

[5] ———-, *Fuzzy r-compactness on fuzzy r-minimal spaces* , Int. J. Fuzzy Logic and Intelligent Systems, *accepted*.

[6] W. K. Min, *Fuzzy almost r-M continuity on fuzzy r-minimal structures*, submitted.

[7] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(1965), 338–353.

Department of Mathematics
Kangwon National University
Chuncheon, 200-701, Korea
E-mail: wkmin@kangwon.ac.kr

Department of Mathematics
Kangwon National University
Chuncheon, 200-701, Korea
E-mail: kimmw@kangwon.ac.kr