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THE CAPABILITY OF PERIODIC NEURAL NETWORK APPROXIMATION

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Abstract. In this paper, we investigate the possibility of 2π-periodic continuous function approximation by periodic neural networks. Using the Riemann sum and the quadrature formula, we show the capability of a periodic neural network approximation.

1. Introduction

There has been growing interest in neural network approximation in recent years ([3, 5, 6, 8]) because it has many important applications in signal processing, robotics, sequential decision making, time series prediction and modeling, etc. A neural network computes functions that are linear combinations of a single simple nonlinear function composed with affine functional. A general form of feedforward neural network with one hidden layer is

$$\sum_{i=1}^{n} a_i \sigma(b_i x + c_i), \tag{1.1}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ is a univariate activation function. The Gaussian function $\sigma(x) = e^{-x^2}$, the squashing function $\sigma(x) = (1 + e^{-x})^{-1}$ and the generalized multiquadrics $\sigma(x) = (1 + x^2)^{\alpha}, \alpha \notin \mathbb{Z}$, are examples of an activation function. In most papers related to the approximation capability by neural network, they investigated the approximation of continuous functions on a compact set and target functions were not periodic functions.

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In this paper, we study the approximation of continuous 2π-periodic functions by periodic neural network.

Note that the class of all trigonometric functions of order at most n is denoted by $\mathbb{T}_n$. The element in $\mathbb{T}_n$ has a complex expression of the form

$$\sum_{|j|\le n} c_j e^{ijx}, \tag{1.2}$$

where $c_j \in \mathbb{C}$.

For a continuous 2π-periodic function f, the norm of f is given by

$$\|f\|_\infty := \sup\{|f(x)| : x \in [-\pi, \pi]\}.$$

In addition, we use the symbol $S_n(f)$ to denote the nth partial sum of Fourier series of a periodic function f and

$$G_n(f) := \frac{1}{n}\sum_{k=1}^{n} S_k(f) \tag{1.3}$$

represents the nth Fejer sum of f. Note that $G_n(f) \in \mathbb{T}_n$.

2. Main results

The following lemma explains the uniform approximation by trigonometric polynomials for continuous 2π-periodic functions. This approximation is constructive since trigonometric polynomials are obtained from the partial sums of Fourier series. The proof of this lemma is in [1, 7].

LEMMA 2.1. *Let f be a continuous 2π-periodic function on $[-\pi, \pi]$. Then*

$$\|f - G_n(f)\|_\infty \to 0,$$

as $n \to 0$.

From (1.2) and (1.3), $G_n(f, x) = \sum_{|j|\le n} c_j(f) e^{ijx}$ for some $c_j(f) \in \mathbb{C}$. Thus we need to show that a periodic neural network with a minimal constraint approximates e^{ijx} arbitrarily closed for any $j \in \mathbb{Z}$. First of all, we investigate the periodic neural network approximation using a Riemann sum.

LEMMA 2.2. *Let σ be a continuous 2π-periodic function with $\Gamma := \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma(t)e^{-it}dt \neq 0$. Then, for a given $j \in \mathbb{Z}$,*

$$\|e^{ij\cdot} - \frac{1}{m\Gamma}\sum_{k=1}^{m} e^{i\frac{\pi(2k-m)}{m}}\sigma(j\cdot - \frac{\pi(2k-m)}{m})\|_\infty \to 0 \tag{2.1}$$

as $m \to \infty$.

Proof. By the substitution rule of integration, we have, for $x \in [-\pi, \pi]$,

$$\begin{aligned}\Gamma \cdot e^{ijx} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma(t)e^{-it}dt \cdot e^{ijx}\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma(t)e^{i(jx-t)}dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma(jx-t)e^{it}dt.\end{aligned}$$

Thus

$$e^{ijx} = \frac{1}{2\pi\Gamma}\int_{-\pi}^{\pi}\sigma(jx-t)e^{it}dt. \tag{2.2}$$

Note that

$$N_m(\sigma, x) := \sum_{k=1}^{m} e^{i\frac{\pi(2k-m)}{m}}\sigma(jx - \frac{\pi(2k-m)}{m})\frac{2\pi}{m} \tag{2.3}$$

is a Riemann sum for $\int_{-\pi}^{\pi}\sigma(jx-t)e^{-it}dt$. Thus

$$\begin{aligned}&\|e^{ij\cdot} - \frac{1}{m\Gamma}\sum_{k=1}^{m} e^{i\frac{\pi(2k-m)}{m}}\sigma(j\cdot - \frac{\pi(2k-m)}{m})\|_\infty\\ = \; &\|\frac{1}{2\pi\Gamma}\int_{-\pi}^{\pi}\sigma(j\cdot - t)e^{it}dt - \frac{1}{m\Gamma}\sum_{k=1}^{m} e^{i\frac{\pi(2k-m)}{m}}\sigma(j\cdot - \frac{\pi(2k-m)}{m})\|_\infty\\ \to \; &0\end{aligned}$$

as $m \to \infty$. Thus we complete the proof. □

From Lemma 2.1 and Lemma 2.2, we obtain the following.

THEOREM 2.3. *Let σ be a continuous 2π-periodic function with $\Gamma := \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma(t)e^{-it}dt \neq 0$. For a given $\epsilon > 0$ and any continuous 2π-periodic function f, there exists a periodic neural network*

$$N_{n,m}(\sigma,x) := \sum_{|j|\leq n}\sum_{k=1}^{m} c_j(f)\frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(jx - \frac{\pi(2k-m)}{m}) \tag{2.4}$$

such that

$$\|f - N_{n,m}(\sigma)\|_\infty < \epsilon,$$

where $c_j(f)$'s are coefficients of Fejer sum of f .

Proof. Let $\epsilon > 0$ be given. By Lemmma 2.1, there exists a Fejer sum $G_n(f)$ of f such that

$$\|f - G_n(f)\|_\infty < \frac{\epsilon}{2}.$$

By Lemma 2.2, there exists $m_j \in \mathbb{N}$ such that

$$\|e^{ij\cdot} - \frac{1}{m_j\Gamma}\sum_{k=1}^{m_j} e^{i\frac{\pi(2k-m_j)}{m_j}}\sigma(j\cdot - \frac{\pi(2k-m_j)}{m_j})\|_\infty < \frac{\epsilon}{2^{j^2+1}(|c_j(f)|+1)} \tag{2.5}$$

for each $j \in \mathbb{Z}$ with $|j| \leq n$.
Let $m = \max\{m_j : |j| \leq n\}$. By Lemma 2.2 and (2.5), we have

$$\begin{aligned}
& \|G_n(f) - N_{n,m}(\sigma)\|_\infty \\
= \ & \|\sum_{|j|\leq n} c_j(f)e^{ij\cdot} - \sum_{|j|\leq n}\sum_{k=1}^{m} c_j(f)\frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(j\cdot - \frac{\pi(2k-m)}{m})\|_\infty \\
\leq \ & \sum_{|j|\leq n} |c_j(f)|\cdot\|e^{ij\cdot} - \sum_{k=1}^{m}\frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(j\cdot - \frac{\pi(2k-m)}{m})\|_\infty \\
< \ & \frac{\epsilon}{2}.
\end{aligned}$$

Therefore a periodic neural network which is defined by

$$N_{n,m}(\sigma,x) := \sum_{|j|\leq n}\sum_{k=1}^{m} c_j(f)\frac{1}{m\Gamma}e^{i\frac{\pi(2k-m)}{m}}\sigma(jx - \frac{\pi(2k-m)}{m})$$

satisfies

$$\begin{aligned}\|f - N_{n,m}(\sigma)\|_\infty &\leq \|f - G_n(f)\|_\infty + \|G_n(f) - N_{n,m}(\sigma)\|_\infty \\ &< \epsilon.\end{aligned}$$

Thus we complete the proof. □

In Lemma 2.2 and Theorem 2.4, we have to use a different number of neurons to approximate different e^{ijx} for j with $|j| \leq n$ since we used the Riemann sums. If we add a minimal constraint on an activation function σ of periodic neural networks, we have the exact periodic neural network approximation for trigonometric polynomials.

The following lemma explains that the quadrature formula of a trigonometric function is exact. This gives us a simple periodic neural network approximation for continuous 2π-periodic functions.

LEMMA 2.4. *For any $P \in \mathbb{T}_n$, we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(x)dx = \frac{1}{n+1}\sum_{l=0}^{n} P(\frac{2\pi l}{n+1}). \tag{2.6}$$

Proof. It is enough to show that (2.6) is true for $P(x) = e^{irx}$, where $r = 0, 1, \ldots, n$. When $r = 0$, then $P(x) = e^0 = 1$ and so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(x)dx = 1 = \frac{1}{n+1}\sum_{l=0}^{n} P(\frac{2\pi l}{n+1}).$$

For any r with $1 \leq r \leq n$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{irx}dx = \frac{1}{2\pi ir}(e^{ir\pi} - e^{-ir\pi}) = 0$$

and

$$\begin{aligned}\frac{1}{n+1}\sum_{l=0}^{n} P(\frac{2\pi l}{n+1}) &= \frac{1}{n+1}\sum_{l=0}^{n} e^{ir(\frac{2\pi l}{n+1})} \\ &= \frac{e^{2\pi ir} - 1}{(n+1)(e^{\frac{2\pi ir}{n+1}} - 1)} \\ &= 0.\end{aligned}$$

Thus we complete the proof. □

From Lemma 2.4, we obtain the following.

THEOREM 2.5. *Let σ be a continuous 2π-periodic polynomial of degree m with $\Gamma := \frac{1}{2\pi}\int_{-\pi}^{\pi}\sigma(t)e^{-it}dt \neq 0$. For a given $\epsilon > 0$ and any continuous 2π-periodic function f, there exists a neural network*

$$N_{n,m}(\sigma, x) := \sum_{|j|\leq n}\sum_{k=0}^{m+1} c_j(f)\frac{1}{(m+2)\Gamma}e^{i\frac{2\pi k}{m+1}}\sigma(jx - \frac{2\pi k}{m+1}) \tag{2.7}$$

such that

$$\|f - N_{n,m}\|_\infty < \epsilon,$$

where $c_j(f)$'s are coefficients of Fejer sum of f .

Proof. Let $\epsilon > 0$ be given. From (2.2), we have

$$e^{ijx} = \frac{1}{2\pi\Gamma}\int_{-\pi}^{\pi}\sigma(jx - t)e^{it}dt \tag{2.8}$$

for $x \in [\pi, \pi]$ and j with $|j| \leq n$. Note that $\sigma(jx - t)e^{it} \in \mathbb{T}_{m+1}$ as a function of t. By Lemma 2.4, we get

$$\frac{1}{2\pi\Gamma}\int_{-\pi}^{\pi}\sigma(jx - t)e^{it}dt = \frac{1}{(m+2)\Gamma}\sum_{k=0}^{m+1}e^{i\frac{2\pi k}{m+1}}\sigma(jx - \frac{2\pi k}{m+1}). \tag{2.9}$$

By Lemma 2.1, (2.8) and (2.9), we have

$$\begin{aligned} & \|f - G_n(f)\|_\infty \\ = \; & \|f - \sum_{|j|\leq n}\sum_{k=0}^{m+1} c_j(f)\frac{1}{(m+2)\Gamma}e^{i\frac{2\pi k}{m+1}}\sigma(jx - \frac{2\pi k}{m+1})\|_\infty \\ < \; & \epsilon. \end{aligned}$$

for sufficiently large $n \in \mathbb{N}$. Thus we complete the proof. □

3. Discussions

In our proofs, we showed a possibility of periodic function approximation by neural networks. Unlike other results by neural network approximation, we suggested that the weights in periodic neural networks could be restricted to integers. On the other hand, one of the main topics in neural network approximation is the complexity problem. The complexity problem is almost the same as the problem of degree of approximation.

In this paper, we used the Fejer sums. They approximate the continuous 2π-periodic function uniformly on $[-\pi, \pi]$ but they do not give us any information of approximation order. In order to investigate the approximation order by periodic neural network approximation, we may use different trigonometric polynomials which satisfies the following.

Let $P \in \mathbb{T}_n$ be an even and non-negative trigonometric polynomial which satisfies
(1) $\int_{-\pi}^{\pi} P(x)dx = 1$
(2) $\int_0^{\pi} x^k P(x) \leq cn^{-k}, k = 0, 1, 2,$
where c is a constant. According to [2, 4, 9], if we use the integral

$$L_n(f) := \int_{-\pi}^{\pi} f(t)P(x-t)dt, \tag{3.1}$$

then we have

$$\|f - L_n(f)\|_\infty \leq C\omega_2(f, \frac{1}{n}), \tag{3.2}$$

where ω_2 denotes the second modulus of smoothness of f.

In order to obtain an approximation order by periodic neural networks, we need to show that $L_n(f)$ in (3.1) can be approximated uniformly by periodic neural networks or it is approximated by periodic neural networks with some approximation order. We will study this in the future.

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