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ON ALMOST PSEUDO-VALUATION DOMAINS

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Abstract. Let D be an integral domain, and let $\bar{D}$ be the integral closure of D. We show that if D is an almost pseudo-valuation domain (APVD), then D is a quasi-Prüfer domain if and only if D/P is a quasi-Prüfer domain for each prime ideal P of D, if and only if $\bar{D}$ is a valuation domain. We also show that $D(X)$, the Nagata ring of D, is a locally APVD if and only if D is a locally APVD and $\bar{D}$ is a Prüfer domain.

1. Introduction

Let D be an integral domain, K be the quotient field of D, and $\bar{D}$ be the integral closure of D in K. An *overring* of D is a ring between D and K. As in [11], we say that a prime ideal P of D is *strongly prime* if $xy \in P$ and $x, y \in K$ imply $x \in P$ or $y \in P$, while D is a *pseudo-valuation domain* (PVD) if every prime ideal of D is strongly prime; equivalently, if D is quasi-local whose maximal ideal is strongly prime. It is well known that D is a PVD if and only if there exists a valuation overring V of D such that $\text{Spec}(V) = \text{Spec}(D)$ [11, Theorem 2.7]; so if D is a PVD, then $\text{Spec}(D)$ is linearly ordered under inclusion [11, Corollary 1.3]. Let D be a PVD with maximal ideal M. It is also known that if D is not a valuation domain, then $M^{-1} = \{x \in K | xM \subseteq D\}$ is a valuation domain such that $\text{Spec}(M^{-1}) = \text{Spec}(D)$ (in particular, M is the maximal ideal of M^{-1}) [11, Theorem 2.10].

As generalizations of "strongly prime" and "PVD", Badawi and Houston [2] introduced the notion of "strongly primary" and "almost PVD";

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(i) An ideal I of D is *strongly primary* if $xy \in I$ with $x, y \in K$ implies $x \in I$ or $y^n \in I$ for some integer $n \geq 1$.

(ii) D is an *almost PVD* (APVD) if each prime ideal of D is strongly primary.

Clearly, a strongly prime ideal is strongly primary, and thus a PVD is an APVD. It is known that if D is quasi-local with maximal ideal M, then D is an APVD if and only if there exists a valuation overring V of D such that $M = MV$ and $\sqrt{MV}$ is the maximal ideal of V [2, Theorem 3.4]. It is also known that if D is an APVD, then Spec(D) is linearly ordered under inclusion (and hence D is quasi-local) and $\bar{D}$ is a PVD [2, Propositions 3.2 and 3.7].

Let X be an indeterminate over D, and let $D[X]$ be the polynomial ring over D. For each $f \in D[X]$, we denote by $c(f)$ the ideal of D generated by the coefficients of f. Let $N = \{f \in D[X] | c(f) = D\}$; then N is a saturated multiplicative subset of $D[X]$. The quotient ring $D[X]_N$, denoted by $D(X)$, is called the *Nagata ring of* D. It is known that D is a Prüfer domain if and only if $D(X)$ is a Prüfer domain [1, Theorem 4]. As in [9], we say that D is a *quasi-Prüfer domain* if for each prime ideal P of D, if Q is a prime ideal of $D[X]$ with $Q \subseteq P[X]$, then $Q = (Q \cap D)[X]$. It is well known that D is a Prüfer domain if and only if D is integrally closed and quasi-Prüfer [10, Theorem 19.15].

Following [8], we say that D is a *locally pseudo-valuation domain* (LPVD) if D_M is a PVD for each maximal ideal M of D. Clearly, an LPVD is a global part of PVDs, and thus it is natural to call D a *locally APVD* (LAPVD) if D_M is an APVD for each maximal ideal M of D. Note that a PVD is an APVD, and thus an LPVD is an LAPVD. In [4, Corollary 3.9], Chang showed that $D(X)$ is an LPVD if and only if D is an LPVD and $\bar{D}$ is a Prüfer domain. In this paper, we study some properties of an LAPVD. More precisely, we show that if D is an APVD with maximal ideal M, then D is a quasi-Prüfer domain if and only if D/P is a quasi-Prüfer domain for each prime ideal P of D, if and only if $\bar{D}$ is a valuation domain and that if M is finitely generated, then D is a quasi-Prüfer domain. We also show that $D(X)$ is an LAPVD if and only if D is an LAPVD and a quasi-Prüfer domain, if and only if D is an LAPVD and $\bar{D}$ is a Prüfer domain.

2. Almost pseudo-valuation domains

Throughout this paper D is an integral domain with quotient field K, $\bar{D}$ is the integral closure of D in K, X is an indeterminate over D, $D[X]$ is the polynomial ring over D, and $D(X)$ is the Nagata ring of D.

We know that both PVDs and APVDs are quasi-local; so we will say that (D, M) is a PVD or an APVD if D is a PVD or an APVD with maximal ideal M. We begin this paper by recalling some known results for APVDs and quasi-Prüfer domains (Lemmas 1-3).

Lemma 1. *If D is not a valuation domain, then the following statements are equivalent.*

(1) *D is an APVD.*
(2) *D has a strongly primary maximal ideal.*
(3) *D is quasi-local and the maximal ideal M of D is such that $(M : M)$ is a valuation domain with M primary to the maximal ideal of $(M : M)$.*
(4) *D is quasi-local, and there is a valuation overring of D in which the maximal ideal of D is primary.*

Proof. [2, Theorem 3.4]. □

Lemma 2. *Let (D, M) be an APVD that is not a valuation domain, and let $P \subsetneq M$ be a prime ideal of D.*

(1) *$\bar{D}$ is a PVD with maximal ideal $\sqrt{M\bar{D}}$.*
(2) *P is a strongly prime ideal of D; so $P = PD_P$.*
(3) *D_P is a valuation domain; so $D_P = \bar{D}_{D\setminus P}$ and P is a strongly prime ideal of $\bar{D}$.*
(4) *D/P is an APVD.*

Proof. (1) [2, Proposition 3.7]. (2) Since P is not maximal, P is a strongly prime of D [2, Proposition 3.2], and thus $P = PD_P$. (3) Since P is not maximal, D_P is a valuation domain [6, Lemma 3.1]; so $D_P = \bar{D}_{D\setminus P}$ because a valuation domain is integrally closed. Hence $P = \bar{D} \cap PD_P$, and thus P is a strongly prime ideal of $\bar{D}$ by (2). (4) This follows directly from [6, Theorem 3.4] since $P = PD_P$ and D_P is a valuation domain by (2) and (3). □

Lemma 3. *The following statements are equivalent:*

(1) *D is a quasi-Prüfer domain.*
(2) *$\bar{D}$ is a Prüfer domain.*

(3) *Each overring of D is a quasi-Prüfer domain.*
(4) *$D(X)$ is a quasi-Prüfer domain.*
(5) *D_M is a quasi-Prüfer domain for each maximal ideal M of D.*

Proof. This appears in [5, Theorem 1.1]. □

It is known that if (D, M) is a PVD which is not a valuation domain, then every overring of D is a PVD if and only if $\bar{D}$ is a valuation domain [12, Proposition 2.7]. Also, if each overring of D is an APVD, then $\bar{D}$ is a valuation domain [2, Proposition 3.8], while $\bar{D}$ being a valuation domain does not imply that each overring of D is an APVD [2, Example 3.9]. Our next result shows that if each overring of D is an APVD, then D is quasi-Prüfer.

Theorem 4. (cf. [4, Theorem 2.3]) *The following statements are equivalent for an APVD (D, M).*

(1) *D is a quais-Prüfer domain.*
(2) *$\bar{D}$ is a valuation domain.*
(3) *$\bar{D} = (M : M)$.*
(4) *Each overring of D is a quasi-Prüfer domain.*
(5) *There is an integral overring of D which is a quasi-Prüfer domain.*
(6) *$\bar{D}$ is a quasi-Prüfer domain.*
(7) *Each integrally closed overring of D is a valuation domain.*

Moreover, if dim$(D) = 1$, then the above conditions are equivalent to
(8) *Each overring $R(\neq K)$ of D is integral over D.*

Proof. Let $V = (M : M)$. Then V is a valuation domain and M is a primary ideal of V by Lemma 1.

(1) ⇔ (2) This is an immediate consequence of Lemma 3 because $\bar{D}$ is quasi-local (Lemma 2(1)). (2) ⇒ (3) Note that $\sqrt{M\bar{D}}$ is the maximal ideal of $\bar{D}$ and $\bar{D} \subseteq V$. Also, note that $M\bar{D} \subseteq MV = M$ since M is an ideal of V; so $(\sqrt{M\bar{D}})V \subsetneq V$. Thus $\bar{D} = V$ [10, Theorem 17.6]. (2) ⇒ (4) and (7) Let R be an overring of D, and let $\bar{R}$ be the integral closure of R. Then $\bar{D} \subseteq \bar{R}$; so $\bar{R}$ is a valuation domain [10, Theorem 17.6]. Thus R is a quasi-Prüfer domain by Lemma 3. (3) ⇒ (6); (4) ⇒ (5); (6) ⇒ (5); and (7) ⇒ (2) Clear. (5) ⇒ (2) Let R be a quasi-Prüfer domain such that $D \subseteq R \subseteq \bar{D}$. Then $\bar{D}$ is the integral closure of R, and hence $\bar{D}$ is a valuation domain by Lemma 3. (2) ⇒ (8) Let R be an overring of D, and let $\bar{R}$ be the integral closure of R. Then $\bar{D} \subseteq \bar{R}$, and since

$\dim(\bar{D}) = 1$, we have $\bar{D} = \bar{R}$ (cf. [10, Theorem 17.6]). (8) $\Rightarrow$ (3) This follows because $D \subseteq \bar{D} \subseteq V$. □

Let (D, M) be an APVD such that $\bar{D}$ is a valuation domain. Then each overring R of D is comparable to $\bar{D}$, i.e., either $R \subseteq \bar{D}$ or $\bar{D} \subseteq R$. For if $R \nsubseteq \bar{D}$, then $\bar{D} \subsetneq \bar{R}$, the integral closure of R, and hence $\bar{R}$ is a valuation domain [10, Theorem 17.6]. Let Q be the maximal ideal of $\bar{R}$. Then $Q \cap D$ is not a maximal ideal of D, and hence $\bar{R} = D_{Q\cap D}$ (cf. [10, Theorem 17.6] and Lemma 2(3)). Since $\bar{R}$ is quasi-local, R is quasi-local with maximal ideal $Q \cap R$. Hence $\bar{R} = D_{Q\cap D} \subseteq R_{Q\cap R} = R$, and thus $\bar{D} \subsetneq \bar{R} = R$. Thus if $\bar{D}$ is a valuation domain, then each overring of an APVD D is an APVD if and only if each integral overring of D is an APVD [2, Proposition 3.10].

Corollary 5. *Let (D, M) be an APVD, and let $P \subsetneq M$ be a prime ideal of D. Then D is a quasi-Prüfer domain if and only if D/P is a quasi-Prüfer domain.*

Proof. Note that $P = PD_P$, P is a strongly prime ideal of both D and $\bar{D}$, $D_P = \bar{D}_{D\setminus P}$ is a valuation domain, and D/P is an APVD (Lemma 2). Also, note that $(D/P)_{P/P} \cong D_P/PD_P = D_P/P$; hence D_P/P (resp., $\bar{D}/P$) can be considered as the quotient field (resp., integral closure) of D/P. Moreover, since $\bar{D}_{D\setminus P}$ is a valuation domain, it follows that $\bar{D}$ is a valuation domain if and only if $\bar{D}/P$ is a valuation domain [7, Lemma 4.5(v)]. Thus by Theorem 4, D is a quasi-Prüfer domain if and only if $\bar{D}$ is a valuation domain, if and only if $\bar{D}/P$ is a valuation domain, if and only if D/P is a quasi-Prüfer domain. □

Corollary 6. (cf. [4, Corollary 2.5]) *Let (D, M) be an APVD such that M is finitely generated, and let $\{P_\alpha\}$ be the set of prime ideals of D properly contained in M. Then*

(1) *$P := \cup P_\alpha$ is a prime ideal of D.*
(2) *$P \subsetneq M$, and hence D/P is a one-dimensional local Noetherian domain.*
(3) *D is a quasi-Prüfer domain, and hence $\bar{D}$ is a valuation domain.*
(4) *An overring R of D has a prime ideal lying over M (if and) only if R is integral over D.*

Proof. (1) and (2) Clearly, P is a prime ideal because $\{P_\alpha\}$ is linearly ordered under inclusion. Also, since M is finitely generated, $P \subsetneq M$.

Next, note that D/P is one-dimensional quasi-local, and thus D/P is Noetherian by Cohen's theorem

(3) By (2), D/P is a one-dimensional local Noetherian domain, and hence $\overline{D/P}$ is a Dedekind domain (so Prüfer domain). Also, since D/P is an APVD by Lemma 2(4), $\overline{D/P}$ is quasi-local, and thus $\overline{D/P}$ is a valuation domain. Thus D/P is quasi-Prüfer by Theorem 4, and so D is quasi-Prüfer by Corollary 5.

(4) First, note that $\bar{D}$ is a valuation domain by (3), and hence $\bar{R}$ is also a valuation domain [10, Theorem 17.6]. Let Q be a prime ideal of R such that $Q \cap D = M$. Then $M \subseteq Q \subseteq Q\bar{R} \subsetneq \bar{R}$, and since $\sqrt{M\bar{D}}$ is a maximal ideal of $\bar{D}$ by Lemma 2(1), $\bar{R} \subseteq \bar{D}$ [10, Theorem 17.6]. Thus $\bar{R} = \bar{D}$. □

We next give the main result of this paper.

THEOREM 7. *The following statements are equivalent.*

(1) *D is an APVD and a quasi-Prüfer domain.*
(2) *D is an APVD and $\bar{D}$ is a valuation domain.*
(3) *$D(X)$ is an APVD.*

Proof. Let M be the maximal ideal of D, and note that $D(X)$ is quasi-local with maximal ideal $M(X) = MD(X)$ [10, Proposition 33.1].

(1) ⇔ (2) Theorem 4.

(2) ⇒ (3) Assume that D is an APVD such that $\bar{D}$ is a valuation domain. Then M is a primary ideal of $\bar{D}$ by Lemma 2(1); so the Nagata ring $\bar{D}(X)$ of $\bar{D}$ is a valuation domain [10, Proposition 18.7] and $M(X) = M\bar{D}(X)$ is a primary ideal of $\bar{D}(X)$ [10, Proposition 33.1(4)]. Thus $D(X)$ is an APVD by Lemma 1.

(3) ⇒ (1) Let $D(X)$ be an APVD. Then $M(X)$ is a strongly primary ideal of $D(X)$ by Lemma 1, and since $M(X) \cap K = M$ (cf. [10, Proposition 33.1(4)]), M is strongly primary. Thus D is an APVD by Lemma 1.

Next, assume to the contrary that D is not a quasi-Prüfer domain. Then, by Lemma 3, there exists a prime ideal Q of $D[X]$ such that $Q \subseteq M[X]$ and $(Q \cap D)[X] \subsetneq Q$. Let $N = \{g \in D[X] | c(g) = D\}$; then $Q_N \subsetneq M(X)$, and so Q_N is strongly prime by Lemma 2(2). Choose $f \in Q \setminus (Q \cap D)[X]$, and let a be a coefficient of f that is not in $Q \cap D$. Then $a \notin Q_N$ and $f = a\frac{f}{a} \in Q_N$, and hence $\frac{f}{a} \in Q_N \subseteq M(X)$, a contradiction. □

Corollary 8. *The following statements are equivalent.*

(1) *D is an LAPVD and a quasi-Prüfer domain.*

(2) *D is an LAPVD and $\bar{D}$ is a Prüfer domain.*

(3) *$D(X)$ is an LAPVD.*

Proof. (1) $\Leftrightarrow$ (2) Lemma 3.

(1) $\Rightarrow$ (3) Assume that D is an LAPVD and a quasi-Prüfer domain. Let Q be a maximal ideal of $D(X)$; then $Q = (Q \cap D)(X)$ with $Q \cap D$ maximal ideal of D [10, Proposition 33.1]. Note that $D_{Q\cap D}$ is an APVD and a quasi-Prüfer domain by Lemma 3. Thus $D(X)_Q = (D_{Q\cap D}[X])_{Q_{Q\cap D}} = D_{Q\cap D}(X)$, the Nagata ring of $D_{Q\cap D}$, is an APVD by Theorem 7.

(3) $\Rightarrow$ (1) Let P be a maximal ideal of D. Then $P(X)$ is a maximal ideal of $D(X)$ [10, Proposition 33.1]. Hence $(D(X))_{P(X)} = D[X]_{P[X]} = D_P(X)$, the Nagata ring of D_P, is an APVD. Thus D_P is an APVD and D_P is a quasi-Prüfer domain by Theorem 7. Thus D is an LAPVD and a quasi-Prüfer domain by Lemma 3. □

Proposition 9. *Let P be a prime ideal of D such that $P = PD_P$ and D_P is a valuation domain. Then D/P is an LAPVD if and only if D is an LAPVD.*

Proof. First, note that P is strongly prime, and if P is a maximal ideal, then D is a PVD. Hence we may assume that P is not maximal. Next, note that each maximal ideal of D/P is of the form M/P for some maximal ideal M of D such that $(D/P)_{M/P} \cong D_M/PD_M$ and $(D_M)_{PD_M} = D_P$. Thus D is an LAPVD if and only if D_M is an APVD for each maximal ideal M of D, if and only if D_M/PD_M is an APVD for each maximal ideal M of D [6, Theorem 3.4], if and only if D/P is an LAPVD. □

Corollary 10. *Let $V = F + M$ be a valuation domain and $R = D + M$, where F is a field, M is a nonzero maximal ideal of V, and D is a proper subring of F. Then R is an LAPVD if and only if either D is an LAPVD with $F = K$ or D is a field.*

Proof. ($\Rightarrow$) Let R be an LAPVD, and assume that D is not a field. Then M is not a maximal ideal of R. So $R_M = K + M$ is a valuation domain by Lemma 2(3); hence $F = K$ [3, Theroem 2.1]. Moreover, since $D \cong R/M$, D is an LAPVD by Proposition 9. ($\Leftarrow$) If D is a field, then R is a PVD [7, Proposition 4.9], and hence R is an LAPVD. Next, assume that D is an LAPVD with $F = K$. Note that $M = MR_M$, $D \cong R/M$, and $R_M = K + M$ is a valuation domain [3, Theroem 2.1]. Thus R is an LAPVD by Proposition 9. □

We end this paper by constructing an LAPVD that is neither an APVD nor an LPVD.

Example 11. (1) Let $\mathbb{Q}[\![t]\!]$ be the power series ring over the field $\mathbb{Q}$ of rational numbers, and let $D = \mathbb{Q}[\![t^2, t^3]\!]$. Then D is an APVD that is not a PVD and $\bar{D} = \mathbb{Q}[\![t]\!]$ [6, Example 2.1]. Clearly, $\bar{D}$ is a valuation domain, and thus $D(X)$ is an APVD by Theorem 7.

(2) Let the notation be as in (1) above. Let M be the maximal ideal of D. Let $S = D[X^2, X^3] \setminus (MD[X^2, X^3] \cup X^2D[X])$, and set $R = D[X^2, X^3]_S$. Then R is a one-dimensional semi-local Noetherian domain with maximal ideals MR and $X^2D[X]_S$. Note that $R_{MR} = D(X)$; so R_{MR} is an APVD by (1). Also, note that $R_{X^2D[X]_S} = D[X^2, X^3]_{X^2D[X]}$; $X^2D[X]_{X^2D[X]}$ is a maximal ideal of $D[X^2, X^3]_{X^2D[X]}$; and $D[X]_{XD[X]}$ is a one-dimensional valuation domain; Note that

$$(X^2D[X]_{X^2D[X]})D[X]_{XD[X]} = X^2D[X]_{XD[X]};$$

so $X^2D[X]_{XD[X]}$ is a primary ideal of $D[X]_{XD[X]}$. Thus $R_{X^2D[X]_S}$ is an APVD by Lemma 1. Therefore R is an LAPVD but not an LPVD.

References

[1] J. Arnold, *On the ideal theory of the Kronecker function ring and the domain $D(X)$*, Canad. J. Math. **21**(1969), 558-563.

[2] A. Badawi and E. G. Houston, *Powerful ideals, strongly primary ideals, almost pseudo-valuation domains, and conductive domains*, Comm. Algebra **30**(2002), 1591-1606.

[3] E. Bastida and R. Gilmer, *Overrings and divisorial ideals of the form $D + M$*, Michigan Math. J. **20**(1973), 79-95.

[4] G. W. Chang, *Locally pseudo-valuation domain of the form $D[X]_{N_v}$*, J. Korean Math. Soc. **45**(2008), 1405-1416.

[5] G. W. Chang and M. Fontana *Uppers to zero in polynomial rings and prüfer-like domains*, Comm. Algebra **37**(2009), 164-192.

[6] G. W. Chang, H. Nam and J. Park, *Strongly primary ideals*, in Arithmetical Properties of Commutative Rings and Monoids, Lecture Notes in Pure and Appl. Math., Chapman and Hall, **241**(2005), 378-386.

[7] D. E. Dobbs, *Coherence, ascent of going-down, and pseudo-valuation domains*, Houston J. Math. **4**(1978), 551-567.

[8] D. E. Dobbs and M. Fontana, *Locally pseudo-valuation domains*, Ann. Mat. Pura Appl.(4) **134**(1983), 147-168.

[9] M. Fontana, J. A. Huckaba and I. J. Papick, *Prüfer domains*, Marcel Dekker, New York, 1997.

[10] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, New York, 1972.

[11] J. R. Hedstrom and E. G. Housotn, *Pseudo-valuation domains*, Pacific J. Math. **75**(1978), 137-147.

[12] J. R. Hedstrom and E. G. Housotn, *Pseudo-valuation domains* (II), Houston J. Math. **4**(1978), 199-207.

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