RELATIVE ISOPERIMETRIC INEQUALITY FOR MINIMAL SUBMANIFOLDS IN SPACE FORMS

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Abstract. Let $C$ be a closed convex set in $\mathbb{S}^m$ or $\mathbb{H}^m$. Assume that $\Sigma$ is an $n$-dimensional compact minimal submanifold outside $C$ such that $\Sigma$ is orthogonal to $\partial C$ along $\partial\Sigma \cap \partial C$ and $\partial\Sigma$ lies on a geodesic sphere centered at a fixed point $p \in \partial\Sigma \cap \partial C$ and that $r$ is the distance in $\mathbb{S}^m$ or $\mathbb{H}^m$ from $p$. We make use of a modified volume $M_p(\Sigma)$ of $\Sigma$ and obtain a sharp relative isoperimetric inequality

$$\frac{1}{2}n^2 \omega_n M_p(\Sigma)^{n-1} \leq \text{Vol}(\partial\Sigma \sim \partial C)^n,$$

where $\omega_n$ is the volume of a unit ball in $\mathbb{R}^n$. Equality holds if and only if $\Sigma$ is a totally geodesic half ball centered at $p$.

1. Introduction

Let $\Sigma$ be a domain in a complete simply connected surface with constant Gaussian curvature $K$. The classical isoperimetric inequality says that

$$4\pi \text{Area}(\Sigma) - K \text{Area}(\Sigma)^2 \leq \text{Length}(\partial\Sigma)^2,$$

where equality holds if and only if $\Sigma$ is a geodesic disk. One natural way to extend this optimal inequality is to find the corresponding relative isoperimetric inequality. Let $C$ be a closed convex set in a complete simply connected surface $S$ with constant Gaussian curvature $K \leq 0$. It has been known that if $\Sigma$ is a relatively compact subset in $S \sim C$, then

$$(1.1) \quad 2\pi \text{Area}(\Sigma) - K \text{Area}(\Sigma)^2 \leq \text{Length}(\partial\Sigma \sim \partial C)^2,$$

where equality holds if and only if $\Sigma$ is a geodesic half disk [1]. Here $\sim$ denotes the set minus operator. The inequality (1.1) is called the
relative isoperimetric inequality for $\Sigma$. Recently this inequality has been generalized to various directions. (See [2, 3, 4, 6, 7, 8, 9].)

In this paper we study relative isoperimetric inequalities for an $n$-dimensional minimal submanifold $\Sigma$ outside a closed convex set $C$ in space forms. Under the assumption that the relative boundary $\partial \Sigma \sim \partial C$ lies on a geodesic sphere centered at $p \in \partial \Sigma \cap \partial C$, we prove a sharp relative isoperimetric inequality. More precisely our main theorem is stated as follows.

**Theorem.** Let $C$ be a closed convex set in $S^m$ or $H^m$. Assume that $\Sigma$ is an $n$-dimensional compact minimal submanifold outside $C$ such that $\Sigma$ is orthogonal to $\partial C$ along $\partial \Sigma \cap \partial C$ and $\partial \Sigma$ lies on a geodesic sphere centered at a fixed point $p \in \partial \Sigma \cap \partial C$ and that $r$ is the distance in $S^m$ or $H^m$ from $p$. Furthermore, in case of $\Sigma \subset S^m$, assume $r \leq \frac{\pi}{2}$. Then

$$\frac{1}{2} n^\ast \omega_n M_p(\Sigma)^{n-1} \leq \text{Vol}(\partial \Sigma \sim \partial C)^n,$$

where $\omega_n$ is the volume of a unit ball in $R^n$. Equality holds if and only if $\Sigma$ is a totally geodesic half ball centered at $p$.

2. Proof of main theorem

Let $p$ be a point in the $m$-dimensional sphere $S^m \subset R^{m+1}$ and let $r(x)$ be the distance from $p$ to $x$ in $S^m$. Choe and Gulliver [5] defined the modified volume $M_p(\Sigma)$ of $\Sigma$ with center at $p$ as

$$M_p(\Sigma) = \int_\Sigma \cos r.$$

Similarly for $\Sigma$ in the $m$-dimensional hyperbolic space $H^m$, they defined the modified volume of $\Sigma$ by

$$M_p(\Sigma) = \int_\Sigma \cosh r.$$

Using the concept of the modified volume, they were able to prove the isoperimetric inequalities for minimal submanifolds in $S^m$ or $H^m$. In order to prove our main theorem, we need the following monotonicity property which holds for minimal submanifolds outside a closed convex set in space forms.
Relative isoperimetric inequality for minimal submanifolds

Lemma 1. Let $C$ be a closed convex set in $\mathbb{S}^m$ or $\mathbb{H}^m$. Assume that $\Sigma$ is an $n$-dimensional compact minimal submanifold outside $C$ such that $\Sigma$ is orthogonal to $\partial C$ along $\partial \Sigma \cap \partial C$. Suppose that $r(\cdot) = \text{dist}(p, \cdot)$ in $\mathbb{S}^m$ or $\mathbb{H}^m$ for any $p \in \partial \Sigma \cap \partial C$. Denote by $B(p, r)$ the geodesic ball of radius $r$ centered at $p$.

(a) In case of $\Sigma$ in $\mathbb{S}^m$, for $0 < r < \min\{\frac{\pi}{2}, \text{dist}(p, \partial \Sigma \sim \partial C)\}$

$$\frac{M_p(\Sigma \cap B(p, r))}{\sin^n r}$$

is monotonically nondecreasing function of $r$.

(b) In case of $\Sigma$ in $\mathbb{H}^m$, for $0 < r < \text{dist}(p, \partial \Sigma \sim \partial C)$

$$\frac{M_p(\Sigma \cap B(p, r))}{\sinh^n r}$$

is monotonically nondecreasing function of $r$.

Proof. For (a), define $\Sigma_r = \Sigma \cap B(p, r)$. Then

$$M_p(\Sigma_r) = \int_{\Sigma_r} \cos r \leq -\frac{1}{n} \int_{\Sigma_r} \Delta \cos r$$

$$= \frac{1}{n} \int_{\partial \Sigma_r \sim \partial C} \sin r \frac{\partial r}{\partial \nu} + \frac{1}{n} \int_{\partial \Sigma_r \cap \partial C} \sin r \frac{\partial r}{\partial \nu}.$$ 

Since $\frac{\partial r}{\partial \nu} = \langle \nabla r, \nu \rangle \leq 0$ on $\partial \Sigma_r \cap \partial C$ by the orthogonality condition, one sees that

$$M_p(\Sigma_r) \leq \frac{1}{n} \int_{\partial \Sigma_r \sim \partial C} \sin r \frac{\partial r}{\partial \nu} = \frac{\sin r}{n} \int_{\partial \Sigma_r \sim \partial C} |\nabla r|.$$ 

Denote the volume forms on $\Sigma$ and $\partial \Sigma_r$ by $dv$ and $d\Sigma_r$, respectively. Then

$$dv = \frac{1}{|\nabla r|} d\Sigma_r dr.$$ 

Thus

$$\frac{d}{dr} \int_{\Sigma_r} \cos r |\nabla r|^2 dv = \frac{d}{dr} \int_0^r \int_{\partial \Sigma_r} \cos r |\nabla r| d\Sigma_r dr = \cos r \int_{\partial \Sigma_r} |\nabla r|.$$
Using the fact that \( r \leq \frac{\pi}{2} \) and \( |\nabla r| \leq 1 \) on \( \Sigma \), we get

\[
M_p(\Sigma_r) \leq \frac{1}{n} \sin r \cos r \int_{\Sigma_r} |\nabla r| \leq \frac{1}{n} \sin r \cos r \frac{d}{dr} \int_{\Sigma_r} \cos r |\nabla r|^2
\leq \frac{1}{n} \sin r \frac{d}{dr} \int_{\Sigma_r} \cos r
\leq \frac{1}{n} \sin r \frac{d}{dr} M_p(\Sigma_r).
\]

Therefore

\[
\frac{d}{dr} \log \left( \frac{M_p(\Sigma_r)}{\sin^n r} \right) \geq 0,
\]

which implies that the function \( \frac{M_p(\Sigma_r)}{\sin^n r} \) is monotonically nondecreasing. A similar proof holds for (b).

From the above monotonicity property, we can prove our main result about relative isoperimetric inequality for minimal submanifolds outside a convex set satisfying that the relative boundary \( \partial \Sigma \sim \partial C \) lies on a geodesic sphere.

**Theorem 2.** Let \( C \) be a closed convex set in \( S^m \) or \( \mathbb{H}^m \). Assume that \( \Sigma \) is an \( n \)-dimensional compact minimal submanifold outside \( C \) such that \( \Sigma \) is orthogonal to \( \partial C \) along \( \partial \Sigma \cap \partial C \) and \( \partial \Sigma \) lies on a geodesic sphere centered at a fixed point \( p \in \partial \Sigma \cap \partial C \) and that \( r \) is the distance in \( S^m \) or \( \mathbb{H}^m \) from \( p \). Furthermore, in case of \( \Sigma \subset S^m \), assume \( r \leq \frac{\pi}{2} \). Then

\[
\frac{1}{2} n \omega_n M_p(\Sigma)^{n-1} \leq \text{Vol}(\partial \Sigma \sim \partial C)^n,
\]

where \( \omega_n \) is the volume of a unit ball in \( \mathbb{R}^n \). Equality holds if and only if \( \Sigma \) is a totally geodesic half ball centered at \( p \).
Proof. Assume that $\Sigma \subset S^m$. Let $r(\cdot) = \text{dist}(p, \cdot)$ in $M$. Let $R$ be the radius of the geodesic sphere on which $\partial \Sigma \sim \partial C$ lies. It follows that

$$M_p(\Sigma) \leq -\frac{1}{n} \int_{\Sigma} \Delta \cos r$$

$$= \frac{1}{n} \int_{\partial \Sigma \sim \partial C} \sin r \frac{\partial r}{\partial \nu} + \frac{1}{n} \int_{\partial \Sigma \sim \partial C} \sin r \frac{\partial r}{\partial \nu}$$

$$\leq \frac{1}{n} \int_{\partial \Sigma \sim \partial C} \sin r \frac{\partial r}{\partial \nu}$$

$$= \frac{\sin R}{n} \int_{\partial \Sigma \sim \partial C} \frac{\partial r}{\partial \nu}$$

$$\leq \frac{\sin R}{n} \text{Vol}(\partial \Sigma \sim \partial C).$$

Since

$$\lim_{r \to 0} \frac{M_p(\Sigma \cap B(p, r))}{\sin^n r} = \frac{\omega_n}{2},$$

we see from Lemma 1 that

$$\frac{\omega_n}{2} \leq \frac{M_p(\Sigma)}{\sin^n R}.$$

Thus we have

$$M_p(\Sigma) \leq \left( \frac{2}{\omega_n} \right)^{\frac{1}{n}} (M_p(\Sigma))^\frac{1}{n} \frac{1}{n} \text{Vol}(\partial \Sigma \sim \partial C),$$

which gives the desired inequality. Moreover, equality holds if and only if $\Sigma$ is a cone with density at $p$ equal to 1 with constant sectional curvature 1 and $\partial \Sigma \cap \partial C$ is totally geodesic, or equivalently $\Sigma$ is a totally geodesic half ball.

Similarly one can prove the above theorem in case of $\Sigma \subset \mathbb{H}^m$. \qed

References


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