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RELATIVE ISOPERIMETRIC INEQUALITY FOR MINIMAL SUBMANIFOLDS IN SPACE FORMS

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Abstract. Let C be a closed convex set in $\mathbb{S}^m$ or $\mathbb{H}^m$. Assume that Σ is an n-dimensional compact minimal submanifold outside C such that Σ is orthogonal to ∂C along $\partial\Sigma \cap \partial C$ and $\partial\Sigma$ lies on a geodesic sphere centered at a fixed point $p \in \partial\Sigma \cap \partial C$ and that r is the distance in $\mathbb{S}^m$ or $\mathbb{H}^m$ from p. We make use of a modified volume $M_p(\Sigma)$ of Σ and obtain a sharp relative isoperimetric inequality

$$\frac{1}{2}n^n\omega_n M_p(\Sigma)^{n-1} \leq \mathrm{Vol}(\partial\Sigma \sim \partial C)^n,$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$. Equality holds if and only if Σ is a totally geodesic half ball centered at p.

1. Introduction

Let Σ be a domain in a complete simply connected surface with constant Gaussian curvature K. The classical isoperimetric inequality says that

$$4\pi\mathrm{Area}(\Sigma) - K\mathrm{Area}(\Sigma)^2 \leq \mathrm{Length}(\partial\Sigma)^2,$$

where equality holds if and only if Σ is a geodesic disk. One natural way to extend this optimal inequality is to find the corresponding relative isoperimetric inequality. Let C be a closed convex set in a complete simply connected surface S with constant Gaussian curvature $K \leq 0$. It has been known that if Σ is a relatively compact subset in $S \sim C$, then

$$2\pi\mathrm{Area}(\Sigma) - K\mathrm{Area}(\Sigma)^2 \leq \mathrm{Length}(\partial\Sigma \sim \partial C)^2, \tag{1.1}$$

where equality holds if and only if Σ is a geodesic half disk [1]. Here $\sim$ denotes the set minus operator. The inequality (1.1) is called the

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relative isoperimetric inequality for Σ. Recently this inequality has been generalized to various directions. (See [2, 3, 4, 6, 7, 8, 9].)

In this paper we study relative isoperimetric inequalities for an n-dimensional minimal submanifold Σ outside a closed convex set C in space forms. Under the assumption that the relative boundary $\partial\Sigma \sim \partial C$ lies on a geodesic sphere centered at $p \in \partial\Sigma \cap \partial C$, we prove a sharp relative isoperimetric inequality. More precisely our main theorem is stated as follows.

Theorem. *Let C be a closed convex set in $\mathbb{S}^m$ or $\mathbb{H}^m$. Assume that Σ is an n-dimensional compact minimal submanifold outside C such that Σ is orthogonal to ∂C along $\partial\Sigma \cap \partial C$ and $\partial\Sigma$ lies on a geodesic sphere centered at a fixed point $p \in \partial\Sigma \cap \partial C$ and that r is the distance in $\mathbb{S}^m$ or $\mathbb{H}^m$ from p. Furthermore, in case of $\Sigma \subset \mathbb{S}^m$, assume $r \leq \frac{\pi}{2}$. Then*

$$\frac{1}{2}n^n\omega_n M_p(\Sigma)^{n-1} \leq \mathrm{Vol}(\partial\Sigma \sim \partial C)^n,$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$. Equality holds if and only if Σ is a totally geodesic half ball centered at p.

2. Proof of main theorem

Let p be a point in the m-dimensional sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ and let $r(x)$ be the distance from p to x in $\mathbb{S}^m$. Choe and Gulliver [5] defined the *modified volume* $M_p(\Sigma)$ of Σ with center at p as

$$M_p(\Sigma) = \int_\Sigma \cos r.$$

Similarly for Σ in the m-dimensional hyperbolic space $\mathbb{H}^m$, they defined the *modified volume* of Σ by

$$M_p(\Sigma) = \int_\Sigma \cosh r.$$

Using the concept of the modified volume, they were able to prove the isoperimetric inequalities for minimal submanifolds in $\mathbb{S}^m$ or $\mathbb{H}^m$. In order to prove our main theorem, we need the following monotonicity property which holds for minimal submanifolds outside a closed convex set in space forms.

LEMMA 1. *Let C be a closed convex set in $\mathbb{S}^m$ or $\mathbb{H}^m$. Assume that Σ is an n-dimensional compact minimal submanifold outside C such that Σ is orthogonal to ∂C along $\partial\Sigma \cap \partial C$. Suppose that $r(\cdot) = \operatorname{dist}(p, \cdot)$ in $\mathbb{S}^m$ or $\mathbb{H}^m$ for any $p \in \partial\Sigma \cap \partial C$. Denote by $B(p,r)$ the geodesic ball of radius r centered at p.*
(a) *In case of Σ in $\mathbb{S}^m$, for $0 < r < \min\{\frac{\pi}{2}, \operatorname{dist}(p, \partial\Sigma \sim \partial C)\}$*

$$\frac{M_p(\Sigma \cap B(p,r))}{\sin^n r}$$

is monotonically nondecreasing function of r.
(b) *In case of Σ in $\mathbb{H}^m$, for $0 < r < \operatorname{dist}(p, \partial\Sigma \sim \partial C)$*

$$\frac{M_p(\Sigma \cap B(p,r))}{\sinh^n r}$$

is monotonically nondecreasing function of r.

Proof. For (a), define $\Sigma_r = \Sigma \cap B(p,r)$. Then

$$\begin{aligned} M_p(\Sigma_r) = \int_{\Sigma_r} \cos r &\le -\frac{1}{n}\int_{\Sigma_r} \Delta \cos r \\ &= \frac{1}{n}\int_{\partial\Sigma_r \sim \partial C} \sin r \frac{\partial r}{\partial \nu} + \frac{1}{n}\int_{\partial\Sigma_r \cap \partial C} \sin r \frac{\partial r}{\partial \nu}. \end{aligned}$$

Since $\frac{\partial r}{\partial \nu} = \langle \nabla r, \nu \rangle \le 0$ on $\partial\Sigma_r \cap \partial C$ by the orthogonality condition, one sees that

$$M_p(\Sigma_r) \le \frac{1}{n}\int_{\partial\Sigma_r \sim \partial C} \sin r \frac{\partial r}{\partial \nu} = \frac{\sin r}{n}\int_{\partial\Sigma_r \sim \partial C} |\nabla r|.$$

Denote the volume forms on Σ and $\partial\Sigma_r$ by dv and $d\Sigma_r$, respectively. Then

$$dv = \frac{1}{|\nabla r|} d\Sigma_r dr.$$

Thus

$$\frac{d}{dr}\int_{\Sigma_r} \cos r |\nabla r|^2 dv = \frac{d}{dr}\int_0^r \int_{\partial\Sigma_r} \cos r |\nabla r| d\Sigma_r dr = \cos r \int_{\partial\Sigma_r} |\nabla r|.$$

Using the fact that $r \le \frac{\pi}{2}$ and $|\nabla r| \le 1$ on Σ, we get

$$\begin{aligned}
M_p(\Sigma_r) &\le \frac{1}{n}\frac{\sin r}{\cos r}\cos r\int_{\partial\Sigma_r}|\nabla r| \\
&= \frac{1}{n}\frac{\sin r}{\cos r}\cos r\frac{d}{dr}\int_{\Sigma_r}\cos r|\nabla r|^2 \\
&\le \frac{1}{n}\frac{\sin r}{\cos r}\frac{d}{dr}\int_{\Sigma_r}\cos r \\
&= \frac{1}{n}\frac{\sin r}{\cos r}\frac{d}{dr}M_p(\Sigma_r).
\end{aligned}$$

Therefore

$$\frac{d}{dr}\log\left(\frac{M_p(\Sigma_r)}{\sin^n r}\right) \ge 0,$$

which implies that the function $\dfrac{M_p(\Sigma_r)}{\sin^n r}$ is monotonically nondecreasing. A similar proof holds for (b). □

From the above monotonicity property, we can prove our main result about relative isoperimetric inequality for minimal submanifolds outside a convex set satisfying that the relative boundary $\partial\Sigma \sim \partial C$ lies on a geodesic sphere.

THEOREM 2. *Let C be a closed convex set in $\mathbb{S}^m$ or $\mathbb{H}^m$. Assume that Σ is an n-dimensional compact minimal submanifold outside C such that Σ is orthogonal to ∂C along $\partial\Sigma \cap \partial C$ and $\partial\Sigma$ lies on a geodesic sphere centered at a fixed point $p \in \partial\Sigma \cap \partial C$ and that r is the distance in $\mathbb{S}^m$ or $\mathbb{H}^m$ from p. Furthermore, in case of $\Sigma \subset \mathbb{S}^m$, assume $r \le \frac{\pi}{2}$. Then*

$$\frac{1}{2}n^n\omega_n M_p(\Sigma)^{n-1} \le \mathrm{Vol}(\partial\Sigma \sim \partial C)^n,$$

where ω_n is the volume of a unit ball in $\mathbb{R}^n$. Equality holds if and only if Σ is a totally geodesic half ball centered at p.

Proof. Assume that $\Sigma \subset \mathbb{S}^m$. Let $r(\cdot) = \mathrm{dist}(p, \cdot)$ in M. Let R be the radius of the geodesic sphere on which $\partial\Sigma \sim \partial C$ lies. It follows that

$$\begin{aligned}
M_p(\Sigma) &\leq -\frac{1}{n}\int_\Sigma \Delta \cos r \\
&= \frac{1}{n}\int_{\partial\Sigma\sim\partial C} \sin r \frac{\partial r}{\partial \nu} + \frac{1}{n}\int_{\partial\Sigma\cap\partial C} \sin r \frac{\partial r}{\partial \nu} \\
&\leq \frac{1}{n}\int_{\partial\Sigma\sim\partial C} \sin r \frac{\partial r}{\partial \nu} \\
&= \frac{\sin R}{n}\int_{\partial\Sigma\sim\partial C} \frac{\partial r}{\partial \nu} \\
&\leq \frac{\sin R}{n}\mathrm{Vol}(\partial\Sigma \sim \partial C).
\end{aligned}$$

Since

$$\lim_{r\to 0} \frac{M_p(\Sigma \cap B(p,r))}{\sin^n r} = \frac{\omega_n}{2},$$

we see from Lemma 1 that

$$\frac{\omega_n}{2} \leq \frac{M_p(\Sigma)}{\sin^n R}.$$

Thus we have

$$M_p(\Sigma) \leq \left(\frac{2}{\omega_n}\right)^{\frac{1}{n}} (M_p(\Sigma))^{\frac{1}{n}} \frac{1}{n}\mathrm{Vol}(\partial\Sigma \sim \partial C),$$

which gives the desired inequality. Moreover, equality holds if and only if Σ is a cone with density at p equal to 1 with constant sectional curvature 1 and $\partial\Sigma \cap \partial C$ is totally geodesic, or equivalently Σ is a totally geodesic half ball.

Similarly one can prove the above theorem in case of $\Sigma \subset \mathbb{H}^m$. $\square$

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