PROJECTIVE PROPERTIES OF REPRESENTATIONS
OF A QUIVER $Q = \bullet \to \bullet$ AS $R[x]$-MODULES

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Abstract. In this paper we extend the projective properties of
representations of a quiver $Q = \bullet \to \bullet$ as left $R$-modules to the
projective properties of representations of quiver $Q = \bullet \to \bullet$ as left
$R[x]$-modules. We show that if $P$ is a projective left $R$-module then
$0 \to P[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as
$R[x]$-modules. And we show $0 \to L$ is a projective representation of
$Q = \bullet \to \bullet$ as $R$-module if and only if $0 \to L[x]$ is a projective rep-
resentation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules. Then we show
if $P$ is a projective left $R$-module then $P[x] \overset{id}{\longrightarrow} P[x]$ is a projec-
tive representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules. We also
show that if $L \overset{id}{\longrightarrow} L$ is a projective representation of $Q = \bullet \to \bullet$
as $R$-module, then $L[x] \overset{id}{\longrightarrow} L[x]$ is a projective representation of
a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules.

1. Introduction

A quiver is just a directed graph with vertices and edges (arrows) ([1]).
We may consider many different types of quivers. We allow multiple ed-
ges and multiple arrows, and edges going from a vertex back to the same
vertex. Originally a representation of quiver assigned a vector space to
each vertex - and a linear map to each edge (or arrow) - with the linear
map going from the vector space assigned to the initial vertex to the one
assigned to the terminal vertex. For example, a representation of the

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quiver $Q = \bullet \rightarrow \bullet$ is $V_1 \xrightarrow{f} V_2$, $V_1$ and $V_2$ are vector spaces and $f$ is a linear map (morphism). Then we extend this representation to the left $R$-modules, a representation of the quiver $Q = \bullet \rightarrow \bullet$ is $M_1 \xrightarrow{\phi} M_2$, $M_1$ and $M_2$ are left $R$-modules and $\phi$ is an $R$-linear map.

If $M$ is a left $R$-module, then the polynomial $M[x]$ is a left $R[x]$-module defined by

$$r(m_0 + m_1x + m_2x^2 + \cdots + m_ix^i) = rm_0 + rm_1x + rm_2x^2 + \cdots + rm_ix^i$$

$$x(m_0 + m_1x + m_2x^2 + \cdots + m_ix^i) = m_0x + m_1x^2 + m_2x^3 + \cdots + m_ix^{i+1}.$$  

We call $M[x]$ as a polynomial module. Similarly we can define the power series $M[[x]]$ as a left $R[x]$-module and we call a power series module.

Now we can define $R$-linear maps between these two representations. $R$-linear maps of $M_1 \xrightarrow{f} M_2$ to $N_1 \xrightarrow{g} N_2$ are given by a commutative diagram

$$
\begin{array}{ccc}
M_1 & \xrightarrow{f} & M_2 \\
\downarrow{s_1} & & \downarrow{s_2} \\
N_1 & \xrightarrow{g} & N_2
\end{array}
$$

with $s_1, s_2$ $R$-linear maps.

In (3) a homotopy of quivers was developed and in (2) cyclic quiver ring was studied. The theory of projective representations was developed in (4) and the theory of injective representation was studied in (5). Recently, in (7) injective covers and envelopes of representations of linear quivers was studied, and in (6) properties of multiple edges of quivers was studied.

**Definition 1.1.** (8) A left $R$-module $P$ is said to be projective if given any surjective linear map $\sigma : M' \rightarrow M$ and any linear map $h : P \rightarrow M$, there is a linear map $g : P \rightarrow M'$ such that $\sigma \circ g = h$. That is

$$
\begin{array}{ccc}
\ & P & \\
\ & \downarrow{g} & \\
M' & \xrightarrow{\sigma} & M \\
\ & \downarrow{h} & \\
& & 0
\end{array}
$$
can always be completed to a commutative diagram.

**Definition 1.2.** ([4]) Let $P_1, P_2, M_1, M_2, N_1$ and $N_2$ be left $R$-modules. A representation $P_1 \rightarrow P_2$ of a quiver $Q = \bullet \rightarrow \bullet$ is called a projective representation if every diagram of representations

$$(P_1 \xrightarrow{g} P_2)$$

$$(M_1 \xrightarrow{\alpha} M_2) \rightarrow (N_1 \xrightarrow{\beta} N_2) \rightarrow (0 \rightarrow 0)$$

can be completed to a commutative diagram as follows:

$$(P_1 \xrightarrow{g} P_2)$$

$$(M_1 \xrightarrow{\alpha} M_2) \rightarrow (N_1 \xrightarrow{\beta} N_2) \rightarrow (0 \rightarrow 0)$$

**2. Projective representation of a quiver $Q = \bullet \rightarrow \bullet$ as $R[x]$-modules**

**Theorem 2.1.** If $P$ is a projective left $R$-module, then $0 \rightarrow P[x]$ is a projective representation of a quiver $Q = \bullet \rightarrow \bullet$ as $R[x]$-modules.

**Proof.** Let $M_1, M_2, N_1$ and $N_2$ be left $R[x]$-modules, and $\alpha : M_1 \rightarrow N_1$ and $\beta : M_2 \rightarrow N_2$ be onto $R[x]$-linear maps, and $f : R[x] \rightarrow N_2$ be a $R[x]$-linear map. Consider the following diagram

$$(0 \rightarrow P[x])$$

$$(M_1 \xrightarrow{\tilde{g}} M_2) \rightarrow (N_1 \xrightarrow{\tilde{h}} N_2) \rightarrow (0 \rightarrow 0).$$

Since $P$ is a projective left $R$-module, there exists an $R$-linear map $t : P \rightarrow M_2$ such that $\beta \circ t = \tilde{f}|_P$.

Define $\tilde{t} : P[x] \rightarrow M_2$ by $\tilde{t}(p_0 + p_1x + \cdots + p_nx^n) = t(p_0) + t(p_1)x + \cdots + t(p_n)x^n$. Then
\[ \beta \circ t(p_0 + p_1 x + \cdots + p_n x^n) \]
\[ = \beta(t(p_0) + t(p_1)x + \cdots + t(p_n)x^n) \]
\[ = (\beta \circ t)(p_0) + (\beta \circ t)(p_1)x + \cdots + (\beta \circ t)(p_n)x^n \]
\[ = \bar{f}_p(p_0) + \bar{f}_p(p_1)x + \cdots + \bar{f}_p(p_n)x^n \]
\[ = \bar{f}(p_0 + p_1 x + \cdots + p_n x^n). \]

So we have \( \beta \circ \bar{t} = \bar{f} \). Therefore, we can complete the following diagram as a commutative diagram. Hence, \( 0 \rightarrow P[x] \) is a projective representation of a quiver \( Q = \bullet \rightarrow \bullet \) as \( R[x] \)-modules.

We can extend above result to the power series modules.

**Corollary 2.2.** If \( P \) is a projective left \( R \)-module, then \( 0 \rightarrow P[[x]] \) is a projective representation of a quiver \( Q = \bullet \rightarrow \bullet \) as \( R[x] \)-modules.

**Example 2.3.** Let \( R = \mathbb{Z}_6 \), then \( P = \mathbb{Z}_2 \) is a projective \( \mathbb{Z}_6 \)-module. \( 0 \rightarrow \mathbb{Z}_2[x] \) is a projective representation of a quiver \( Q = \bullet \rightarrow \bullet \) as \( \mathbb{Z}_6[x] \)-modules.

**Theorem 2.4.** \( 0 \rightarrow L \) is a projective representation of \( Q = \bullet \rightarrow \bullet \) as \( R \)-modules if and only if \( 0 \rightarrow L[x] \) is a projective representation of a quiver \( Q = \bullet \rightarrow \bullet \) as \( R[x] \)-modules.

**Proof.** Let \( M_1, M_2, N_1 \) and \( N_2 \) be left \( R[x] \)-modules, and \( \alpha : M_1 \rightarrow N_1 \) and \( \beta : M_2 \rightarrow N_2 \) be onto \( R[x] \)-linear maps, and \( f : L[x] \rightarrow N_2 \) be a \( R[x] \)-linear map.

Consider the following diagram

\[
\begin{array}{cccccccccccc}
0 & \\
\downarrow & & & & & & & & & & & & \downarrow \beta \\
M_1 & \rightarrow & M_2 & \rightarrow & N_1 & \rightarrow & N_2 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
(0 & \rightarrow & L[x]) & \\
\downarrow & & & & & & & & & & & & \downarrow \alpha \\
M_1 & \rightarrow & M_2 & \rightarrow & N_1 & \rightarrow & N_2 & \rightarrow & 0 & \rightarrow & 0. \\
\end{array}
\]
Since $0 \to L$ is a projective representation, we can complete the following diagram:

\[
\begin{array}{c}
0 \to L \\
(M_1 \to M_2) \to (N_1 \to N_2) \to (0 \to 0)
\end{array}
\]

Define $\bar{t}: L[x] \to M_2$ by $\bar{t}(n_0 + n_1 x + \cdots + n_n x^n) = t(n_0) + t(n_1) x + \cdots + t(n_n) x^n$. Then

\[
\begin{align*}
\beta \circ \bar{t}(n_0 + n_1 x + \cdots + n_n x^n) \\
= \beta(t(n_0) + t(n_1) x + \cdots + t(n_n) x^n) \\
= (\beta \circ t)(p_0) + (\beta \circ t)(p_1)x + \cdots + (\beta \circ t)(p_n)x^n \\
= f(n_0) + f(n_1)x + \cdots + f(n_n)x^n \\
= f(n_0 + n_1 x + \cdots + n_n x^n).
\end{align*}
\]

So we have the following diagram by $\bar{t}: L[x] \to M_2$:

\[
\begin{array}{c}
0 \to L[x] \\
(M_1 \to M_2) \to (N_1 \to N_2) \to (0 \to 0)
\end{array}
\]

as a commutative diagram. Hence, $0 \to L[x]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules.

Conversely, Let $M_1, M_2, N_1$ and $N_2$ be left $R$-modules, and $\alpha: M_1 \to N_1$ and $\beta: M_2 \to N_2$ be onto $R$-linear maps, and $f: L \to N_2$ be a $R$-linear map. Consider the following diagram:

\[
\begin{array}{c}
0 \to L \\
(M_1 \to M_2) \to (N_1 \to N_2) \to (0 \to 0)
\end{array}
\]

Since $0 \to L[x]$ is a projective representation, we can complete the following diagram by $\tilde{t}: L[x] \to M_2[x]$

\[
\begin{array}{c}
0 \quad (0 \to L) \\
(M_1 \to M_2) \quad (N_1 \to N_2) \quad (0 \to 0)
\end{array}
\]

where $\tilde{f}: L[x] \to M_2[x]$ by $\tilde{f}(n_0 + n_1 x + \cdots + n_j x^j) = f(n_0) + f(n_1)x + \cdots + f(n_j)x^j$. and $\tilde{\beta}: M_2[x] \to N_2[x]$ by $\tilde{\beta}(m_0 + m_1 x + \cdots + m_i x^i) = \beta(m_0) + \beta(m_1)x + \cdots + \beta(m_i)x^i$. Define $t: L \to M_2$ by $t(n_0) = m_0$. Let $n_0 \in L$ then $\beta \circ t(n_0) = \beta(m_0)$. Since

\[
\begin{align*}
\tilde{\beta} \circ \tilde{t}(n_0) &= \tilde{\beta}(m_0 + m_1 x + \cdots + m_i x^i) \\
&= \beta(m_0) + \beta(m_1)x + \cdots + \beta(m_i)x^i \\
&= \tilde{f}(n_0) = f(n_0),
\end{align*}
\]

$\beta(m_1), \cdots, \beta(m_i) = 0$. So $\beta(m_0) = f(n_0)$. Therefore $\beta \circ t(n_0) = f(n_0)$.

So we have the following diagram

\[
\begin{array}{c}
0 \quad (0 \to L) \\
(M_1 \to M_2) \quad (N_1 \to N_2) \quad (0 \to 0)
\end{array}
\]

as a commutative diagram. Hence, $0 \to L$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R$-modules.

\[\square\]

Corollary 2.5. $0 \to L$ is a projective representation of $Q = \bullet \to \bullet$ as $R$-modules if and only if $0 \to L[[x]]$ is a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[[x]]$-modules.

Remark 2.6. $P[x] \to 0$ is not a projective representation of a quiver $Q = \bullet \to \bullet$ as $R[x]$-modules if $P \neq 0$, because the following diagram
If \( t \) is a projective left \( R \)-module, then \( P[x] \xrightarrow{id} P[x] \) is a projective representation of a quiver \( Q = \bullet \to \bullet \) as \( R[x]\)-modules if \( P \neq 0 \).

**Theorem 2.7.** If \( P \) is a projective left \( R \)-module, then \( P[x] \xrightarrow{id} P[x] \) is a projective representation of a quiver \( Q = \bullet \to \bullet \) as \( R[x]\)-modules.

**Proof.** Let \( M_1, M_2, N_1 \) and \( N_2 \) be left \( R[x]\)-modules and let \( g : M_1 \to M_2 \) and \( h : N_1 \to N_2 \) be \( R[x]\)-linear maps. Let \( \alpha : M_1 \to N_1 \), \( \beta : M_2 \to N_2 \) be onto \( R[x]\)-linear maps. Let \( k : P[x] \to N_1 \) be an \( R[x]\)-linear map and choose \( h \circ k : P[x] \to N_2 \) as an \( R[x]\)-linear map. And consider the following diagram:

\[
\begin{array}{c}
(M_1 \xrightarrow{g} M_2) \\
\downarrow k \\
(N_1 \xrightarrow{h} N_2)
\end{array}
\to
\begin{array}{c}
(P[x] \xrightarrow{id} P[x]) \\
\downarrow h \circ k \\
(0 \xrightarrow{0} 0)
\end{array}
\]

Since \( P \) is a projective left \( R \)-module, there exist \( R \)-linear maps \( s : P \to M_1 \) and \( t : P \to M_2 \) such that \( \alpha \circ s = k|_P \) and \( \beta \circ t = h \circ k|_P \).

Define \( \bar{s} : P[x] \to M_1 \) by \( \bar{s}(p_0 + p_1 x + \cdots + p_n x^n) = s(p_0) + s(p_1)x + \cdots + s(p_n)x^n \). Then

\[
\begin{align*}
\alpha \circ \bar{s}(p_0 + p_1 x + \cdots + p_n x^n) & = \alpha(s(p_0) + s(p_1)x + \cdots + s(p_n)x^n) \\
& = (\alpha \circ s)(p_0) + (\alpha \circ s)(p_1)x + \cdots + (\alpha \circ s)(p_n)x^n \\
& = k|_P(p_0) + k|_P(p_1)x + \cdots + k|_P(p_n)x^n \\
& = k(p_0 + p_1 x + \cdots + p_n x^n).
\end{align*}
\]

Define \( \bar{t} : P[x] \to M_2 \) by \( \bar{t}(p_0 + p_1 x + \cdots + p_n x^n) = t(p_0) + t(p_1)x + \cdots + t(p_n)x^n \). Then similarly we have \( \beta \circ \bar{t}(p_0 + p_1 x + \cdots + p_n x^n) = \)
\[ (h \circ k)(p_0 + p_1x + \cdots + p_nx^n). \] So we have the following diagram by \( \tilde{s} : P[x] \to M_1 \) and \( \tilde{t} : P[x] \to M_2 \)

\[
\begin{array}{cccc}
\ gens & \text{id} & P[x] \\
\downarrow \tilde{s} & \downarrow k & \downarrow h & \downarrow \text{id} \\
\mathcal{P}[x] & M_1 & N_1 & \mathcal{P}[x] \\
\downarrow g & \downarrow h & \downarrow \text{id} & \downarrow \text{id} \\
\mathcal{P}[x] & M_2 & N_2 & \mathcal{P}[x] \\
\end{array}
\]

as a commutative diagram. Hence, \( P[x] \xrightarrow{\text{id}} P[x] \) is a projective representation of a quiver \( Q = \bullet \to \bullet \) as \( \mathbb{R}[x] \)-modules.

**Corollary 2.8.** If \( P \) is a projective left \( \mathbb{R} \)-module, then \( P[[x]] \xrightarrow{\text{id}} P[[x]] \) is a projective representation of a quiver \( Q = \bullet \to \bullet \) as \( \mathbb{R}[x] \)-modules.

**Example 2.9.** Let \( R = \mathbb{Z}_6 \), then \( P = \mathbb{Z}_2 \) is a projective \( \mathbb{Z}_6 \)-module. \( \mathbb{Z}_2[x] \xrightarrow{\text{id}} \mathbb{Z}_2[x] \) is a projective representation of a quiver \( Q = \bullet \to \bullet \) as \( \mathbb{Z}_6[x] \)-modules.

**Theorem 2.10.** If \( L \xrightarrow{\text{id}} L \) is a projective representation of \( Q = \bullet \to \bullet \) as \( \mathbb{R} \)-modules, then \( L[x] \xrightarrow{\text{id}} L[x] \) is a projective representation of a quiver \( Q = \bullet \to \bullet \) as \( \mathbb{R}[x] \)-modules.

**Proof.** Let \( M_1, M_2, N_1, N_2 \) be left \( \mathbb{R}[x] \)-modules, and \( \alpha : M_1 \to N_1 \) and \( \beta : M_2 \to N_2 \) be onto \( \mathbb{R}[x] \)-linear maps, and consider the following diagram

\[
\begin{array}{cccc}
\ gens & \text{id} & L[x] \\
\downarrow k & \downarrow \text{id} & \downarrow \text{id} \\
\mathcal{P}[x] & M_1 & N_1 & \mathcal{P}[x] \\
\downarrow g & \downarrow h & \downarrow \text{id} & \downarrow \text{id} \\
\mathcal{P}[x] & M_2 & N_2 & \mathcal{P}[x] \\
\end{array}
\]

Since \( L \xrightarrow{\text{id}} L \) is a projective representation, we can complete the following diagram
Quiver $Q = \bullet \rightarrow \bullet$

\[ \begin{array}{c}
\begin{tikzpicture}
\node (L) at (0,0) {$L$};
\node (M) at (-2,-2) {$(M_1 \xrightarrow{g} M_2)$};
\node (N) at (2,-2) {$(N_1 \xrightarrow{h} N_2)$};
\node (00) at (4,-2) {$(0 \rightarrow 0)$};
\node (id) at (0,-0.5) {$(L \xrightarrow{id} L)$};
\node (s) at (-1.5,-1) {$(s)$};
\node (t) at (1.5,-1) {$(t)$};
\node (id_p) at (2.5,-0.5) {$(\text{id}_p)$};
\node (hok_p) at (4.5,-0.5) {$(h \text{id}_p)$};
\draw[->] (L) -- (M) node [midway, above] {$s$};
\draw[->] (M) -- (N) node [midway, above] {$t$};
\draw[->] (N) -- (00) node [midway, above] {$\text{id}_p$};
\draw[->] (L) -- (id);\draw[->] (id) -- (00) node [midway, above] {$\text{id}$};
\draw[->] (id_p) -- (hok_p);\draw[->] (hok_p) -- (00);\draw[->] (s) -- (id_p);\draw[->] (t) -- (hok_p);
\end{tikzpicture}
\end{array} \]

as a commutative diagram.

Define $\bar{s} : L[x] \rightarrow M_1$ by $\bar{s}(n_0 + n_1 x + \cdots + n_i x^i) = s(n_0) + s(n_1) x + \cdots + s(n_i) x^i$ and $\bar{t} : L[x] \rightarrow M_2$ by $\bar{t}(n_0 + n_1 x + \cdots + n_j x^j) = t(n_0) + t(n_1) x + \cdots + t(n_j) x^j$. Then we see that the following by $\bar{s} : L[x] \rightarrow M_1$ and $\bar{t} : L[x] \rightarrow M_2$

\[ \begin{array}{c}
\begin{tikzpicture}
\node (Lx) at (0,0) {$L[x]$};
\node (Mx) at (-2,-2) {$(M_1 \xrightarrow{g} M_2)$};
\node (Nx) at (2,-2) {$(N_1 \xrightarrow{h} N_2)$};
\node (00) at (4,-2) {$(0 \rightarrow 0)$};
\node (idx) at (0,-0.5) {$(L[x] \xrightarrow{id} L[x])$};
\node (sb) at (-1.5,-1) {$(\bar{s})$};
\node (tb) at (1.5,-1) {$(\bar{t})$};
\node (hok) at (2.5,-0.5) {$(h \text{id})$};
\draw[->] (Lx) -- (Mx) node [midway, above] {$\bar{s}$};
\draw[->] (Mx) -- (Nx) node [midway, above] {$\bar{t}$};
\draw[->] (Nx) -- (00) node [midway, above] {$h \text{id}$};
\draw[->] (Lx) -- (idx);\draw[->] (idx) -- (00) node [midway, above] {$\text{id}$};
\draw[->] (hok) -- (00);\draw[->] (hok) -- (00);\draw[->] (sb) -- (idx);\draw[->] (tb) -- (hok);
\end{tikzpicture}
\end{array} \]

as a commutative diagram. Hence, $L[x] \rightarrow L[x]$ is a projective representation of a quiver $Q = \bullet \rightarrow \bullet$ as $R[x]$-modules.

**Corollary 2.11.** If $L \xrightarrow{id} L$ is a projective representation of $Q = \bullet \rightarrow \bullet$ as $R$-module, then $L[[x]] \xrightarrow{id} L[[x]]$ is a projective representation of a quiver $Q = \bullet \rightarrow \bullet$ as $R[x]$-modules.

\[ \square \]

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