Korean J. Math. **18** (2010), No. 4, pp. 335–342

ON CHARACTERIZATIONS OF PRÜFER v-MULTIPLICATION DOMAINS

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Abstract. Let D be an integral domain with quotient field K, $\mathcal{I}(D)$ be the set of nonzero ideals of D, and w be the star-operation on D defined by $I_w = \{x \in K | xJ \subseteq I$ for some $J \in \mathcal{I}(D)$ such that J is finitely generated and $J^{-1} = D\}$. The D is called a Prüfer v-multiplication domain if $(II^{-1})_w = D$ for all nonzero finitely generated ideals I of D. In this paper, we show that D is a Prüfer v-multiplication domain if and only if $(A \cap (B + C))_w = ((A \cap B) + (A \cap C))_w$ for all $A, B, C \in \mathcal{I}(D)$, if and only if $(A(B \cap C))_w = (AB \cap AC)_w$ for all $A, B, C \in \mathcal{I}(D)$, if and only if $((A+B)(A \cap B))_w = (AB)_w$ for all $A, B \in \mathcal{I}(D)$, if and only if $((A + B) : C)_w = ((A : C) + (B : C))_w$ for all $A, B, C \in \mathcal{I}(D)$ with C finitely generated, if and only if $((a : b) + (b : a))_w = D$ for all nonzero $a, b \in D$, if and only if $(A : (B \cap C))_w = ((A : B) + (A : C))_w$ for all $A, B, C \in \mathcal{I}(D)$ with B, C finitely generated.

1. Introduction

Let D be an integral domain with quotient field K. Let $\mathcal{I}(D)$ be the set of nonzero ideals of D and let $F(D)$ be the set of nonzero fractional ideals of D; so $\mathcal{I}(D) = \{I \in F(D) | I \subseteq D\}$. A map $* : F(D) \to F(D)$, $I \mapsto I_*$, is called a *star-operation on* D if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in F(D)$;

(1) $(aD)_* = aD$ and $(aI)_* = aI_*$,
(2) $I \subseteq I_*$ and if $I \subseteq J$, then $I_* \subseteq J_*$, and
(3) $(I_*)_* = I_*$.

Given a star-operation $*$ on D, we can construct two new star-operations $*_f$ and $*_w$ on D as follows; for each $I \in F(D)$, $I_{*_f} = \cup\{J_* | J \subseteq I$ and J

Received July 12, 2010. Revised November 6, 2010. Accepted November 11, 2010.
2000 Mathematics Subject Classification: 13A15, 13F05, 13G05.
Key words and phrases: Prüfer domain, Prüfer v-multiplication domain, the w-operation.

is a nonzero finitely generated ideal of $D\}$ and $I_{*_w} = \{x \in K | xJ \subseteq I$ for J a nonzero finitely generated ideal of D with $J_* = D\}$. An $I \in F(D)$ is said to be $*$-invertible if $(II^{-1})_* = D$, where $I^{-1} = \{x \in K | xI \subseteq D\}$. An $I \in F(D)$ is called a $*$-ideal if $I_* = I$; so I_* is a $*$-ideal, while a $*$-ideal is a maximal $*$-ideal if it is maximal among proper integral $*$-ideals. Let $*$-Max(D) denote the set of all maximal $*$-ideals of D. It is well known that $*_f$-Max$(D) \neq \emptyset$ if D is not a field; $*_f$-Max$(D) = *_w$-Max(D) [1, Theorem 2.1]; each maximal $*$-ideal is a prime ideal; and $D = \cap_{P \in *_f\text{-Max}(D)} D_P$.

The v-operation is a star-operation defined by $I_v = (I^{-1})^{-1}$, the t-operation is defined by $t = v_f$, and the w-operation is by $w = v_w$. The d-operation is just the identity map on $F(D)$, i.e., $I_d = I$ for all $I \in F(D)$; so $d = d_f = d_w$. Clearly, $I_d \subseteq I_w \subseteq I_t \subseteq I_v$ for all $I \in F(D)$. An overring of D means a ring between D and K. We say that an overring R of D is t-linked over D if $I_v = D$ implies $(IR)_v = R$ for all nonzero finitely generated ideals I of D.

We say that D is a *valuation domain* if either $x \in D$ or $x^{-1} \in D$ for all nonzero $x \in K$. Hence if $A = (a_1, \ldots, a_n)$ is an ideal of a valuation domain D, then $\{a_iD\}$ is linearly ordered under inclusion; so $A = a_iV$ for some i. Thus, each finitely generated ideal of a valuation domain is principal. Also, if A, B are ideals of a valuation domain, then either $A \subseteq B$ or $B \subseteq A$. The D is called a *Prüfer domain* (resp., *Prüfer v-multiplication domain* (PvMD)) if each nonzero finitely generated ideal I of D is invertible (resp., t-invertible), i.e., $II^{-1} = D$ (resp., $(II^{-1})_t = D$). Clearly, $I \in F(D)$ is t-invertible if and only if $II^{-1} \nsubseteq P$ for all maximal t-ideals P of D. Hence t-Max$(D) = w$-Max(D) implies that I is t-invertible if and only if I is w-invertible. Thus, D is a PvMD if and only if each nonzero finitely generated ideal of D is w-invertible.

It is well known that D is a Prüfer domain if and only if D_M is a valuation domain for all maximal ideals M of D, if and only if two generated nonzero ideal of D is invertible, if and only if each overring of D is integrally closed, if and only if each overring of D is a Prüfer domain (see, for example, [5, Sections 22 - 28]). The theory of PvMDs runs along lines parallel to that of Prüfer domains. For example, D is a PvMD if and only if D_P is a valuation domain for all maximal t-ideals P of D, if and only if two generated nonzero ideal of D is t-invertible, if and only if each t-linked overring of D is integrally closed, if and only if each t-linked overring of D is a PvMD (cf. [3, 4, 6, 9] and Lemma 4).

Let X be an indeterminate over D. For each polynomial $f \in K[X]$, the content of f, denoted by A_f, is the fractional ideal of D generated by the coefficients of f. We know that D is a Prüfer domain if and only if $A_f A_g = A_{fg}$ for all $0 \neq f, g \in D[X]$ and that D is integrally closed if and only if $(A_f A_g)_t = (A_{fg})_t$ for all $0 \neq f, g \in D[X]$. Clearly, if D is a PvMD, then $(A_f A_g)_t = (A_{fg})_t$ for all $0 \neq f, g \in D[X]$, but since an integrally closed domain need not be a PvMD, the converse does not hold. In [3, Corollary 3.7], Chang proved that D is a PvMD if and only if $(A_f A_g)_w = (A_{fg})_w$ for all $0 \neq f, g \in D[X]$.

The followings are other characterizations of Prüfer domains, which are due to Jensen 1963 [7]. (Note that $(A : B) = \{x \in D | xB \subseteq A\}$ for ideals A and B of D.)

Theorem 1. *The following statements are equivalent for an integral domain D.*

(1) *D is a Prüfer domain.*
(2) $A \cap (B + C) = (A \cap B) + (A \cap C)$ *for all* $A, B, C \in \mathcal{I}(D)$.
(3) $A(B \cap C) = AB \cap AC$ *for all* $A, B, C \in \mathcal{I}(D)$.
(4) $(A + B)(A \cap B) = AB$ *for all* $A, B \in \mathcal{I}(D)$.
(5) $(A+B) : C = A : C + B : C$ *for all* $A, B, C \in \mathcal{I}(D)$ *with C finitely generated.*
(6) $(a : b) + (b : a) = D$ *for all* $0 \neq a, b \in D$.
(7) $(A : (B \cap C)) = (A : B) + (A : C)$ *for all* $A, B, C \in \mathcal{I}(D)$ *with B, C finitely generated.*

The purpose of this paper is to give the PvMD analog of Theorem 1, which also gives new characterizations of PvMDs.

2. Characterizations of PvMDs

Let D denote an integral domain with quotient field K. In this section, we use the w-operation to characterize PvMDs. We first need some lemmas (Lemmas 2-4), which are already well known, but we give their proofs for the completeness of this paper.

Lemma 2. [1, Corollary 2.13] *Let I and J be nonzero ideals of D.*

(1) $I_w = \cap_{P \in t\text{-Max}(D)} ID_P$.
(2) $I_w D_P = ID_P$ *for all maximal t-ideals P of D.*
(3) $I_w = J_w$ *if and only if* $ID_P = JD_P$ *for all maximal t-ideals P of D.*

Proof. (1) ($\subseteq$) If $x \in I_w$, then there exists a nonzero finitely generated ideal A of D such that $A^{-1} = D$ and $xA \subseteq I$. Hence $x \in xD_P = xAD_P \subseteq ID_P$ for all maximal t-ideals P of D. ($\supseteq$) For $a \in \cap_{P \in t\text{-Max}(D)} ID_P$, let $A = \{b \in D | ba \in I\}$. Then $aA \subseteq I$ and $A \nsubseteq P$ for all $P \in t$-Max(D). So $A_t = D$, and hence there is a nonzero finitely generated ideal B of D such that $B \subseteq A$ and $B^{-1} = B_v = D$. Thus, $a \in I_w$, because $aB \subseteq aA \subseteq I$.

(2) and (3) These are immediate consequences of (1). □

Lemma 3. [5, Theorems 4.3 and 4.4] *Let S be a multiplicative subset of D, and let A, B be nonzero ideals of D.*

(1) $(A + B)D_S = AD_S + BD_S$.
(2) $(AB)D_S = (AD_S)(BD_S)$.
(3) $(A \cap B)D_S = AD_S \cap BD_S$.
(4) *If I is an ideal of D_S, then $I = (I \cap D)D_S$.*
(5) *If B is finitely generated, then $(A : B)D_S = (AD_S : BD_S)$.*

Proof. (1) and (2) are clear. (3) Since $A \cap B \subseteq AD_S \cap BD_S$, we have $(A \cap B)D_S \subseteq (AD_S \cap BD_S)D_S = AD_S \cap BD_S$. Conversely, if $x = \frac{a}{s} = \frac{b}{t} \in AD_S \cap BD_S$, where $a \in A, b \in B$ and $s, t \in S$, then $at = bs \in A \cap B$; hence $x = \frac{at}{st} = \frac{bs}{st} \in (A \cap B)D_S$.

(4) If $x \in I \subseteq D_S$, then there is an $s \in S$ such that $sx \in I \cap D$. Hence $x = \frac{sx}{s} \in (I \cap D)D_S$. Conversely, since $I \cap D \subseteq I$, we have $(I \cap D)D_S \subseteq ID_S = I$.

(5) If $x \in (A : B)$, then $xB \subseteq A$; hence $xBD_S \subseteq AD_S$. So $x \in (AD_S : BD_S)$. Hence $(A : B) \subseteq (AD_S : BD_S)$, and thus $(A : B)D_S \subseteq (AD_S : BD_S)$. Conversely, if $y \in (AD_S : BD_S)$, then $yB \subseteq yBD_S \subseteq AD_S$. Note that, since B is finitely generated, there exists an $s \in S$ such that $syB \subseteq A$. Hence $sy \in (A : B)$, and thus $y \in (A : B)D_S$. □

Lemma 4. [6, Theorem 5] *The following statements are equivalent for D.*

(1) *D is a PvMD.*
(2) *Each nonzero two generated ideal of D is t-invertible.*
(3) *D_P is a valuation domain for all maximal t-ideals P of D.*

Proof. (1) $\Rightarrow$ (2) Clear. (2) $\Rightarrow$ (3) Let $x = \frac{a}{b} \in K$ be nonzero, where $a, b \in D$, and let P be a maximal t-ideal of D. Since $((a,b)(a,b)^{-1})_t = D$, we have $(a,b)(a,b)^{-1} \nsubseteq P$. Hence

$$D_P = ((a,b)(a,b)^{-1})D_P = ((a,b)D_P)((a,b)^{-1}D_P)$$
$$\subseteq ((a,b)D_P)((a,b)D_P)^{-1} \subseteq D_P,$$

and so $((a,b)D_P)((a,b)D_P)^{-1} = D_P$. So $(a,b)D_P$ is invertible, and since D_P is quasi-local, $(a,b)D_P = aD_P$ or $(a,b)D_P = bD_P$ [5, Proposition 7.4]. Thus, $x = \frac{a}{b} \in D_P$ or $x^{-1} = \frac{b}{a} \in D_P$.

(3) $\Rightarrow$ (1) Let I be a nonzero finitely generated ideal of D, and let P be a maximal t-ideal of D. Then ID_P is principal, and hence $D_P = (ID_P)(ID_P)^{-1} = (ID_P)(I^{-1}D_P) = (II^{-1})D_P$ [9, Lemma 1.4], or $II^{-1} \nsubseteq P$. Thus $(II^{-1})_t = D$. □

Let $N_v = \{f \in D[X] | (A_f)_v = D\}$. Then N_v is a saturated multiplicative subset of $D[X]$, and the ring $D[X]_{N_v}$, called the ($v-$)Nagata ring, has many interesting ring-theoretic properties (cf. [8, 2]). For example, each invertible ideal of $D[X]_{N_v}$ is principal [8, Theorem 2.14] and $I[X]_{N_v} \cap K = I_w$ and $I_w[X]_{N_v} = I[X]_{N_v}$ for all nonzero fractional ideals I of D [2, Lemma 2.1]. Also, D is a PvMD if and only if $D[X]_{N_v}$ is a Prüfer domain [8, Theorem 3.7].

Lemma 5. *If $A, B \in \mathcal{I}(D)$, then*

(1) $A[X]_{N_v} + B[X]_{N_v} = (A+B)[X]_{N_v}$,
(2) $A[X]_{N_v} \cap B[X]_{N_v} = (A \cap B)[X]_{N_v}$,
(3) $(A[X]_{N_v}) \cdot (B[X]_{N_v}) = (AB)[X]_{N_v}$, *and*
(4) $(A[X]_{N_v} : B[X]_{N_v}) = (A : B)[X]_{N_v}$ *if B is finitely generated.*

Proof. (1), (2) and (3) Clear (cf. Lemma 3). (4) If $a \in (A : B)$, then $aB \subseteq A$, and hence $aB[X]_{N_v} \subseteq A[X]_{N_v}$. Thus, $a \in (A[X]_{N_v} : B[X]_{N_v})$, and so $(A : B)[X]_{N_v} \subseteq (A[X]_{N_v} : B[X]_{N_v})$. For the reverse containment, let $B = (a_1, \ldots, a_n)$. If $u \in D[X]$ such that $uB[X]_{N_v} \subseteq A[X]_{N_v}$, then $ua_i \in A[X]_{N_v}$ for $i = 1, \ldots, n$. Hence there is an $f_i \in N_v$ with $uf_ia_i \in A[X]$. So if we set $f = f_1 \cdots f_n$, then $ufB \subseteq A[X]$, and so $A_{uf}B \subseteq A$. Hence $A_{uf} \subseteq (A : B) \Rightarrow uf \in (A : B)[X]$, $\Rightarrow u \in (A : B)[X]_{N_v}$. Thus, $(A[X]_{N_v} : B[X]_{N_v}) \subseteq (A : B)[X]_{N_v}$. □

We next give the main result of this paper, whose proofs heavily depend on the proofs of [5, Theorem 25.2], and in its proofs we use the results of Lemmas 2, 3, and 4 without any comment.

Theorem 6. *The following statements are equivalent for an integral domain D.*

(1) *D is a PvMD.*
(2) $(A \cap (B+C))_w = ((A \cap B) + (A \cap C))_w$ *for all* $A, B, C \in \mathcal{I}(D)$.
(3) $(A(B \cap C))_w = (AB \cap AC)_w$ *for all* $A, B, C \in \mathcal{I}(D)$.
(4) $((A+B)(A \cap B))_w = (AB)_w$ *for all* $A, B \in \mathcal{I}(D)$.
(5) $((A+B) : C)_w = ((A : C) + (B : C))_w$ *for all* $A, B, C \in \mathcal{I}(D)$ *with* C *finitely generated.*
(6) $((a : b) + (b : a))_w = D$ *for all nonzero* $a, b \in D$.
(7) $(A : (B \cap C))_w = ((A : B) + (A : C))_w$ *for all* $A, B, C \in \mathcal{I}(D)$ *with* B, C *finitely generated.*
(8) $A[X]_{N_v} \cap (B[X]_{N_v} + C[X]_{N_v}) = (A[X]_{N_v} \cap B[X]_{N_v}) + (A[X]_{N_v} \cap C[X]_{N_v})$ *for all* $A, B, C \in \mathcal{I}(D)$.
(9) $A[X]_{N_v} \cdot (B[X]_{N_v} \cap C[X]_{N_v}) = (A[X]_{N_v} \cdot B[X]_{N_v}) \cap (A[X]_{N_v} \cdot C[X]_{N_v})$ *for all* $A, B, C \in \mathcal{I}(D)$.
(10) $(A[X]_{N_v} + B[X]_{N_v})(A[X]_{N_v} \cap B[X]_{N_v}) = A[X]_{N_v} \cdot B[X]_{N_v}$ *for all* $A, B \in \mathcal{I}(D)$.
(11) $((A[X]_{N_v} + B[X]_{N_v}) : C[X]_{N_v}) = (A[X]_{N_v} : C[X]_{N_v}) + (B[X]_{N_v} : C[X]_{N_v})$ *for all* $A, B, C \in \mathcal{I}(D)$ *with* C *finitely generated.*
(12) $(aD[X]_{N_v} : bD[X]_{N_v}) + (bD[X]_{N_v} : aD[X]_{N_v}) = D[X]_{N_v}$ *for all nonzero* $a, b \in D$.

Proof. Let P be a maximal t-ideal of D. Hence if D is a PvMD, the D_P is a valuation domain by Lemma 4.

(1) $\Rightarrow$ (2) We may assume $BD_P \subseteq CD_P$, because D_P is a valuation domain. Hence $(A \cap (B+C))D_P = AD_P \cap (BD_P + CD_P) = AD_P \cap CD_P = (AD_P \cap BD_P) + (AD_P \cap CD_P) = ((A \cap B) + (A \cap C))D_P$. Thus, by Lemma 2, we have $(A \cap (B+C))_w = ((A \cap B) + (A \cap C))_w$.

(2) $\Rightarrow$ (6) For any nonzero $a, b \in D$, we have $a \in aD_P \cap ((a-b)D_P + bD_P) = ((a) \cap ((a-b) + (b)))D_P = ((a) \cap ((a-b) + (b)))_w D_P = (((a) \cap (a-b)) + ((a) \cap (b)))_w D_P = (((a) \cap (a-b)) + ((a) \cap (b)))D_P = (aD_P \cap (a-b)D_P) + (aD_P \cap bD_P)$. Hence $a = (a-b)x + y$, or $xb = a(x-1) + y$ for some $x \in D_P$ and $y \in aD_P \cap bD_P$. Thus, $x \in (aD_P : bD_P)$, while $a(1-x) = y - bx \in bD_P$; so $1 - x \in (bD_P : aD_P)$. Hence $1 = x + (1-x) \in (aD_P : bD_P) + (bD_P : aD_P) = ((a : b) + (b : a))D_P$. Thus, $1 \in \cap_{P \in t\text{-Max}(D)} ((a : b) + (b : a))D_P = ((a : b) + (b : a))_w$, and so $((a : b) + (b : a))_w = D$.

(6) $\Rightarrow$ (1) Let $a, b \in D$ be nonzero. Since $((a : b) + (b : a))_w = D$, we have $(a : b) \nsubseteq P$ or $(b : a) \nsubseteq P$. If $(a : b) \nsubseteq P$, then $D_P = (a : b)D_P = (aD_P : bD_P)$; so $b \in aD_P$. Similarly, $(b : a) \nsubseteq P$ implies $a \in bD_P$. Hence D_P is a valuation domain, and thus D is a PvMD.

(1) $\Rightarrow$ (3) Assume $BD_P \subseteq CD_P$, because D_P is a valuation domain. Hence $(A(B \cap C))D_P = AD_P(BD_P \cap CD_P) = (AD_P)(BD_P) = (AD_P)(BD_P) \cap (AD_P)(CD_P) = (AB \cap AC)D_P$. Thus, $(A(B \cap C))_w = (AB \cap AC)_w$.

(3) $\Rightarrow$ (4) By (3), we have $((A + B)(A \cap B))_w = ((A + B)A \cap (A + B)B)_w \supseteq (AB)_w$. Conversely, $(A+B)(A\cap B) = A(A\cap B)+B(A\cap B) \subseteq AB$, and hence $((A+B)(A\cap B))_w \subseteq (AB)_w$. Thus, $((A+B)(A\cap B))_w = (AB)_w$

(4) $\Rightarrow$ (1) For any nonzero $a, b \in D$, we have $((a, b)((a) \cap (b))_w = (ab)_w = (ab)$ by (4), and since (ab) is t-invertible, (a, b) is also t-invertible. Thus, D is a PvMD.

(1) $\Rightarrow$ (5) Assume $AD_P \subseteq BD_P$, because D_P is a valuation domain. Then $(AD_P : CD_P) \subseteq (BD_P : CD_P)$, and hence $((A + B) : C)D_P = ((AD_P + BD_P) : CD_P) = (BD_P : CD_P) = (AD_P : CD_P) + (BD_P : CD_P) = ((A : C) + (B : C))D_P$. Thus, $((A+B) : C)_w = ((A : C) + (B : C))_w$.

(5) $\Rightarrow$ (6) $((a : b) + (b : a))_w = (((a) : (a, b)) + ((b) : (a, b)))_w = (((a) + (b)) : (a, b))_w = D$.

(1) $\Rightarrow$ (7) First, since $B\cap C \subseteq B$ and $B\cap C \subseteq C$, we have $(A : B)+(A : C) \subseteq (A : (B\cap C))$; hence $((A : B)+(A : C))_w \subseteq (A : (B\cap C))_w$. For the reverse containment, assume $BD_P \subseteq CD_P$, because D_P is a valuation domain. Hence $(A : (B \cap C))D_P \subseteq (AD_P : (B \cap C)D_P) = (AD_P : (BD_P \cap CD_P)) = (AD_P : BD_P) = (AD_P : BD_P) + (AD_P : CD_P) = ((A : B) + (A : C))D_P$. Thus $(A : (B \cap C))_w = ((A : B) + (A : C))_w$.

(7) $\Rightarrow$ (6) $D = (((a)\cap(b)) : ((a)\cap(b)))_w = ((((a)\cap(b)) : (a))+(((a)\cap(b)) : (b)))_w = (((a) : (b)) + ((b) : (a)))_w$.

(1) $\Rightarrow$ (8), (9), (10), (11) and (12). These follow directly from Theorem 1, because $D[X]_{N_v}$ is a Prüfer domain.

(8) $\Rightarrow$ (2) By (8) and Lemma 5, $(A \cap (B + C))[X]_{N_v} = ((A \cap B) + (A \cap C))[X]_{N_v}$. Thus, $(A \cap (B + C))_w = (A \cap (B + C))[X]_{N_v} \cap K = ((A \cap B) + (A \cap C))[X]_{N_v} \cap K = ((A \cap B) + (A \cap C))_w$ [2, Lemma 2.1].

(9) $\Rightarrow$ (3), (10) $\Rightarrow$ (4), (11) $\Rightarrow$ (5), (12) $\Rightarrow$ (6). These can be proved by the same way as the proof of (8) $\Rightarrow$ (2) above. □

Remark 7. Let $(i)_w$ denote the condition $(i)X_w = Y_w$ of Theorem 6, and let $(i)_t$ be the condition $X_t = Y_t$.

(1) It is known that D is a PvMD if and only if D is integrally closed and $t = w$ [8, Theorem 3.4]. Hence if D is a PvMD, then the $(2)_t$ holds,

i.e., $(A \cap (B+C))_t = ((A \cap B) + (A \cap C))_t$ for all $A, B, C \in \mathcal{I}(D)$. (Also, D being a PvMD implies the $(3)_t$, $(4)_t$, $(5)_t$, $(6)_t$ and $(7)_t$.)

(2) Since t-Max$(D) = w$-Max(D), we have $A_w = D \Leftrightarrow A_t = D$ for $A \in \mathcal{I}(D)$. Thus by the (1) $\Leftrightarrow$ (6) of Theorem 6, D is a PvMD if and only if the $(6)_t$ holds, i.e., $((a : b) + (b : a))_t = D$ for all $a, b \in D$. However, we don't know if the $(2)_t, (3)_t$, $(4)_t$, $(5)_t$ or $(7)_t$ imply that D is a PvMD.

Acknowledgement The author would like to thank the referee for his/her useful suggestions.

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